Computational and Applied Mathematics (2021) 40:262
https://doi.org/10.1007/s40314-021-01653-w

A different monotone iterative technique for a class of nonlinear three-point BVPs

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Received: 15 April 2021 / Revised: 7 September 2021 / Accepted: 20 September 2021 /

Abstract
This work examines the existence of the solutions of a class of three-point nonlinear boundary value problems that arise in bridge design due to its nonlinear behavior. A maximum and anti-maximum principles are derived with the support of Green's function and their constant sign. A different monotone iterative technique is developed with the use of lower solution $x(z)$ and upper solution $y(z)$. We have also discussed the classification of well ordered $(x \leq y)$ and reverse ordered $(y \leq x)$ cases for both positive and negative values of $\sup\left(\frac{\partial f}{\partial w}\right)$. Established results are verified with the help of some examples.

Keywords Monotone iterative technique · Reversed ordered upper–lower solutions · Three point BVPs · Bridge design · Nonlinear ODEs · Green's function

Mathematics Subject Classification 34L30 · 34B27 · 34B15

1 Introduction

The study of bridge designs have their own importance, like suspension bridge has nonlinear behaviors (such as large oscillation, traveling wave) that are very difficult to analyze. McKenna and Lazer (1990) show that linear model is inadequate to describe this type of nonlinear behavior. Several nonlinear models are summed up in Drábek et al. (2003) (see

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Communicated by Jose Alberto Cuminato.

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also references therein) that describes the behavior of suspension bridges. In past years, there are several collapse of suspension bridge, e.g., Tacoma Narrows Bridge, Golden Gate Bridge, Brooklyn Bridge etc. If we mention the collapse of Tacoma Narrows Bridge, it happened due to a jumping or asymmetric type of non-linearity. Drábek et al. (2003) further concluded that if a system is asymmetric and having a large uni-directional load then the system shows multiple oscillatory solutions, i.e., oscillations are directly proportional to asymmetry and unidirectional load.

The motivation came from paper (McKenna and Lazer 1990), where authors have studied the existence and multiplicity of periodic solutions of possible mathematical models for the nonlinear behavior of a suspension bridge. In this work, they considered the road-bed as a one-dimensional vibrating beam which is governed by the following equations

$$\begin{aligned} &w_{tt} + EIw_{zzzz} + \delta w_t = -kw^+ + W(z) + \varepsilon f(z,t),\\ &w(0,t) = w(L,t) = w_{zz}(0,t) = w_{zz}(L,t) = 0. \end{aligned}$$

Here L is length of beam, $w(z,t)$ is downward deflection, k is spring constant, $W(z)$ is weight per unit length of the bridge pushing it down, and $\varepsilon f(z,t)$ is the external forcing term. If $W(z) = W_0 \sin(z/L)$, $f(z,t) = f(t)\sin(z/L)$, and $w(z,t) = w(t)\sin(z/L)$, then we get

$$w''(z) = f(z, w, w'), \tag{1.1}$$

where

$$f(z,w,w') = \begin{cases} \delta w' + EI\left(\frac{\pi}{L}\right)^4 w + kw - W_0 - \varepsilon, & w > 0,\\ \delta w' + EI\left(\frac{\pi}{L}\right)^4 w - W_0 - \varepsilon, & w < 0. \end{cases}$$

In Geng and Cui (2010), Verma and Singh (2014), Zou et al. (2007), authors have discussed that large size bridges are often constructed with multi-point supports which refers to multi-point boundary conditions. To highlight the position or angle of the bridge, different types of boundary conditions can be taken near the endpoints.

Nonlinear boundary value problems (NLBVPs) have been discussed by many researchers in recent decades (Verma et al. 2020), like shooting method (Taliaferro 1979), topological degree method (Lloyd 1978), topological transversality (Granas 1976), theory of fixed point index (Webb 2012), upper–lower solutions method (Coster and Habets 2006), monotone iterative techniques (MIT) (Cherpion et al. 2001), Quasilinearization (O'Regan and El-Gebeily 2008) etc.

Literature shows that the coupled technique, monotone iterative technique (MIT) in the presence of upper–lower solutions is an efficient method for the study of two point as well as multi-points NLBVPs (Cherpion et al. 2001). The concept of MIT was introduced by Picard (1893). Later Gendzojan (1964), discussed the coupled technique for the following second-order two-point BVPs,

$$w''(z) + f(z, w, w') = 0, \quad w(a) = 0, \quad w(b) = 0. \tag{1.2}$$

Here nonlinear function f is dependent on the derivative of solution w. The approximating scheme for the above problem (1.2) is as follows,

$$\begin{aligned} &-x_n'' + \mu(z)x_n' + \gamma(z)x_n = f(z, x_{n-1}, x_{n-1}') + \mu(z)x_{n-1}' + \gamma(z)x_{n-1},\\ &\quad x_n(a) = 0, \quad x_n(b) = 0, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &-y_n'' + \mu(z)y_n' + \gamma(z)y_n = f(z, y_{n-1}, y_{n-1}') + \mu(z)y_{n-1}' + \gamma(z)y_{n-1},\\ &\quad y_n(a) = 0, \quad y_n(b) = 0, \end{aligned} \tag{1.4}$$

where $\mu(z)$ and $\gamma(z)$ are functions of z related to f. For Dirichlet BVPs Bernfeld and Chandra (1977), considered the following iterative scheme,

$$-w_n''(z) + \lambda w_n(z) = f(z, w_{n-1}, w_n') + \lambda w_{n-1}(z), \quad w_n(a) = w_n(b) = 0.$$

A different concept was introduced by Omari (1986) for Dirichlet BVPs, where he assumed that the nonlinear function $f(z, w, w')$ is one sided Lipschitz in w and Lipschitz in w'. Omari used the following approximation scheme

$$\begin{aligned} &w_n''(z) - 2k|w_n'(z) - w_{n-1}'(z)| + \lambda w_n(z) = f(z, w_{n-1}, w_{n-1}') + \lambda w_{n-1}(z), \\ &w_n(a) = w_n(b) = 0. \end{aligned}$$

MIT in the presence of upper and lower solutions are studied by several researchers (see book Coster and Habets 2006). In all the studies, the usual order $(x \leq y)$ is considered, where x and y are lower and upper solutions respectively. In reverse order $(x \geq y)$ case for two point BVPs, first study was done by Amann et al. (1978). Omari and Trombetta (1992) used MIT with upper and lower solutions, when they appear in reverse order. In this study, they have considered the following periodic BVPs

$$-w''(z) + cw'(z) + f(z, w) = 0, \quad w(a) = w(b), \quad w'(a) = w'(b),$$

and the following approximations scheme

$$\begin{aligned} &-w_n''(z) + cw_n'(z) + Kw_n(z) = -f(z, w_{n-1}) + Kw_{n-1}(z), \\ &w_n(a) = w_n(b), \quad w_n'(a) = w_n'(b). \end{aligned}$$

Cabada et al. (2001) studied the existence and approximation of solutions for the Neumann two-point BVPs by using lower and upper solutions in reverse ordered case. They have developed the following approximation scheme,

$$\begin{aligned} &w_n'' - 2k|w_n' - w_{n-1}'| + \gamma w_n = f(z, w_{n-1}, w_{n-1}') + \gamma w_{n-1}, \\ &w_n'(a) = w_n'(b) = 0. \end{aligned} \tag{1.5}$$

Recently, this coupled technique is also successively used for the existence of solution of three point or multi-point BVPs, like Li et al. (2008) discussed the existence and uniqueness results for the class of three point NLBVPs with the help of MIT and upper–lower solutions. Recently, authors (Singh and Verma 2013), used the following approximation scheme for the second order differential equation with different types of boundary conditions

$$-w_{n+1}''(z) - \lambda w_{n+1}(z) = f(z, w_n, w_n') - \lambda w_n.$$

In this article, we introduce a different type of monotone iterative technique, as follows

$$\begin{aligned} &-w_{n+1}''(z) - \mu w_{n+1}'(z) - \gamma w_{n+1}(z) = f(z, w_n, w_n') - \mu w_n' - \gamma w_n, \\ &w_{n+1}'(0) = 0, \quad w_{n+1}(1) = \delta w_{n+1}(\eta), \end{aligned}$$

and establish the existence of solution for the following three-point NLBVPs

$$w''(z) + f(z, w, w') = 0, \quad z \in I_0 = (0, 1), \tag{1.6}$$

$$w'(0) = 0, \quad w(1) = \delta w(\eta), \tag{1.7}$$

where $I = [0, 1]$, $\eta \in I_0$, $\delta > 0$, and $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Making use of some sufficient conditions, uniform convergent sequences are generated with the support of maximum and anti-maximum principles. The distinction of well and reverse ordered cases are also analyzed for $\sup\left(\frac{\partial f}{\partial w}\right) > 0$ and $\sup\left(\frac{\partial f}{\partial w}\right) < 0$.

This paper is organized in the following manner. In Sect. 2, we describe some preliminary results. Section 3 deals the Green's function and its constant sign. In Sect. 4, maximum, anti maximum principle and upper–lower solutions are discussed. Sections 5 and 6 are used to establish the main result and for the construction of examples. In Sect. 7, we have concluded the paper and in the appendix, we have drawn a flow chart (Fig. 6) which simplifies the results we have obtained in this article.

2 Preliminaries

This section describes the linear model of three-point NLBVPs (1.6) and (1.7). Consider the non-homogeneous three-point linear BVPs,

$$-w''(z) - \mu w'(z) - \gamma w(z) = h(z), \quad z \in I_0, \tag{2.1}$$

$$w'(0) = 0, \quad w(1) = \delta w(\eta) + b, \tag{2.2}$$

where b is any constant, $h \in C(I)$, $\gamma \in \mathbb{R}$, and μ is some positive real number.

To solve this problem, we consider the following Cauchy problem

$$w''(z) + \mu w'(z) + \gamma w(z) = 0, \quad z \in I_0, \tag{2.3}$$

$$w'(0) = 0, \quad w(1) = \delta w(\eta). \tag{2.4}$$

The solutions of problem (2.3) and (2.4) are described as follows:

1. If $\mu^2 - 4\gamma = -k^2 < 0$, then the solution will be
$$w(z) = \mathrm{e}^{-\frac{\mu z}{2}}\left[c_1 \cos\left(\frac{kz}{2}\right) + c_2 \sin\left(\frac{kz}{2}\right)\right]. \tag{2.5}$$
2. If $\mu^2 - 4\gamma = k^2 > 0$, then the solution will be
$$w(z) = \mathrm{e}^{-\frac{\mu z}{2}}\left[c_1 \cosh\left(\frac{kz}{2}\right) + c_2 \sinh\left(\frac{kz}{2}\right)\right]. \tag{2.6}$$
3. If $\mu^2 - 4\gamma = k^2 = 0$, then the solution will be
$$w(z) = \mathrm{e}^{-\frac{\mu z}{2}}\left[c_1 + c_2 z\right]. \tag{2.7}$$

   Here k is some positive real number.

3 Derivation of solution for linear BVPs

This section provides the solution of linear three point BVPs (2.1) and (2.2). Based on the solutions of the problem (2.3) and (2.4), we derive the Green's function $g(z, t)$ for the following three cases

(I) $\mu^2 - 4\gamma = -k^2 < 0$,
(II) $\mu^2 - 4\gamma = k^2 > 0$,
(III) $\mu^2 - 4\gamma = k^2 = 0$.

3.1 Case I: $\mu^2 - 4\gamma = -k^2 < 0$

(A_0): Suppose that the following inequalities hold

(a) there exists $\gamma > 0$ and $k \in (0, \frac{\pi}{2})$ such that $\mu^2 - 4\gamma = -k^2 < 0$;

(b) $e^{\frac{\eta\mu}{2}} \left(\mu \sin\left(\frac{k}{2}\right) + k\cos\left(\frac{k}{2}\right)\right) - \delta e^{\mu/2}\left(\mu\sin\left(\frac{\eta k}{2}\right) + k\cos\left(\frac{\eta k}{2}\right)\right) < 0, \delta e^{\mu/2}\sin\left(\frac{\eta k}{2}\right) - e^{\frac{\eta\mu}{2}}\sin\left(\frac{k}{2}\right) \leq 0$.

Lemma 3.1 *The solution of nonhomogeneous linear BVPs* (2.1) *and* (2.2) *is given by*

$$w(z) = \frac{b e^{-\frac{\mu z}{2}} e^{\frac{\mu(1+\eta)}{2}} \left(\mu\sin\left(\frac{kz}{2}\right) + k\cos\left(\frac{kz}{2}\right)\right)}{e^{\frac{\eta\mu}{2}}\left(\mu\sin\left(\frac{k}{2}\right) + k\cos\left(\frac{k}{2}\right)\right) - \delta e^{\mu/2}\left(\mu\sin\left(\frac{\eta k}{2}\right) + k\cos\left(\frac{\eta k}{2}\right)\right)} - \int_0^1 g(z,t)h(t)\,dt, \tag{3.1}$$

where $g(z,t)$ is the Green's function of (2.3) *and* (2.4), *which is defined as*

$$g(z,t) = \begin{cases} \dfrac{2e^{\frac{1}{2}\mu(t-z)}\left(\mu\sin\left(\frac{kz}{2}\right)+k\cos\left(\frac{kz}{2}\right)\right)\left(\delta e^{\mu/2}\sin\left(\frac{1}{2}k(\eta-t)\right)+e^{\frac{\eta\mu}{2}}\sin\left(\frac{1}{2}k(t-1)\right)\right)}{k\left(e^{\frac{\eta\mu}{2}}\left(\mu\sin\left(\frac{k}{2}\right)+k\cos\left(\frac{k}{2}\right)\right)-\delta e^{\mu/2}\left(\mu\sin\left(\frac{\eta k}{2}\right)+k\cos\left(\frac{\eta k}{2}\right)\right)\right)}, & 0 \leq z \leq t \leq \eta, \\ \dfrac{2e^{\frac{1}{2}\mu(t-z)}\left(\mu\sin\left(\frac{kt}{2}\right)+k\cos\left(\frac{kt}{2}\right)\right)\left(\delta e^{\mu/2}\sin\left(\frac{1}{2}k(\eta-z)\right)+e^{\frac{\eta\mu}{2}}\sin\left(\frac{1}{2}k(z-1)\right)\right)}{k\left(e^{\frac{\eta\mu}{2}}\left(\mu\sin\left(\frac{k}{2}\right)+k\cos\left(\frac{k}{2}\right)\right)-\delta e^{\mu/2}\left(\mu\sin\left(\frac{\eta k}{2}\right)+k\cos\left(\frac{\eta k}{2}\right)\right)\right)}, & t \leq z,\ t \leq \eta, \\ \dfrac{2\sin\left(\frac{1}{2}k(t-1)\right)e^{\frac{1}{2}\mu(\eta+t-z)}\left(\mu\sin\left(\frac{kz}{2}\right)+k\cos\left(\frac{kz}{2}\right)\right)}{k\left(e^{\frac{\eta\mu}{2}}\left(\mu\sin\left(\frac{k}{2}\right)+k\cos\left(\frac{k}{2}\right)\right)-\delta e^{\mu/2}\left(\mu\sin\left(\frac{\eta k}{2}\right)+k\cos\left(\frac{\eta k}{2}\right)\right)\right)}, & z \leq t,\ \eta \leq t, \\ \dfrac{e^{\frac{1}{2}\mu(t-z)}\left(2\delta e^{\mu/2}\sin\left(\frac{1}{2}k(t-z)\right)\left(\mu\sin\left(\frac{\eta k}{2}\right)+k\cos\left(\frac{\eta k}{2}\right)\right)-2e^{\frac{\eta\mu}{2}}\sin\left(\frac{1}{2}k(1-z)\right)\left(\mu\sin\left(\frac{kt}{2}\right)+k\cos\left(\frac{kt}{2}\right)\right)\right)}{k\left(e^{\frac{\eta\mu}{2}}\left(\mu\sin\left(\frac{k}{2}\right)+k\cos\left(\frac{k}{2}\right)\right)-\delta e^{\mu/2}\left(\mu\sin\left(\frac{\eta k}{2}\right)+k\cos\left(\frac{\eta k}{2}\right)\right)\right)}, & \eta \leq t \leq z \leq 1. \end{cases}$$

If (A_0) holds, then $g(z,t) \geq 0$.

Proof The Green's function for the linear BVPs (2.3) and (2.4), is defined as follows,

$$g(z,t) = \begin{cases} g_1(z,t) = e^{-\frac{\mu z}{2}}\left[a_1\cos\left(\frac{kz}{2}\right) + a_2\sin\left(\frac{kz}{2}\right)\right], & 0 \leq z \leq t \leq \eta; \\ g_2(z,t) = e^{-\frac{\mu z}{2}}\left[a_3\cos\left(\frac{kz}{2}\right) + a_4\sin\left(\frac{kz}{2}\right)\right], & t \leq z,\ t \leq \eta; \\ g_3(z,t) = e^{-\frac{\mu z}{2}}\left[a_5\cos\left(\frac{kz}{2}\right) + a_6\sin\left(\frac{kz}{2}\right)\right], & z \leq t,\ \eta \leq t; \\ g_4(z,t) = e^{-\frac{\mu z}{2}}\left[a_7\cos\left(\frac{kz}{2}\right) + a_8\sin\left(\frac{kz}{2}\right)\right], & \eta \leq t \leq z \leq 1. \end{cases} \tag{3.2}$$

Using the properties of Green's function, we have the following two sets of system of equations

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\mu & k & 0 & 0 \\ 0 & 0 & e^{\frac{\mu\eta}{2}}\cos\left(\frac{k}{2}\right) - \delta e^{\frac{\mu\eta}{2}}\cos\left(\frac{k\eta}{2}\right) & e^{\frac{\mu\eta}{2}}\sin\left(\frac{k}{2}\right) - \delta e^{\frac{\mu\eta}{2}}\sin\left(\frac{k\eta}{2}\right) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{2e^{\frac{\mu t}{2}}\sin\frac{kt}{2}}{k} \\ -\dfrac{2e^{\frac{\mu t}{2}}\cos\frac{kt}{2}}{k} \\ 0 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\mu & k & 0 & 0 \\ \delta \mathrm{e}^{\frac{\mu}{2}} \cos\left(\frac{k\eta}{2}\right) & \delta \mathrm{e}^{\frac{\mu}{2}} \sin\left(\frac{k\eta}{2}\right) & -\mathrm{e}^{\frac{\mu\eta}{2}} \cos\left(\frac{k}{2}\right) & -\mathrm{e}^{\frac{\mu\eta}{2}} \sin\left(\frac{k}{2}\right) \end{pmatrix} \begin{pmatrix} a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} \frac{2\mathrm{e}^{\frac{\mu t}{2}} \sin \frac{kt}{2}}{k} \\ -\frac{2\mathrm{e}^{\frac{\mu t}{2}} \cos \frac{kt}{2}}{k} \\ 0 \\ 0 \end{pmatrix}.$$

Using above set of equations, we can compute the values of all coefficients i.e., a_i's, $i = 1, 2, \ldots 8$. Finally, under the assumption (A_0), we can easily establish the sign of Green's function, i.e., $g(z, t) \geq 0$. Hence the result.

It is easy to see that the three-point linear BVPs (2.1) and (2.2) is equivalent to

$$w(z) = \frac{b\mathrm{e}^{-\frac{\mu z}{2}} \mathrm{e}^{\frac{\mu(1+\eta)}{2}} \left(\mu \sin\left(\frac{kz}{2}\right) + k \cos\left(\frac{kz}{2}\right)\right)}{\mathrm{e}^{\frac{\eta\mu}{2}} \left(\mu \sin\left(\frac{k}{2}\right) + k \cos\left(\frac{k}{2}\right)\right) - \delta \mathrm{e}^{\mu/2} \left(\mu \sin\left(\frac{\eta k}{2}\right) + k \cos\left(\frac{\eta k}{2}\right)\right)} - \int_0^1 g(z, t) h(t) \mathrm{d}t.$$

□

3.2 Case II: $\mu^2 - 4\gamma = k^2 > 0$

(A_1) Assume that

(a) there exist $\gamma \in R$, such that $\mu^2 - 4\gamma = k^2 > 0$;

(b) $\mathrm{e}^{\frac{\eta\mu}{2}} \left(\mu \sinh\left(\frac{k}{2}\right) + k \cosh\left(\frac{k}{2}\right)\right) - \delta \mathrm{e}^{\mu/2} \left(\mu \sinh\left(\frac{\eta k}{2}\right) + k \cosh\left(\frac{\eta k}{2}\right)\right) > 0$ and $\delta \mathrm{e}^{\mu/2} \sinh\left(\frac{\eta k}{2}\right) - \mathrm{e}^{\frac{\eta\mu}{2}} \sinh\left(\frac{k}{2}\right) \leq 0$.

Lemma 3.2 *The solution of nonhomogeneous linear BVPs* (2.1) *and* (2.2) *is given by*,

$$w(z) = \frac{b\mathrm{e}^{-\frac{\mu z}{2}} \mathrm{e}^{\frac{\mu(1+\eta)}{2}} \left(\mu \sinh\left(\frac{kz}{2}\right) + k \cosh\left(\frac{kz}{2}\right)\right)}{\mathrm{e}^{\frac{\eta\mu}{2}} \left(\mu \sinh\left(\frac{k}{2}\right) + k \cosh\left(\frac{k}{2}\right)\right) - \delta \mathrm{e}^{\mu/2} \left(\mu \sinh\left(\frac{\eta k}{2}\right) + k \cosh\left(\frac{\eta k}{2}\right)\right)} - \int_0^1 g(z, t) h(t) \mathrm{d}t, \tag{3.3}$$

where $g(z, t)$ is the Green's function of (2.3) *and* (2.4), *which is defined as*,

$$g(z,t) = \begin{cases} \frac{2\mathrm{e}^{\frac{1}{2}\mu(t-z)}\left(\mu\sinh\left(\frac{kz}{2}\right)+k\cosh\left(\frac{kz}{2}\right)\right)\left(\delta\mathrm{e}^{\mu/2}\sinh\left(\frac{1}{2}k(\eta-t)\right)+\mathrm{e}^{\frac{\eta\mu}{2}}\sinh\left(\frac{1}{2}k(t-1)\right)\right)}{k\left(\mathrm{e}^{\frac{\eta\mu}{2}}\left(\mu\sinh\left(\frac{k}{2}\right)+k\cosh\left(\frac{k}{2}\right)\right)-\delta\mathrm{e}^{\mu/2}\left(\mu\sinh\left(\frac{\eta k}{2}\right)+k\cosh\left(\frac{\eta k}{2}\right)\right)\right)}, & 0 \leq z \leq t \leq \eta, \\ \frac{2\mathrm{e}^{\frac{1}{2}\mu(t-z)}\left(\mu\sinh\left(\frac{kt}{2}\right)+k\cosh\left(\frac{kt}{2}\right)\right)\left(\delta\mathrm{e}^{\mu/2}\sinh\left(\frac{1}{2}k(\eta-z)\right)-\mathrm{e}^{\frac{\eta\mu}{2}}\sinh\left(\frac{1}{2}k(1-z)\right)\right)}{k\left(\mathrm{e}^{\frac{\eta\mu}{2}}\left(\mu\sinh\left(\frac{k}{2}\right)+k\cosh\left(\frac{k}{2}\right)\right)-\delta\mathrm{e}^{\mu/2}\left(\mu\sinh\left(\frac{\eta k}{2}\right)+k\cosh\left(\frac{\eta k}{2}\right)\right)\right)}, & t \leq z,\ t \leq \eta, \\ \frac{2\sinh\left(\frac{1}{2}k(t-1)\right)\mathrm{e}^{\frac{1}{2}\mu(\eta+t-z)}\left(\mu\sinh\left(\frac{kz}{2}\right)+k\cosh\left(\frac{kz}{2}\right)\right)}{k\left(\mathrm{e}^{\frac{\eta\mu}{2}}\left(\mu\sinh\left(\frac{k}{2}\right)+k\cosh\left(\frac{k}{2}\right)\right)-\delta\mathrm{e}^{\mu/2}\left(\mu\sinh\left(\frac{\eta k}{2}\right)+k\cosh\left(\frac{\eta k}{2}\right)\right)\right)}, & z \leq t,\ \eta \leq t, \\ \frac{2\mathrm{e}^{\frac{1}{2}\mu(t-z)}\left(\delta\mathrm{e}^{\mu/2}\sinh\left(\frac{1}{2}k(t-z)\right)\left(\mu\sinh\left(\frac{\eta k}{2}\right)+k\cosh\left(\frac{\eta k}{2}\right)\right)+\mathrm{e}^{\frac{\eta\mu}{2}}\sinh\left(\frac{1}{2}k(z-1)\right)\left(\mu\sinh\left(\frac{kt}{2}\right)+k\cosh\left(\frac{kt}{2}\right)\right)\right)}{k\left(\mathrm{e}^{\frac{\eta\mu}{2}}\left(\mu\sinh\left(\frac{k}{2}\right)+k\cosh\left(\frac{k}{2}\right)\right)-\delta\mathrm{e}^{\mu/2}\left(\mu\sinh\left(\frac{\eta k}{2}\right)+k\cosh\left(\frac{\eta k}{2}\right)\right)\right)}, & \eta \leq t \leq z \leq 1. \end{cases} \tag{3.4}$$

If (A_1) holds, then $g(z, t) \leq 0$.

Proof Proof is similar to the proof given in Lemma 3.1. □

3.3 Case III: $\mu^2 - 4\gamma = 0$

(A_2) Assume that

(a) $\gamma = \frac{\mu^2}{4}$;
(b) $(\mu + 2)\mathrm{e}^{\frac{\eta\mu}{2}} - \delta \mathrm{e}^{\mu/2}(\eta\mu + 2) < 0$ and $\delta\eta \mathrm{e}^{\mu/2} - \mathrm{e}^{\frac{\eta\mu}{2}} \le 0$.

Lemma 3.3 *The solution of linear BVPs (2.1) and (2.2) is given by*

$$w(z) = \frac{b\mathrm{e}^{-\frac{\mu z}{2}}\mathrm{e}^{\frac{\mu(1+\eta)}{2}}(2 + z\mu)}{\mathrm{e}^{\frac{\eta\mu}{2}}(2 + \mu) - \delta \mathrm{e}^{\mu/2}(2 + \mu\eta)} - \int_0^1 g(z, t)h(t)\mathrm{d}t, \tag{3.5}$$

where $g(z, t)$ is the Green's function of (2.3) and (2.4), which is defined as,

$$g(z,t) = \begin{cases} \dfrac{(\mu z+2)\mathrm{e}^{\frac{1}{2}\mu(t-z)}\left(\delta \mathrm{e}^{\mu/2}(\eta-t)+(t-1)\mathrm{e}^{\frac{\eta\mu}{2}}\right)}{(\mu+2)\mathrm{e}^{\frac{\eta\mu}{2}}-\delta \mathrm{e}^{\mu/2}(\eta\mu+2)}, & 0 \le z \le t \le \eta, \\ \dfrac{(\mu t+2)\mathrm{e}^{\frac{1}{2}\mu(t-z)}\left(\delta \mathrm{e}^{\mu/2}(\eta-z)+(z-1)\mathrm{e}^{\frac{\eta\mu}{2}}\right)}{(\mu+2)\mathrm{e}^{\frac{\eta\mu}{2}}-\delta \mathrm{e}^{\mu/2}(\eta\mu+2)}, & t \le z,\ t \le \eta, \\ \dfrac{(t-1)(\mu z+2)\mathrm{e}^{\frac{1}{2}\mu(\eta+t-z)}}{(\mu+2)\mathrm{e}^{\frac{\eta\mu}{2}}-\delta \mathrm{e}^{\mu/2}(\eta\mu+2)}, & z \le t,\ \eta \le t, \\ \dfrac{\mathrm{e}^{\frac{1}{2}\mu(t-z)}\left(\delta \mathrm{e}^{\mu/2}(\eta\mu+2)(t-z)+(z-1)\mathrm{e}^{\frac{\eta\mu}{2}}(\mu t+2)\right)}{(\mu+2)\mathrm{e}^{\frac{\eta\mu}{2}}-\delta \mathrm{e}^{\mu/2}(\eta\mu+2)}, & \eta \le t \le z \le 1. \end{cases}$$

If (A_2) holds, then $g(z, t) \ge 0$.

Proof Proof is similar to the proof given in Lemma 3.1. □

4 An approximation scheme

In this section, we derive maximum and anti maximum principle to prove monotonicity. We also define upper and lower solutions and establish a new approximation scheme for three-point BVPs.

Proposition 4.1 *Assume that (A_0), (A_2) hold, and $w \in C^2(I)$ satisfies*

$$\begin{aligned} &-w''(z) - \mu w'(z) - \gamma w(z) \ge 0, \quad z \in I_0, \\ &w'(0) = 0, \quad w(1) \ge \delta w(\eta). \end{aligned}$$

Then $w(z)$ is non positive, $\forall z \in I$.

Proof Using Eqs. (3.1) and (3.5), and conditions (A_0), (A_2), we can show effortlessly that $w(z)$ is non positive, $\forall z \in I$. □

Proposition 4.2 *Suppose that (A_1) holds and $w \in C^2(I)$ satisfies*

$$\begin{aligned} &-w''(z) - \mu w'(z) - \gamma w(z) \ge 0, \quad z \in I_0, \\ &w'(0) = 0, \quad w(1) \ge \delta w(\eta). \end{aligned}$$

Then $w(z)$ is non negative, $\forall z \in I$.

Proof Proof of this proposition is similar to the proof of above Proposition 4.1. □

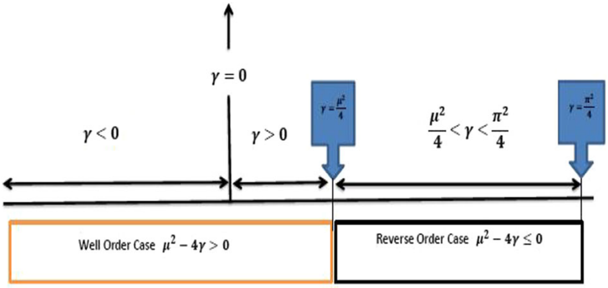


Fig. 1 Well and reverse order case

Definition 4.1 The function $x(z) \in C^2(I)$ is called a lower solution of the NLBVPs (1.6) and (1.7), if

$$\begin{aligned} L_0(z, x) &= -x''(z) - f(z, x, x') \leq 0, \quad z \in I_0, \\ x'(0) &= 0, \quad x(1) \leq \delta x(\eta), \end{aligned}$$

and the function $y(z) \in C^2(I)$ is called an upper solution of the NLBVPs (1.6) and (1.7), if

$$\begin{aligned} U_0(z, y) &= -y''(z) - f(z, y, y') \geq 0, \quad z \in I_0, \\ y'(0) &= 0, \quad y(1) \geq \delta y(\eta). \end{aligned}$$

Here, we introduce a different approximation scheme (for three point NLBVPs) (1.6) and (1.7), which is defined as

$$-w''_{n+1}(z) - \mu w'_{n+1}(z) - \gamma w_{n+1}(z) = f(z, w_n, w'_n) - \mu w'_n - \gamma w_n, \tag{4.1}$$

$$w'_{n+1}(0) = 0, \quad w_{n+1}(1) = \delta w_{n+1}(\eta). \tag{4.2}$$

The sequences of lower solution $(x_n)_n$, (with $x_0 = x$), and upper solution $(y_n)_n$, (with $y_0 = y$), are defined using the above said approximation scheme (4.1) and (4.2), as follows,

$$-x''_{n+1}(z) - \mu x'_{n+1}(z) - \gamma x_{n+1}(z) = f(z, x_n, x'_n) - \mu x'_n - \gamma x_n, \tag{4.3}$$

$$x'_{n+1}(0) = 0, \quad x_{n+1}(1) = \delta x_{n+1}(\eta). \tag{4.4}$$

$$-y''_{n+1}(z) - \mu y'_{n+1}(z) - \gamma y_{n+1}(z) = f(z, y_n, w'_n) - \mu y'_n - \gamma y_n, \tag{4.5}$$

$$y'_{n+1}(0) = 0, \quad y_{n+1}(1) = \delta y_{n+1}(\eta). \tag{4.6}$$

5 Main results

This section gives the main results, i.e., existence results for NLBVPs (1.6) and (1.7). This section is divided into the following two subsections based on reverse and well ordered lower and upper solutions (see Fig. 1)

(I) Reverse order case: $\mu^2 - 4\gamma \leq 0$, i.e., $\mu^2 - 4\gamma = -k^2$, or $\mu^2 - 4\gamma = 0$.
(II) Well order case: $\mu^2 - 4\gamma > 0$, i.e., $\mu^2 - 4\gamma = k^2$.

5.1 Lower and upper solutions in reverse ordered ($x \geq y$)

This subsection deals with the reverse order lower and upper solutions i.e. $x \geq y$. The following results help us to establish the existence results for the three-point NLBVPs (1.6) and (1.7).

Lemma 5.1 *Let $\gamma > 0$ be such that $\mu^2 - 4\gamma = -k^2 < 0$, $\gamma - L \geq 0$, and $2N - \mu \leq 0$, then for all $z \in I$,*

$$(\gamma - L)\left(\mu \sin\left(\frac{kz}{2}\right) + k\cos\left(\frac{kz}{2}\right)\right) - 2\gamma(N(\operatorname{sign} w') + \mu)\sin\left(\frac{kz}{2}\right) \geq 0, \quad (5.1)$$

whenever

$$H_1 = (\gamma - L)\left(\mu \sin\left(\frac{k}{2}\right) + k\cos\left(\frac{k}{2}\right)\right) - 2\gamma(N(\operatorname{sign} w' + \mu)\sin\left(\frac{k}{2}\right) \geq 0, \quad (5.2)$$

where $L, N \in \mathbb{R}^+$ and $0 < \mu \leq \frac{\pi}{2}$.

Proof We can represent the inequality (5.1) in the following two ways

$$(\gamma - L)\left(\mu \sin\left(\frac{kz}{2}\right) + k\cos\left(\frac{kz}{2}\right)\right) - 2\gamma(N + \mu)\sin\left(\frac{kz}{2}\right) \geq 0, \quad \text{when } w' \geq 0. \quad (5.3)$$

$$(\gamma - L)\left(\mu \sin\left(\frac{kz}{2}\right) + k\cos\left(\frac{kz}{2}\right)\right) + 2\gamma(N - \mu)\sin\left(\frac{kz}{2}\right) \geq 0, \quad \text{when } w' \leq 0. \quad (5.4)$$

To begin with inequality (5.1), we have to prove the inequalities (5.3) and (5.4) separately.

For the inequality (5.3): We consider the function,

$$(\gamma - L)\left(\mu \sin\left(\frac{kz}{2}\right) + k\cos\left(\frac{kz}{2}\right)\right) - 2\gamma(N + \mu)\sin\left(\frac{kz}{2}\right),$$

which is non-increasing. Thus for all $z \in I$, we have

$$\begin{aligned}&(\gamma - L)\left(\mu \sin\left(\frac{kz}{2}\right) + k\cos\left(\frac{kz}{2}\right)\right) - 2\gamma(N + \mu)\sin\left(\frac{kz}{2}\right)\\ &\geq (\gamma - L)\left(\mu \sin\left(\frac{k}{2}\right) + k\cos\left(\frac{k}{2}\right)\right) - 2\gamma(N + \mu)\sin\left(\frac{k}{2}\right) \geq 0.\end{aligned}$$

Hence the result.

Making use of similar analysis, we can prove the inequality (5.4). □

Lemma 5.2 *Let $\gamma > 0$ be such that $\mu^2 - 4\gamma = 0$, $\gamma - L \geq 0$, $2N - \mu \leq 0$, and $(\gamma - L) - \gamma\left(N(\operatorname{sign} w') + \mu\right) \geq 0$, then for all $z \in I$,*

$$H_2 = (\gamma - L)(2 + \mu z) - 2\gamma(N(\operatorname{sign} w') + \mu)z \geq 0, \quad (5.5)$$

where $L, N \in R^+$ and $0 < \mu \leq \frac{\pi}{2}$.

Proof See the proof of Lemma 5.1. □

5.1.1 Inequalities based on Green's function

Here, we prove some inequalities based on Green's function.

Lemma 5.3 *Let* (A_0) *be true and* $\gamma - L \geq 0$, $2N - \mu \leq 0$, *and* $H_1 \geq 0$ (*defined in Eq.* (5.1)) *hold, then for any* $z, t \in I$ *and* $z \neq t$, *we have*

$$(\gamma - L)g(z,t) + (N(\operatorname{sign} w') + \mu)\frac{\partial g(z,t)}{\partial z} \geq 0, \tag{5.6}$$

where $L, N \in R^+$ *and* $0 < \mu \leq \frac{\pi}{2}$.

Proof The inequality (5.6) can be written in the following ways

$$(\gamma - L)g(z,t) + (N + \mu)\frac{\partial g(z,t)}{\partial z} \geq 0, \quad \text{when } w' \geq 0. \tag{5.7}$$

$$(\gamma - L)g(z,t) - (N - \mu)\frac{\partial g(z,t)}{\partial z} \geq 0, \quad \text{when } w' \leq 0. \tag{5.8}$$

The inequalities (5.7) and (5.8) must be shown independently to prove the inequality (5.6). For the inequality (5.7): Making use of Eq. (3.2), we substitute the values of $g_i(z,t)$ and $\frac{\partial g_i(z,t)}{\partial z}$, $i = 1, \ldots, 4$, in Eq. (5.7). Now applying the Lemma 5.1, we get,

$$(\gamma - L)g_i(z,t) + (N + \mu)\frac{\partial g_i(z,t)}{\partial z} \geq 0, \quad \text{for all } i = 1, \ldots, 4.$$

Similarly, we can prove the inequality (5.8). □

Lemma 5.4 *Let* (A_2) *be true and* $\gamma - L \geq 0$, $2N - \mu \leq 0$, *and* $H_2 \geq 0$ (*defined in Eq.* (5.5)), *then for any* $z, t \in I$ *and* $z \neq t$, *we get*

$$(\gamma - L)g(z,t) + (N(\operatorname{sign} w') + \mu)\frac{\partial g(z,t)}{\partial z} \geq 0, \tag{5.9}$$

where $L, N \in R^+$ *and* $0 < \mu \leq \frac{\pi}{2}$.

Proof Using Lemma 5.2 and following the similar analysis of Lemma 5.3, we get the required proof. □

5.1.2 Existence theorem for three-point NLBVPs (reverse ordered case)

Throughout this subsubsection, we consider the following assumptions

(R_O) Assume that

(a) there exists x and $y \in C^2(I)$ given by definition 4.1, such that $\forall z \in I,\ x \geq y$;
(b) $f : U \to \mathbb{R}$ such that f is continuous on U, where $U := \{(z, w, v) \in I \times \mathbb{R}^2 : y(z) \leq w \leq x(z)\}$;
(c) there exists $L \geq 0$ such that $\forall\ (z, w_1, v), (z, w_2, v) \in U\ w_1 \leq w_2 \Rightarrow f(z, w_2, v) - f(z, w_1, v) \leq L(w_2 - w_1)$;
(d) $\exists\ N \geq 0$ such that $\forall\ (z, w, v_1), (z, w, v_2) \in U\ |f(z, w, v_2) - f(z, w, v_1)| \leq N|v_2 - v_1|$;

where $\mu^2 - 4\gamma \leq 0$, i.e. $\mu^2 - 4\gamma = -k^2$, or $\mu^2 - 4\gamma = 0$. Based on these assumptions, we further divide this subsubsection into the following two cases

5.1.3 Case I: $\mu^2 - 4\gamma = -k^2 < 0$

Theorem 5.1 *Let* (A_0) *and* (R_O) *be true. Further assume that* $\gamma - L \geq 0$, $2N - \mu \leq 0$, *and* $H_1 \geq 0$, *defined in Lemma* 5.1 *and*

$$F(z, x, y, x', y') = f(z, y(z), y'(z)) - f(z, x(z), x'(z)) - \mu(y-x)' - \gamma(y-x) \\ \geq 0, \quad \text{for all } z \in I, \tag{5.10}$$

then $(x_n)_n$ *and* $(y_n)_n$ *introduced in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively converge in* $C^1(I)$ *monotonically such that*

$$y \leq u \leq v \leq x, \quad \forall z \in I,$$

where v *and* u *are solutions of NLBVPs* (1.6) *and* (1.7).

To demonstrate the above theorem, we require to prove various consequences which are as follows.

Proposition 5.1 *Let* $\mu^2 - 4\gamma = -k^2 < 0$. *Further assume that*

(i) (A_0) *and* (R_O) *are true;*
(ii) $\exists\, \gamma > 0$ *and* $0 < \mu \leq \frac{\pi}{2}$ *such that* $\gamma - L \geq 0$, $2N - \mu \leq 0$, *and* $H_1 \geq 0$. *Then the functions* $(x_n)_n$ *and* $(y_n)_n$ *defined recursively by* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, such that for all* $n \in N$
    (a) $x_{n+1} \leq x_n$,
    (b) $y_{n+1} \geq y_n$.

Proof Let x_n be a lower solution of (1.6) and (1.7) and x_{n+1} is given by (4.3) and (4.4). We observe that $w(z) = x_{n+1} - x_n$ satisfy (2.1) and (2.2), where $h(z) \geq 0$, and $b \geq 0$. Hence by making use of Proposition 4.1, it can be written as $x_{n+1} \leq x_n$. Similarly we can get $y_{n+1} \geq y_n$.

For proving the claim (a) for $n = 0$ we need to show $x_1 \leq x_0$, which comes after the above discussion, i.e., claim (a) holds for $n = 0$. Now we show if it is true for $n - 1$, then it will be true for every n.

Let $w = x_n - x_{n-1}$, where x_{n-1} is a lower solution of (1.6) and (1.7) and $x_n \leq x_{n-1}$. We have

$$-x_n'' - f(z, x_n, x_n') \leq L(x_{n-1} - x_n) + N|x_n' - x_{n-1}'| + \mu(x_n - x_{n-1})' + \gamma(x_n - x_{n-1}), \\ = (\gamma - L)w + (N(\operatorname{sign} w') + \mu)w'.$$

As w satisfies $-w'' - \mu w' - \gamma w = x_{n-1}'' + f(z, x_{n-1}, x_{n-1}') \geq 0$, $w'(0) = 0$, $w(1) \geq \delta w(\eta)$, with $h(z) = x_{n-1}'' + f(z, x_{n-1}, x_{n-1}') \geq 0$, Now to prove claim, we need to show $(\gamma - L)w + (N\ (\operatorname{sign} w') + \mu)w' \leq 0$. And for this it is adequate to demonstrate the following

$$(\gamma - L)\left(\mu \sin\left(\frac{kz}{2}\right) + k\cos\left(\frac{kz}{2}\right)\right) - 2\gamma(N(\operatorname{sign} w') + \mu)\sin\left(\frac{kz}{2}\right) \geq 0,$$
$$\text{and,} \quad (\gamma - L)g(z,t) + (N(\operatorname{sign} w') + \mu)\frac{\partial g(z,t)}{\partial z} \geq 0, \quad z \neq t, \quad \forall z \in I.$$

Using Lemmas 5.1 and 5.3, we can obtain the requirements. Thus we deduce that $x_{n+1} \leq x_n$. Making use of a similar process we can prove $y_{n+1} \geq y_n$. □

Proposition 5.2 *Assume that*

- (i) (A_0) *and* (R_O) *are true;*
- (ii) $\exists\, 0 < \mu \leq \frac{\pi}{2}$ *such that* $\gamma - L \geq 0$, $2N - \mu \leq 0$. *Also if* $H_1 \geq 0$, *and* $F(z, x, y, x', y') \geq 0$, *defined in* Lemma 5.1 *and Eq.* (5.10) *respectively, are valid. Then the sequences* $(x_n)_n$ *and* $(y_n)_n$ *introduced in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, are such that* $x_n \geq y_n$, $\forall n \in N$.

Proof See Proposition 3.3 of Singh and Verma (2013). □

5.1.4 Case II: $\mu^2 - 4\gamma = 0$

To prove the existence result for this case, we follow the same analysis as we did in Theorem 5.1.

Theorem 5.2 *Let* (A_2) *and* (R_O) *be true. Further assume that* $\gamma = \frac{\mu^2}{4}$ *such that* $\gamma - L \geq 0$, $2N - \mu \leq 0$, *and* $H_2 \geq 0$ *(see Lemma* 5.2*) and* $F(z, x, y, x', y') \geq 0$ *(see Eq.* (5.10)*), then the sequences* $(x_n)_n$ *and* $(y_n)_n$ *introduced in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively converge in* $C^1(I)$ *monotonically such that*

$$y \leq u \leq v \leq x, \quad \forall z \in I,$$

where v *and* u *are solutions of NLBVPs* (1.6) *and* (1.7).

Proposition 5.3 *Let* $\mu^2 - 4\gamma = 0$. *Further assume that*

- (i) (A_2) *and* (R_O) *are true;*
- (ii) $\exists\, \gamma = \frac{\mu^2}{4}$, *where* $0 < \mu \leq \frac{\pi}{2}$, *such that* $\gamma - L \geq 0$, $2N - \mu \leq 0$, *and* $H_2 \geq 0$. *Then the sequences* $(x_n)_n$ *and* $(y_n)_n$ *introduced in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively are such that,*
  - (a) $x_{n+1} \leq x_n$, $\forall\, n \in N$,
  - (b) $y_{n+1} \geq y_n$, $\forall\, n \in N$.

Proof Making use of the Lemmas 3.3, 5.2 and 5.4 and using the arguments similar to the proof of Proposition 5.1, we can prove this proposition. □

In the similar manner, we can demonstrate the following results.

Proposition 5.4 *Assume that*

- (i) (A_2) *and* (R_O) *are true;*
- (ii) $\exists\, \gamma = \frac{\pi^2}{4}$, *where* $0 < \mu \leq \frac{\pi}{2}$, *such that* $\gamma - L \geq 0$, $2N - \mu \leq 0$, *and* $H_2 \geq 0$. *Also if* $F(z, x, y, x', y') \geq 0$, *defined in Eq.* (5.10)*, is valid. Then* $\forall\, n \in N$, *the sequences* $(x_n)_n$ *and* $(y_n)_n$ *given by* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, are such that* $x_n \geq y_n$.

5.1.5 Priory bound

Lemma 5.5 *If* $f(z, w, w')$ *satisfies the following assumption,*

(H_R) *let* $\varphi : R^+ \to R^+$ *is such that* $\forall (z, w, v) \in U$, $|f(z, w, v)| \leq \varphi(|v|)$, *and*

$$\max_{z\in I} x - \min_{z\in I} y \leq \int_{l_0}^{\infty} \frac{\xi \, \mathrm{d}\xi}{\varphi(\xi)},$$

where φ *is continuous and* $l_0 = 2\max\{\sup_{z\in I}|x(z)|, \sup_{z\in I}|y(z)|\}$, *then* $\exists\, r > 0$ *such that any solution* $w \in [x(z), y(z)]$ *of*

$$U_0(z, w) \geq 0, \quad z \in I_0, \tag{5.11}$$

$$w'(0) = 0, \quad w(1) \geq \delta w(\eta), \tag{5.12}$$

satisfies $\|w'\|_\infty \leq r$, $\forall\, z \in I$.

Proof We prove the above results in the following cases.

Case (i): If $w(z)$ is not monotone on I_0, let us take an interval $(z_0, z] \subset I_0$ such that $w'(z_0) = 0$ and $w'(z) > 0$ for $z > z_0$. Using (H_R), i.e., $|f(z, w, v)| \leq \varphi(|v|)$ in (5.11) and then integrating from the limit z_0 to z, we get

$$\int_0^{w'} \frac{\xi \,\mathrm{d}\xi}{\varphi(\xi)} \leq \max_{z\in I} x - \min_{z\in I} y.$$

Using (H_R), we have $r > 0$, such that

$$\int_0^{w'} \frac{\xi \,\mathrm{d}\xi}{\varphi(\xi)} \leq \max_{z\in I} x - \min_{z\in I} y \leq \int_{l_0}^{r} \frac{\xi \,\mathrm{d}\xi}{\varphi(\xi)} \leq \int_0^{r} \frac{\xi \,\mathrm{d}\xi}{\varphi(\xi)}.$$

This gives $w'(z) \leq r$.
Now if we take the interval in which $w'(z) < 0$ for $z < z_0$ and $w'(z_0) = 0$, the proof is similar to above proof, hence we get $-w'(z) \leq r$ and hence the outcome follows.

Case (ii): If w in I_0 is such that $w'(z) < 0$ in $z \in (0, 1]$, then $\exists\, \tau \in I_0$ such that $-w'(\tau) \leq 2|x(\tau)|$. Now using (H_R) in (5.11) and then integrating from the limit z to τ, we get

$$\int_0^{-w'} \frac{\xi \,\mathrm{d}\xi}{\varphi(\xi)} \leq \max_{z\in I} x - \min_{z\in I} y.$$

Using (H_R), we have r such that

$$\int_0^{-w'} \frac{\xi \,\mathrm{d}\xi}{\varphi(\xi)} \leq \max_{z\in I} x - \min_{z\in I} y \leq \int_0^{r} \frac{\xi \,\mathrm{d}\xi}{\varphi(\xi)}.$$

This gives $-w' \leq r$.

Case (iii): If w increases monotonically in I_0, i.e., $w'(z) > 0$ in $z \in (0, 1]$. Proof of this case is also similar to the case (ii), hence, we get $w' \leq r$. □

Lemma 5.6 *If* $f(z, w, w')$ *satisfies* (H_R), *then* $\exists\, r > 0$ *such that the solution* $w \in [y(z), x(z)]$ *of*

$$L_0(z, w) \leq 0, \quad z \in I_0, \tag{5.13}$$

$$w'(0) = 0, \quad w(1) \leq \delta w(\eta), \tag{5.14}$$

satisfies $\|w'\|_\infty \leq r$, $\forall\, z \in I$.

Proof Proof of this lemma follows from the proof of the above lemma. □

Proof of Theorem 5.1 (Theorem 5.2) Using the Propositions 5.1 and 5.2 (5.3 and 5.4 for Theorem 5.2), we can easily show that

$$x = x_0 \geq x_1 \geq \cdots \geq x_n \geq \cdots \geq y_n \geq \cdots \geq y_1 \geq y_0 = y. \tag{5.15}$$

From (5.15), we have, $(x_n)_n$ and $(y_n)_n$ satisfy the conditions of monotone convergence theorem, hence they converge to $v(z)$ and $u(z)$ such that

$$v(z) = \lim_{n\to\infty} x_n(z) \quad \text{and} \quad u(z) = \lim_{n\to\infty} y_n(z),$$

such that $\forall\, n,\ x_n \geq u \geq v \geq y_n$. It follows that $(x_n)_n$ given by (4.3) and (4.4) is equibounded (EB) and equicontinuous (EC) in $C^1(I)$ (using Lemma 5.6 and relation (5.15)). It implies that any subsequence $(x_{n_m})_m$ of $(x_n)_n$ is EB and EC in $C^1(I)$ and due to Arzela–Ascoli theorem we prove that $(x_{n_m})_m$ contains a subsubsequence which converges in $C^1(I)$. From uniqueness of limit and monotonicity, we have $x_n \to v$ in $C^1(I)$. As any $(x_{n_m})_m$ of $(x_n)_n$ contains a subsubsequence, which converges to v in $C^1(I)$ it follows that $x_n \to v$ in $C^1(I)$. Similarly, using Proposition 5.1 and Lemma 5.5, we show that $(y_n)_n$ converges to u in $C^1(I)$.

Using the property of derivative and taking limit in (4.3) and (4.4) and (4.5) and (4.6) respectively along with $(x_n)_n$ and $(y_n)_n$ respectively, it can be easily seen that u and v are solutions of (1.6) and (1.7). □

5.2 Well order lower-upper solutions ($x \leq y$)

In this section, we prove the following inequalities to establish the existence of solution of NLBVPs (1.6) and (1.7). The lower and upper solutions appear in well ordered for the existence results. Throughout this subsection, we consider $\mu^2 - 4\gamma > 0$ i.e., $\mu^2 - 4\gamma = k^2 > 0$.

Lemma 5.7 *If $\gamma < 0$ is such that $\mu^2 - 4\gamma = k^2 > 0$, $L + \gamma \leq 0$, and $N - \mu \leq 0$ then for all $z \in I$,*

$$(\gamma + L)\left(\mu \sinh\left(\frac{kz}{2}\right) + k\cosh\left(\frac{kz}{2}\right)\right) - 2\gamma(N(\text{sign}\, w') + \mu)\sinh\left(\frac{kz}{2}\right) \leq 0,$$

whenever $(\gamma + L)k - 2\gamma(N(\text{sign}\, w') + \mu) \leq 0$, where $L, N \in R^+$ and $0 < \mu \leq \frac{\pi}{2}$.

Proof We observe that

$$\begin{aligned}&(\gamma + L)\left(\mu \sinh\left(\frac{kz}{2}\right) + k\cosh\left(\frac{kz}{2}\right)\right) - 2\gamma(N(\text{sign}\, w') + \mu)\sinh\left(\frac{kz}{2}\right)\\ &\quad\leq \big((\gamma + L)k - 2\gamma(N(\text{sign}\, w') + \mu)\big)\cosh\left(\frac{kz}{2}\right) + \mu(\gamma + L)\sin\left(\frac{kz}{2}\right) \leq 0, \quad z \in I,\end{aligned}$$

only if $(\gamma + L)k - 2\gamma(N(\text{sign}\, w') + \mu) \leq 0$. This completes the proof. □

Lemma 5.8 *If $\gamma > 0$ be such that $\mu^2 - 4\gamma = k^2 > 0$, $\gamma - L \leq 0$, and $(N - \mu) \leq 0$, then for all $z,\ s \in [0, 1]$ such that $s \leq z$, and s is fixed, we have*

$$\begin{aligned}&(\gamma - L)\sinh\frac{k}{2}(z - s) + \frac{1}{2}\left(N\left(\text{sign}\, w'\right) + \mu\right)\\ &\quad\times\left(k\cosh\frac{k}{2}(z - s) - \mu\sinh\frac{k}{2}(z - s)\right) \geq 0. \qquad (5.16)\end{aligned}$$

Whenever

$$H_3 = (\gamma - L)\sinh\frac{k}{2} + \frac{1}{2}\left(N(\text{sign}\, w') + \mu\right)\left(k - \mu\sinh\frac{k}{2}\right) \geq 0, \qquad (5.17)$$

where $L, N \in R^+$, $0 < \mu \leq \frac{\pi}{2}$ and $k - \mu\sinh\frac{k}{2} > 0$.

Proof We can rewrite the inequality (5.16) in the following ways

$$\begin{aligned}&(\gamma - L)\sinh\frac{k}{2}(z-s) + \frac{1}{2}(N+\mu)\\&\quad\times\left(k\cosh\frac{k}{2}(z-s) - \mu\sinh\frac{k}{2}(z-s)\right) \geq 0, \qquad \text{when} \quad w' \geq 0. \qquad (5.18)\\&(\gamma - L)\sinh\frac{k}{2}(z-s) - \frac{1}{2}(N-\mu)\\&\quad\times\left(k\cosh\frac{k}{2}(z-s) - \mu\sinh\frac{k}{2}(z-s)\right) \geq 0, \qquad \text{when} \quad w' \leq 0. \qquad (5.19)\end{aligned}$$

To prove the inequality (5.16), we have to show the inequalities (5.18) and (5.19).
For the inequality (5.18): Consider the function,

$$(\gamma - L)\sinh\frac{k}{2}(z-s) + \frac{1}{2}(N+\mu)\left(k\cosh\frac{k}{2}(z-s) - \mu\sinh\frac{k}{2}(z-s)\right).$$

Since the above expression is non increasing. Thus for all $z,\ s \in [0, 1]$ such that $s \leq z$, we have

$$\begin{aligned}&(\gamma - L)\sinh\frac{k}{2}(z-s) + \frac{1}{2}(N+\mu)\left(k\cosh\frac{k}{2}(z-s) - \mu\sinh\frac{k}{2}(z-s)\right)\\&\geq (\gamma - L)\sinh\frac{k}{2} + \frac{1}{2}(N+\mu)\left(k - \mu\sinh\frac{k}{2}\right) \geq 0.\end{aligned}$$

Hence the result.
Similarly, we can prove for the inequality (5.19) . □

5.2.1 Inequalities based on Green's function

Lemma 5.9 *Let* (A_1) *be true and* $\gamma < 0$ *such that* $\gamma + L \geq 0$, $N - \mu \leq 0$ *and* $(\gamma + L)k - 2\gamma(N(\text{sign } w') + \mu) \leq 0$, *then* $\forall\ z, t \in I$ *and* $z \neq t$, *we have*

$$(\gamma - L)g(z,t) + (N\,\text{sign } w') + \mu)\frac{\partial g(z,t)}{\partial z} \geq 0. \qquad (5.20)$$

Proof The inequality (5.20) can be written in the following ways

$$(\gamma - L)g(z,t) + (N+\mu)\frac{\partial g(z,t)}{\partial z} \geq 0, \quad \text{when } w' \geq 0. \qquad (5.21)$$

$$(\gamma - L)g(z,t) - (N-\mu)\frac{\partial g(z,t)}{\partial z} \geq 0, \quad \text{when } w' \leq 0. \qquad (5.22)$$

To prove the inequality (5.20), we have to show the inequalities (5.21) and (5.22). Making use of Eq. (3.4), we substitute the values of $g_i(z,t)$ and $\frac{\partial g_i(z,t)}{\partial z}$, $i = 1, \ldots, 4$, in Eq. (5.21). Now applying the Lemma 5.7, we get,

$$(\gamma - L)g_i(z,t) + (N+\mu)\frac{\partial g_i(z,t)}{\partial z} \geq 0, \quad \text{for all } i = 1, \ldots, 4.$$

Similarly, we can prove for prove the inequality (5.22). □

Lemma 5.10 *Let* (A_1) *be true and* γ, $L > 0$ *such that* $\gamma - L \leq 0$, $N - \mu \leq 0$ *and* $H_3 \geq 0$, *then we have*

$$(\gamma - L)g(z,t) + (N \operatorname{sign} w') + \mu)\frac{\partial g(z,t)}{\partial z} \geq 0,$$

where $L, N \in \mathbb{R}^+$ *and* H_3 *is defined in Eq.* (5.17).

Proof Proof is similar to the proof of Lemma 5.9. □

5.2.2 Existence theorem for nonlinear three point BVPs (well ordered case)

Throughout this subsection, we consider the following assumptions

(W_O): Assume that

- (a) $\exists\, x$ and $y \in C^2(I)$ given by Definition 4.1 such that $x \leq y, \forall\, z \in I$;
- (b) $f : \widetilde{U} \to R$ such that f is continuous on $\widetilde{U}$, where $\widetilde{\widetilde{U}} := \{(z, w, v) \in I \times R^2 : x(z) \leq w \leq y(z)\}$;
- (c) $\exists\, L \geq 0$ such that $\forall\, (z, w_1, v), (z, w_2, v) \in \widetilde{U}$,
  - (i) when $\gamma < 0$, $w_1 \leq w_2 \Rightarrow f(z, w_2, v) - f(z, w_1, v) \geq -L(w_2 - w_1)$;
  - (ii) when $0 < \gamma < \frac{\mu^2}{4}$, $w_1 \leq w_2 \Rightarrow f(z, w_2, v) - f(z, w_1, v) \geq L(w_2 - w_1)$;
- (d) $\exists\, N \geq 0$ such that for all $(z, w, v_1), (z, w, v_2) \in \widetilde{U}$, $|f(z, w, v_2) - f(z, w, v_1)| \leq N|v_2 - v_1|$.

Where $\mu^2 - 4\gamma = k^2 > 0$. Based on γ sign such that $\mu^2 - 4\gamma > 0$, we further divide, this subsubsection into the following two cases:

5.2.3 Case I: $\gamma < 0$

In this subsection, we mention our main result Theorem 5.3 along with other results. In this case, we consider $\gamma < 0$ so that $\mu^2 - 4\gamma = k^2 > 0$.

Theorem 5.3 *Let* (A_1), (W_O) *are true. Further, assume that* $\gamma < 0$ *such that* $\gamma + L \leq 0$, $N - \mu \leq 0$, $(\gamma + L)k - 2\gamma(N(\operatorname{sign} w') + \mu) \leq 0$ *and* $F(z, x, y, x', y') \geq 0$, $\forall\, z \in I$, *then* $(x_n)_n$ *and* $(y_n)_n$ *which are introduced in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, converge in* $C^1(I)$ *monotonically such that,*

$$x \leq v \leq u \leq y, \quad \forall\, z \in I,$$

where v *and* u *are solutions of* (1.6) *and* (1.7).

Proposition 5.5 *Let* $\gamma < 0$ *be such that* $\mu^2 - 4\gamma = -k^2 > 0$. *Further, assume that*

- (i) (A_1) *and* (W_O) *are true*;
- (ii) *there exists* $\gamma < 0$ *such that* $\gamma + L \leq 0$, $N - \mu \leq 0$, *and* $(\gamma + L)k - 2\gamma(N(\operatorname{sign} w') + \mu) \leq 0$. *Then* $(x_n)_n$ *and* $(y_n)_n$, *defined in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively such that*
  - (a) $x_{n+1} \geq x_n, \forall\, n \in N$,
  - (b) $y_{n+1} \leq y_n, \forall\, n \in N$.

Proof Using the Lemmas 3.2, 5.7, and 5.9, we can see the proof of Proposition 5.1. □

Proposition 5.6 *Let* (A_1) *and* (W_O) *are true. Further, assume that* $\gamma < 0$, *such that* $\gamma + L \leq 0$, $N - \mu \leq 0$, $(\gamma + L)k - 2\gamma(N(\text{sign}\, w') + \mu) \leq 0$ *and* $F(z, x, y, x', y') \geq 0$, $\forall\, z \in I$, *are valid then* $\forall\, n \in N$, $(x_n)_n$ *and* $(y_n)_n$ *given in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, such that* $x_n \leq y_n$.

Proof Proof is same as given in Proposition 5.2. □

5.2.4 Case II: $0 < \gamma < \frac{\mu^2}{4}$

In this subsection, we mention our main result Theorem 5.4 along with some other results which are used to prove the main result.

Theorem 5.4 *Let* (A_1) *and* (W_O) *hold, further if* $\gamma - L \leq 0$, $N - \mu \leq 0$, $H_3 \geq 0$, *and* $F(z, x, y, x', y') \geq 0$, $\forall\, z \in I$, *then* $(x_n)_n$ *and* $(y_n)_n$ *given in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, converge in* $C^1(I)$ *monotonically such that* $x \leq v \leq u \leq y$, $\forall z \in I$, *where* v *and* u *are solutions of* (1.6) *and* (1.7).

Proposition 5.7 *Let* $\gamma, \mu > 0$ *be such that* $\mu^2 - 4\gamma = -k^2 > 0$, $N - \mu \leq 0$, *and* $\gamma - L \leq 0$. *Further, assume that*

(i) (A_1) *and* (W_O) *are true;*
(ii) *there exists* $\gamma \in \min\{1, L\}$ *such that* $H_3 \geq 0$. *Then the functions* $(x_n)_n$ *and* $(y_n)_n$ *given in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, such that*
  (a) $x_{n+1} \geq x_n, \quad \forall n \in N,$
  (b) $y_{n+1} \leq y_n, \quad \forall n \in N.$

Proof Using the Lemmas 3.2, 5.8, and 5.10, we can see the proof of Proposition 5.1. □

Proposition 5.8 *Let* (A_1) *and* (W_O) *hold, further assume* $\gamma - L \leq 0$, $N - \mu \leq 0$, $H_3 \geq 0$, *and* $F(z, x, y, x', y') \geq 0$, $\forall\, z \in I$, *are valid then* $(x_n)_n$ *and* $(y_n)_n$ *introduced in* (4.3) *and* (4.4) *and* (4.5) *and* (4.6) *respectively, such that* $x_n \leq y_n$.

Proof Proof is same as given in Proposition 5.2. □

5.2.5 Priory bound

Lemma 5.11 *If* $f(z, w, w')$ *satisfies the following assumption,*

> (H_W) *let* $\varphi : R^+ \to R^+$ *is continuous such that* $\forall\, (z, w, v) \in U$, $|f(z, w, v)| \leq \varphi(|v|)$; *and satisfies*

$$\max_{z\in I} y - \min_{z\in I} x \leq \int_{l_0}^{\infty} \frac{\xi\, \mathrm{d}\xi}{\varphi(\xi)},$$

where $l_0 = 2\max\{\sup_{z\in I}|x(z)|, \sup_{z\in I}|y(z)|\}$, *then* $\exists\, r > 0$ *such that any solution* $w \in [x(z), y(z)]$ *of*

$$U_0(z, w(z)) \geq 0, \quad z \in I_0, \tag{5.23}$$

$$w'(0) = 0, \quad w(1) \geq \delta w(\eta), \tag{5.24}$$

satisfies $\|w'\|_\infty \leq r$.

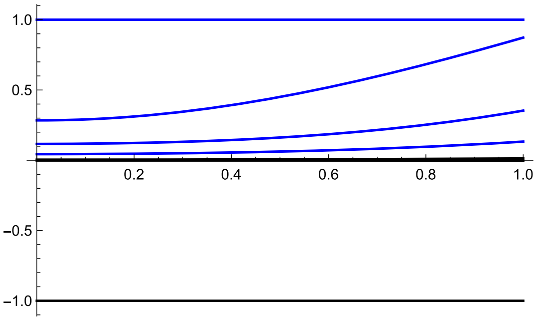


Fig. 2 For Example 6.1, $\gamma = 2,\ \mu = 0.040125,\ x_0$ to x_3 and y_0 to y_3 (reverse order)

Lemma 5.12 *If* $f(z, w, w')$ *satisfies* (H_W), *then* $\exists\ r > 0$ *such that any solution* $w \in [y(z), x(z)]$ *of*

$$L_0(z, w(z)) \geq 0, \quad z \in I_0, \tag{5.25}$$

$$w'(0) = 0, \quad w(1) \leq \delta w(\eta), \tag{5.26}$$

satisfies $\|w'\|_\infty \leq r$.

6 Mathematical demonstration

Example 6.1 Consider the following NLBVPs,

$$-\,w''(z) = \frac{\mathrm{e}^{w} - \mathrm{e}^{w'} + \sin\frac{z}{4}}{32}, \quad z \in I_0, \tag{6.1}$$

$$w'(0) = 0, \quad w(1) = 3w\left(\frac{1}{10}\right), \tag{6.2}$$

where $f(z, w, w') = \frac{\mathrm{e}^{w} - \mathrm{e}^{w'} + \sin\frac{z}{4}}{32}$, $\delta = 3$, $\eta = \frac{1}{10}$. Here $x = 1$ is lower solution and $y = -1$ is upper solution such that $y \leq x$ (reverse order). The Lipschitz constants are $L = \frac{e}{32}$ and $N = \frac{\mathrm{e}^r}{32}$, where $r = \frac{1}{4}$. Choosing $\mu = 2N$, then from Eq. (5.10), we get $\gamma \geq \frac{1}{64}\left(e - \frac{1}{e}\right)$. Since $\max\left\{L, (\mu^2/4), \frac{1}{64}\left(e - \frac{1}{e}\right)\right\} < \gamma < \frac{\pi^2}{4}$, we can choose some values between the above range so that (A_0), and $H_1 \geq 0$, are satisfied. Therefore, Theorem 5.1 is applicable. Thus, the solution of three-point NLBVPs (6.1) and (6.2) exists (Fig. 2).

$$\text{Region of existence (Reverse Order)} = \{(z, w) : 0 \leq z \leq 1, \quad y = -1 \leq w \leq x = 1\}.$$

Example 6.2 Consider the following NLBVPs,

$$-\,w''(z) = \frac{\frac{\mathrm{e}^2}{7} - 2w^3 + w'}{16}, \quad z \in I_0, \tag{6.3}$$

$$w'(0) = 0, \quad w(1) = \frac{1}{3} w\left(\frac{1}{4}\right). \tag{6.4}$$

Here $f(z, w, w') = \frac{1}{16}(\frac{\mathrm{e}^2}{7} - 2w^3 + w')$, $\delta = \frac{1}{3}$, $\eta = \frac{1}{4}$ and $x = -1$, $y = 1$ are in well ordered. The Lipschitz constants are $L = \frac{3}{8}$ and $N = \frac{1}{16}$. We choose $N \leq \mu \leq k$, where

Fig. 3 For Example 6.2, $\gamma = -3,\ \mu = 0.0625,\ x_0$ to x_4 and y_0 to y_3 (well order)

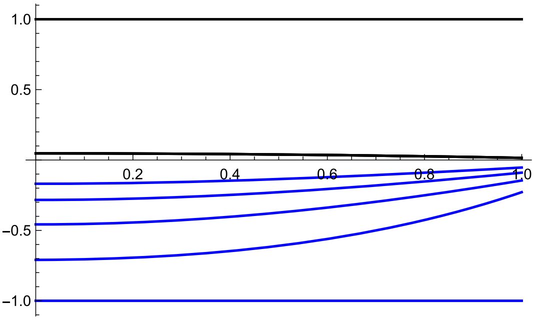
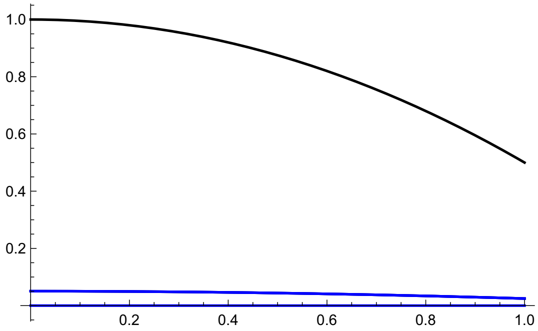


Fig. 4 For Example 6.3, $\gamma = 0.000624,\ \mu = 0.04,\ x_0$ to x_2 and y_0 to y_1 (well order)



$k \in \left[0, \frac{\pi}{2}\right]$. From Eq. (5.10), we have $\gamma \leq \frac{-1}{8}$. Now we can easily obtain a range for $\gamma < \min\{-L, \frac{-1}{8}, \frac{-Lk}{k-2(N+\mu)}\}$. For this range of γ (A_1) and $(\gamma + L)k - 2\gamma(N(\operatorname{sign} w') + \mu) \leq 0$, are satisfied. Therefore, Theorem 5.3 is applicable. Thus, the solution of the three-point BVPs (6.3) and (6.4) exists (Fig. 3).

$$\text{Region of existence (well order)} = \{(z, w) : 0 \leq z \leq 1, \quad x = -1 \leq w \leq y = 1\}.$$

Example 6.3 Consider the following NLBVPs,

$$-w''(z) = \frac{e^w + w'}{20}, \quad z \in I_0, \tag{6.5}$$

$$w'(0) = 0, \quad w(1) = \frac{1}{2} w\left(\frac{1}{5}\right). \tag{6.6}$$

We have $f(z, w, w') = \frac{e^w + w'}{20}$, $\delta = \frac{1}{2}$, $\eta = \frac{1}{5}$. $x = 0$ and $y = \left(1 - \frac{z^2}{2}\right)$ are initial lower and upper solutions arrive at in well ordered. Here, $L = \frac{1}{20}$ and $N = \frac{1}{20}$. If $N \leq \mu \leq \frac{\pi}{2}$, we can choose some sub interval of $0 < \gamma \leq \min\left\{L, \mu^2/4\right\}$ in which the nonlinear conditions, (A_1), $H_3 \geq 0$, and $F(z, x, y, x', y') \geq 0$ are valid. Therefore, Theorem 5.4 is applicable. Thus, the solution of the three-point BVPs (6.5) and (6.6) exists (Fig. 4).

$$\text{Region of existence (well order)} = \left\{(z, w) : 0 \leq z \leq 1, \quad x = 0 \leq w \leq y = \left(1 - \frac{z^2}{2}\right)\right\}.$$

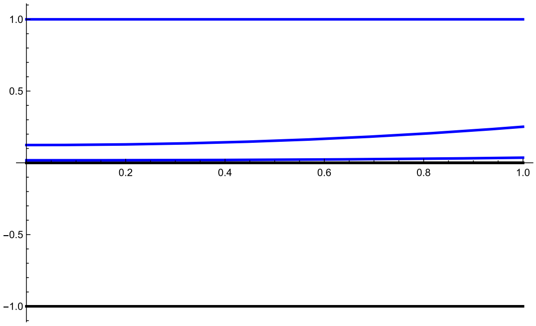


Fig. 5 For Example 6.4, $\gamma = \frac{1}{4},\ \mu = 1,\ x_0$ to x_2 and y_0 to y_2 (reverse order)

Example 6.4 Consider the following NLBVPs,

$$-w''(z) = \frac{w^3}{192} - \frac{w'}{2}, \quad z \in I_0, \tag{6.7}$$

$$w'(0) = 0, \quad w(1) = 2w\left(\frac{1}{8}\right), \tag{6.8}$$

where $f(z, w, w') = \frac{w^3}{192} - \frac{w'}{2}$, $\delta = 2, \eta = \frac{1}{8}$. Consider $y = -1$ is upper solution and $x = 1$ is lower solutions and $L = \frac{1}{64}$ and $N = \frac{1}{2}$. If we have $2N < \mu \leq \frac{\pi}{2}$. We can choose some values of $\gamma = \frac{\mu^2}{4}$ such that conditions (A_2), $\gamma - L \geq 0$, $2N - \mu \leq 0$, and $F(z, x, y, x', y') \geq 0$, and $H_2 \geq 0$ are satisfied. Therefore, Theorem 5.2 is applicable. Thus, the solution of the three-point BVPs (6.7) and (6.8) exists (Fig. 5).

$$\text{Region of existence (reverse order)} = \{(z, w) : 0 \leq z \leq 1, \quad y = -1 \leq w \leq x = 1\}.$$

7 Conclusions

In this article, with the help of different monotone iterative technique (DMIT) an analytical solution of three-point NLBVPs are studied that arises due to oscillating behavior in a suspension bridge. Through this method, we have shown that large size bridge design with m-point boundary conditions, where the nonlinear term includes derivative of solution, can easily be studied. Maximum and anti maximum principles are developed for $k^2 > 0$ and $k^2 \leq 0$ respectively. With the help of lower solution $x(z)$ and upper solution $y(z)$, we have discussed the classification of existence results such that $x \leq y$ (well order) and $y \leq x$ (reverse order). To prove monotonicity of x, y, the following conditions are assumed on the nonlinear function f,

- $f(z, w, w')$ is Lipschitz with respect to w';
- For $k^2 > 0$, if $\gamma < 0$, then $w_1 \leq w_2 \Rightarrow f(z, w_2, v) - f(z, w_1, v) \geq -L(w_2 - w_1)$;
- For $k^2 > 0$, if $0 < \gamma < \frac{\mu^2}{4}$, then $w_1 \leq w_2 \Rightarrow f(z, w_2, v) - f(z, w_1, v) \geq L(w_2 - w_1)$;
- If $-k^2 \leq 0$, then $w_1 \leq w_2 \Rightarrow f(z, w_2, v) - f(z, w_1, v) \leq L(w_2 - w_1)$.

Here μ and γ are taken as constants. We have obtained that DMIT is an efficient method to study the existence of NLBVPs and easy to handle. The existence of solutions of the NLBVPs are shown graphically.

Acknowledgements We are grateful to anonymous reviewers for their valuable comments and suggestions.

Appendix

See Fig. 6.

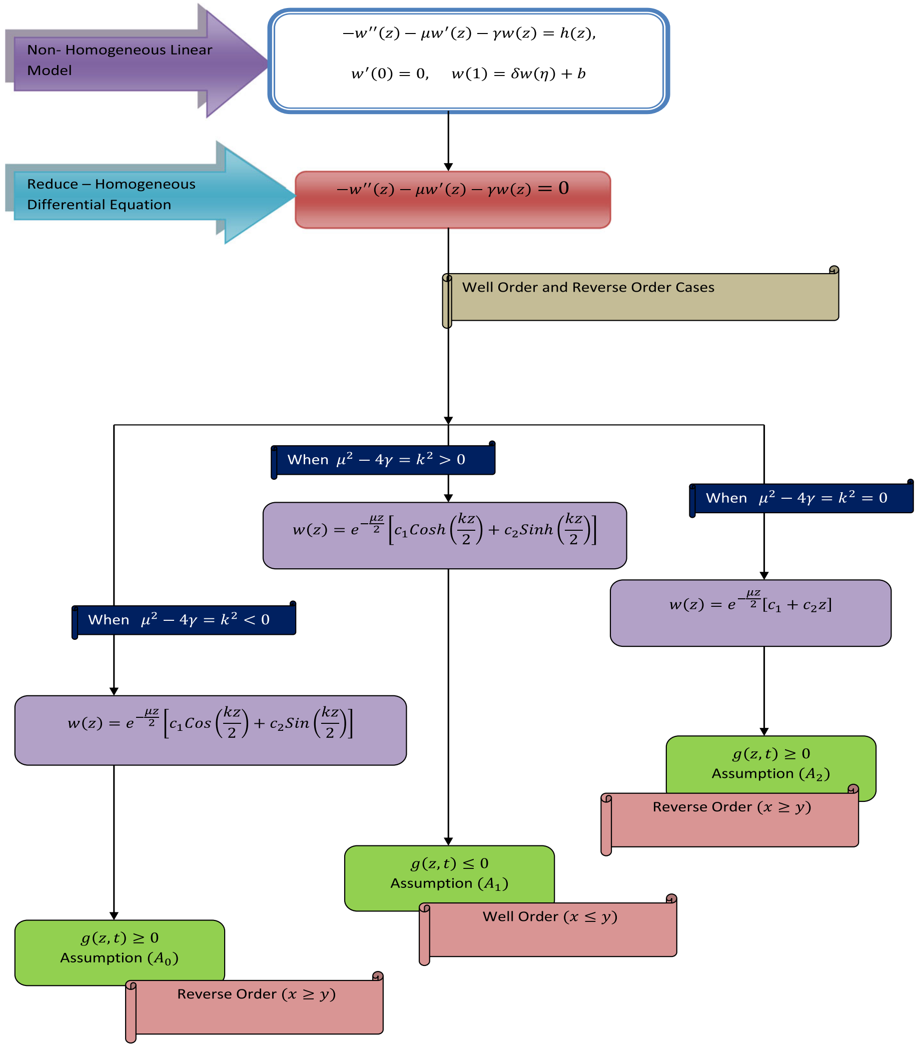


Fig. 6 Well and reverse order cases

References

Amann H, Ambrosetti A, Mancini G (1978) Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities. Math Z 158(2):179–194

Bernfeld SR, Chandra J (1977) Minimal and maximal solutions of nonlinear boundary value problems. Pac J Math 71(1):13–20

Cabada A, Habets P, Lois S (2001) Monotone method for the Neumann problem with lower and upper solutions in the reverse order. Appl Math Comput 117(1):1–14

Cherpion M, De Coster C, Habets P (2001) A constructive monotone iterative method for second-order BVP in the presence of lower and upper solutions. Appl Math Comput 123(1):75–91

Coster CD, Habets P (2006) Two-point boundary value problems: lower and upper solutions, vol 205. Mathematics in science and engineering. Elsevier, Amsterdam

Drábek P, Holubová G, Matas A, Nečesal P (2003) Nonlinear models of suspension bridges: discussion of the results. Appl Math 48(6):497–514

Gendzojan GV (1964) On two-sided Chaplygin approximations to the solution of the two point boundary value problem. Izv SSR Jiz Mate Nauk 17:21–27

Geng F, Cui M (2010) Multi-point boundary value problem for optimal bridge design. Int J Comput Math 87(5):1051–1056

Granas A (1976) Sur la méthode de continuité de poincaré. C R Acad Sci Paris 282:983–985

Li F, Jia M, Liu X, Li C, Li G (2008) Existence and uniqueness of solutions of second-order three-point boundary value problems with upper and lower solutions in the reversed order. Nonlinear Anal TMA 68(8):2381–2388

Lloyd NG (1978) Degree theory. Cambridge University Press, Cambridge

McKenna PJ, Lazer AC (1990) Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. SIAM Rev 32(4):537–578

Omari P (1986) A monotone method for constructing extremal solutions of second order scalar boundary value problems. Appl Math Comput 18(3):257–275

Omari P, Trombetta M (1992) Remarks on the lower and upper solutions method for second- and third-order periodic boundary value problems. Appl Math Comput 50(1):1–21

O'Regan D, El-Gebeily M (2008) Existence, upper and lower solutions and quasilinearization for singular differential equations. IMA J Appl Math 73(1):323–344

Picard (1893) Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires. J Math Pures Appl 9:217–272

Singh M, Verma AK (2013) On a monotone iterative method for a class of three point nonlinear nonsingular BVPs with upper and lower solutions in reverse order. J Appl Math Comput 43(1–2):99–114

Taliaferro SD (1979) A nonlinear singular boundary value problem. Nonlinear Anal TMA 3(6):897–904

Verma AK, Singh M (2014) Existence of solutions for three-point BVPs arising in bridge design. Electron J Differ Equ 2014(173):1–11

Verma AK, Pandit B, Verma L, Agarwal RP (2020) A review on a class of second order nonlinear singular BVPs. Mathematics 8(7):1045

Webb JRL (2012) Existence of positive solutions for a thermostat model. Nonlinear Anal RWA 13(2):923–938

Zou Y, Hu Q, Zhang R (2007) On numerical studies of multi-point boundary value problem and its fold bifurcation. Appl Math Comput 185(1):527–537