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Stochastic Comparisons of Heterogeneous Samples with Homogeneous Exponential Samples

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Abstract—In the present communication, stochastic comparison of a series (parallel) system having heterogeneous components with random lifetimes and series (parallel) system having homogeneous exponential components with random lifetimes has been studied. Further, conditions under which such a comparison is possible has been established.

Keywords—Exponential distribution, Order statistics, Star ordering, Stochastic ordering.

I. Introduction

IN reliability theory and applied probability, order statistics are used to study reliability properties of systems composed of components. Let $X_1, \ldots, X_n$ be independent non-negative random variables representing the lifetimes of compoents with their respective distribution functions as $F_1(\cdot), \ldots, F_n(\cdot)$. The k^{th} smallest of these n, X_i's , $i = 1, \ldots n$ be denoted by $X_{k:n}$. A k-out of n system with n independent components functions if and only if at least k out of these n independent components functions. The lifetime of $(n-k+1)$-out of n system is the same as that of $X_{k:n}$,i.e., the k^{th} order statistic. The lifetime of parallel (or 1-out of-n) and series (or n-out of-n) system are same as that of largest-order statistic $X_{n:n}$ and the smallest order statistic $X_{1:n}$.

Sample range and general sample spacings have been used extensively when observations are independent and identically distributed. On the other hand, in the non-iid case, few results are found in the literature due to the complicated nature of the expressions involved [Refer [1], [2]]. The stochastic comparison of the order statistics and the spacings for non-iid exponential random variables with the corresponding iid exponential random variables were considered by [3], [4], [5], [6], [7], [8],[9] and [10]. In addition, for further review on this topic, one may refer [11].

Let us recall the following definitions which are standard in the literature [see [13], [14] and [15]]. In rest of the paper, increasing and decreasing terms will be used for non-decreasing and non-increasing respectively.

Definition 1: Consider the random variable X (Y) with the probability density function $f(\cdot)$ $(g(\cdot))$, distribution function $F(\cdot)$ $(G(\cdot))$, survival function $\bar{F}(\cdot) = 1 - F(\cdot)$ $(\bar{G}(\cdot) = 1 - G(\cdot))$, failure rate function $r_X(x) = \frac{f(x)}{\bar{F}(x)}$ $\left(r_Y(x) = \frac{g(x)}{\bar{G}(x)}\right)$ and the reversed failure rate function $\tilde{r}_X(x) = \frac{f(x)}{F(x)}$ $\left(\tilde{r}_Y(x) = \frac{g(x)}{G(x)}\right)$. We say that X is smaller than Y in the

(a) likelihood ratio (lr) ordering (written as $X \leq_{lr} Y$) if $\frac{g(x)}{f(x)}$ increases in $x \in \mathbb{R}$.
(b) hazard rate (hr) ordering ($X \leq_{hr} Y$) if $\frac{\bar{G}(x)}{\bar{F}(x)}$ increases in $x \in \mathbb{R}$.
(c) reversed hazard rate (rh) ordering ($X \leq_{rfr} Y$) if $\frac{G(x)}{F(x)}$ increases in $x \in \mathbb{R}$.
(d) usual stochastic (st) ordering ($X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$, for all $x \in \mathbb{R}$.
(e) convex (c) ordering ($X \leq_c Y$) if $G^{-1}F(x)$ convex in $x \in \mathbb{R}$, where G^{-1} denotes the right-continuous inverse.
(f) star ($*$) ordering ($X \leq_* Y$) if $\frac{G^{-1}F(x)}{x}$ increases in $x \in \mathbb{R}$.
(g) dispersive ($disp$) ordering ($X \leq_{disp} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 < \alpha \leq \beta < 1$; where F^{-1} and G^{-1} be the right continuous inverses of F and G respectively.

In the next section, we investigate which series and parallel system ages faster in star ordering on the basis of stochastic comparison. Sufficient conditions under which such a comparison is possible has also been derived.

II. Which Series and Parallel System Ages Faster in Star Ordering?

The following lemma is being used for deriving the main results of the paper:

Lemma 1: Let Z be a random variable with probability density function $h(x)$, survival function $\bar{H}(x)$ and the failure rate function $r_H(x) = h(x)/\bar{H}(x),\ x \geq 0$. Then the function $\psi(x) = -\frac{1}{\lambda x}\ln \bar{H}(x)$ is decreasing (increasing) in x, if the failure rate function $r_H(x)$ is decreasing (increasing) in x.

Proof: It is easy to see

$$\psi'(x) = \frac{1}{\lambda x^2}\ln \bar{H}(x) + \frac{h(x)}{\lambda x \bar{H}(x)}.$$

Clearly,

$$\psi'(x) \leq (\geq) 0 \Leftrightarrow -\frac{\ln \bar{H}(x)}{x} \geq (\leq) \frac{h(x)}{\bar{H}(x)} \tag{1}$$

For $x \geq 0$, consider the function $\phi_1(x) = -\ln \bar{H}(x)$ and $\phi_2(x) = x$. Then $\phi_1'(x) = \frac{h(x)}{\bar{H}(x)}$ and $\phi_2'(x) = 1$. Applying the Lagrange's mean value theorem, for $0 \leq \xi \leq x$, we have

$$\frac{\phi_1(x)}{\phi_2(x)} = \frac{\phi_1'(\xi)}{\phi_2'(\xi)} = \frac{h(\xi)}{\bar{H}(\xi)} \geq (\leq) \frac{h(x)}{\bar{H}(x)}, \tag{2}$$

where the last inequality hold since $r_H(x)$ is decreasing (increasing) in x. Now the assertion follows using (1) and (2). ■

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Definition 2: X is said to be in decreasing failure rate (DFR) if the failure rate function $r_X(x)$ is decreasing function of x.

The following result gives the condition under which a series system with independently distributed components ages faster than independently and identically distributed exponential components in sense of star ordering:

Theorem 1: Let $X_1,\ldots,X_n$ be independent random variables with distribution functions $F_1(\cdot),\ldots,F_n(\cdot)$, respectively. Let $Y_1,\ldots,Y_n$ be a random sample of size n from an exponential distribution with common hazard rate λ. Then

$$Y_{1:n} \leq_* X_{1:n},$$

if $X_1,\ldots,X_n$ has DFR.

Proof: Let $Y_{1:n}$ and $X_{1:n}$ has distribution functions $G_{1:n}$ and $F_{1:n}$, respectively. For $x \geq 0$,

$$G_{1:n}(x) = P(Y_{1:n} \leq x) = \left(1 - e^{-n\lambda x}\right)$$

and

$$F_{1:n}(x) = P(X_{1:n} \leq x) = \left(1 - \prod_{i=1}^{n} \bar{F}_i(x)\right).$$

In order to prove the theorem, it is sufficient to show that $\left(G_{1:n}^{-1}F_{1:n}(x)\right)/x$ is decreasing in $x \geq 0$ (ref definition 1(f)). It may be noted that, for $x \geq 0$,

$$G_{1:n}^{-1}F_{1:n}(x) = -\frac{1}{n\lambda}\ln\left(\prod_{i=1}^{n}\bar{F}_i(x)\right).$$

Also,

$$\frac{G_{1:n}^{-1}F_{1:n}(x)}{x} = -\frac{1}{n\lambda x}\ln\left(\prod_{i=1}^{n}\bar{F}_i(x)\right) = -\frac{1}{n\lambda x}\ln\bar{H}(x),$$

where $\bar{H}(x) = \left(\prod_{i=1}^{n}\bar{F}_i(x)\right)$ is the survival function of the random variable Z. It may be noted that $r_H(x) = \sum_{i=1}^{n} r_{X_i}(x)$. Clearly, if $X_1, X_2,\ldots,X_n$ have DFR, then Z is DFR. Hence, using Lemma 1, $\left(G_{1:n}^{-1}F_{1:n}(x)\right)/x$ is decreasing in $x \geq 0$. ■

Corollary 1: Let $X_1,\ldots,X_n$ be independent exponential random variables with X_i having failure rate $\lambda_i,\ i = 1,\ldots,n$. Let $Y_1,\ldots,Y_n$ be a random sample of size n from an exponential distribution with common hazard rate λ. Then, $Y_{1:n} \leq_* X_{1:n}$.

Proposition 1: Let $X_1,\ldots,X_n$ be independent random variables with distribution functions $F_1(\cdot),\ldots,F_n(\cdot)$, respectively. Let $Y_1,\ldots,Y_n$ be a random sample of size n from an exponential distribution with common hazard rate λ. If $X_1,\ldots,X_n$ have DFR and

(a) $\sum_{i=1}^{n} r_i(x) \geq n\lambda$, then $Y_{1:n} \leq_{disp} X_{1:n}$;

(b) $\sum_{i=1}^{n} r_i(x) \leq n\lambda$, then $Y_{1:n} \leq_{st} X_{1:n}$.

Proof: (a) Under the hypothesis of the proposition, using Theorem 1 we conclude that $Y_{1:n} \leq_* X_{1:n}$. In order to prove the result, it is sufficient to show

$$\lim_{x\to 0}\frac{G_{1:n}^{-1}F_{1:n}(x)}{x} \geq 1$$

(see Theorem 4.B.3, page 215, of [14]).

Consider

$$\lim_{x\to 0}\frac{G_{1:n}^{-1}F_{1:n}(x)}{x} = \lim_{x\to 0}\left[-\frac{1}{n\lambda x}\prod_{i=1}^{n}\bar{F}_i(x)\right] \ \Big|\frac{0}{0} \text{ form}$$

Using L'Hospital Rule,

$$\begin{aligned} &= \lim_{x\to 0}\left[-\frac{\prod_{i=1}^{n}\bar{F}_i(x)\left(\sum_{i=1}^{n} r_i(x)\right)}{n\lambda\prod_{i=1}^{n}\bar{F}_i(x)}\right] \\ &= \lim_{x\to 0}\frac{1}{n\lambda}\sum_{i=1}^{n} r_i(x) \\ &\geq 1, \end{aligned}$$

since $\sum_{i=1}^{n} r_i(x) \geq n\lambda$. Hence the result. ■

Proof: (b) Under the hypothesis of the proposition, using Theorem 1 we conclude that $Y_{1:n} \leq_* X_{1:n}$. [12] shows that if $Y \leq_* X$, then

$$Y \leq_{st} X \Leftrightarrow \lim_{x\to 0^+}\frac{G(x)}{F(x)} \geq 1.$$

In order to prove the result, it is sufficient to show $\lim_{x\to 0^+}\frac{G_{1:n}(x)}{F_{1:n}(x)} \geq 1$.

Consider

$$\begin{aligned} \lim_{x\to 0^+}\frac{G_{1:n}(x)}{F_{1:n}(x)} &= \lim_{x\to 0^+}\frac{1 - e^{-n\lambda x}}{1 - \prod_{i=1}^{n}\bar{F}_i(x)} \\ &= \lim_{x\to 0}\frac{1 - e^{-n\lambda x}}{1 - \prod_{i=1}^{n}\bar{F}_i(x)} \ \Big|\frac{0}{0} \text{ form} \end{aligned}$$

Using L'Hospital Rule,

$$\begin{aligned} &= \lim_{x\to 0}\frac{n\lambda e^{-n\lambda x}}{\left(\prod_{i=1}^{n}\bar{F}_i(x)\right)\left(\sum_{i=1}^{n} r_i(x)\right)} \\ &= \frac{n\lambda}{\lim_{x\to 0}\sum_{i=1}^{n} r_i(x)} \\ &\geq 1, \end{aligned}$$

as $\sum_{i=1}^{n} r_i(x) \leq n\lambda$. Hence the result. ■

The following result gives the condition under which a parallel system with independently distributed components ages faster than independently and identically distributed exponential components in sense of convex ordering:

Theorem 2: Let $X_1,\ldots,X_n$ be independent random variables with distribution functions $F_1(\cdot),\ldots,F_n(\cdot)$, respectively. Let $Y_1,\ldots,Y_n$ be a random sample of size n from an exponential distribution with common hazard rate λ. Then

$$Y_{n:n} \leq_c X_{n:n},$$

if $\frac{\left(\sum_{i=1}^{n}\tilde{r}_i(x)\right)\left(\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)\right)}{1-\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)}$ is decreasing in x.

Proof: Let $Y_{n:n}$ and $X_{n:n}$ has distribution functions $G_{n:n}$ and $F_{n:n}$, respectively. For $x \geq 0$,

$$G_{n:n}(x) = P(Y_{n:n} \leq x) = \left(1 - e^{-\lambda x}\right)^n$$

and

$$F_{n:n}(x) = P(X_{n:n} \leq x) = \prod_{i=1}^{n} F_i(x).$$

In order to prove the theorem, it is sufficient to show that $G_{n:n}^{-1}F_{n:n}(x)$ is concave function of $x \geq 0$ (ref definition 1(e)). It may be noted that for $x \geq 0$,

$$G_{n:n}^{-1}F_{n:n}(x) = -\frac{1}{\lambda}\ln\left(1-\left(\prod_{i=1}^{n}F_i(x)\right)^{\frac{1}{n}}\right).$$

Also,

$$g_{n:n}(G_{n:n}^{-1}F_{n:n}(x)) = n\lambda\left(\prod_{i=1}^{n}F_i^{\frac{n-1}{n}}(x)\right)\left(1-\left(\prod_{i=1}^{n}F_i(x)\right)^{\frac{1}{n}}\right).$$

Differentiating $G_{n:n}^{-1}F_{n:n}(x)$ with respect to x, we have

$$(G_{n:n}^{-1}F_{n:n}(x))' = \frac{f_{n:n}(x)}{g_{n:n}(G_{n:n}^{-1}F_{n:n}(x))} = \frac{\left(\sum_{i=1}^{n}\tilde{r}_i(x)\right)\left(\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)\right)}{n\lambda\left(1-\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)\right)}.$$

Now, if $\frac{\left(\sum_{i=1}^{n}\tilde{r}_i(x)\right)\left(\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)\right)}{1-\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)}$ is decreasing in x, then $Y_{n:n} \leq_c X_{n:n}$. ∎

Corollary 2: Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with distribution functions $F_1(\cdot) = F_2(\cdot)\ldots = F_n(\cdot) = F(\cdot)$. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate λ. Then

$$Y_{n:n} \leq_c X_{n:n},$$

if $X_1, \ldots, X_n$ has DFR.

Proof: Since $\frac{\left(\sum_{i=1}^{n}\tilde{r}_i(x)\right)\left(\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)\right)}{1-\prod_{i=1}^{n}F_i^{\frac{1}{n}}(x)} = nr_X(x)$, therefore the result follows from Theorem 2. ∎

III. Conclusion

Stochastic comparison of a series and parallel systems having heterogeneous components with random lifetimes and series and parallel systems having homogeneous exponential components with random lifetimes has been studied. We find the conditions under which a series system with independently distributed components ages faster than independently and identically distributed exponential components in sense of star ordering. Further, we also find conditions under which a parallel system with independently distributed components ages faster than independently and identically distributed exponential components in sense of convex ordering.

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