On Generalized R-norm Measures of Fuzzy Information

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Abstract

In the present paper we characterize a new parametric generalized R-norm fuzzy entropy and studied. Further, we introduce a new R-norm fuzzy directed divergence measure and discuss its validity and monotonic property with respect to the parameter introduced. New generalized R-norm measures of total fuzzy ambiguity and fuzzy information improvement are also studied.

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1. INTRODUCTION

Fuzzy entropy is an important concept for measuring fuzzy information. Fuzzy entropy of a fuzzy set is a measure of fuzziness which arises from the intrinsic ambiguity or vagueness carried by the fuzzy set. Shannon (1948) was the first to use the word "entropy" to measure uncertain degree of the randomness in a probability distribution. Let X is a discrete random variable with probability distribution $P = (p_1, p_2, ..., p_n)$ in an experiment. According to Shannon, the information contained in this experiment is

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1}$$

which is well known Shannon's entropy (1948).

De Luca and Termini (1972) defined the following fuzzy information measure analogous to the Shannon entropy:

$$H(A) = -\sum_{i=1}^{n} [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))] \qquad (2)$$

On formulating the axioms P1 to P4 mentioned below, which became the essential properties required by the fuzzy information measure:

—P1 (*Sharpness*): $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x) = 0$ or $1; \forall x$.

—P2 (*Maximality*): $H(A)$ is maximum if and only if A is most fuzzy set, i.e. $\mu_A(x) = 0.5; \; \forall x$.

—P3 (*Resolution*): $H(A) \geq H(A^*)$, where A^* is sharpened version of A.

—P4 (*Symmetry*): $H(A) = H(\bar{A})$, where $\bar{A}$ is the complement of A i.e. $\mu_{\bar{A}}(x_i) = 1 - \mu_A(x_i)$.

However, we have other fuzzy information measures but (2) can be regarded as the first correct measure of ambiguity of a fuzzy set. In addition, Yager (1979) also defined an entropy of a fuzzy set based on the distance from the set to its complement set. Similarly, Kosko (1986) introduced another kind of fuzzy entropy by considering the distance from a set to its nearest nonfuzzy set and the distance from the set to its farthest nonfuzzy set. Another kind of fuzzy entropy with an exponential function was introduced by Pal and Pal (1989). Later on, they introduced the concept of higher r^{th} order entropy of a fuzzy set in their paper Pal and Pal (1992). Further, Bhandari and Pal (1993) made a survey on information measures on fuzzy sets and gave some new measures of fuzzy entropy.

Let $\Delta_n = \{P = (p_1, p_2, \ldots, p_n),\ p_i \geq 0,\ i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1\}$ be the set of all probability distributions associated with a discrete random variable X taking finite values $x_1, x_2, \ldots, x_n$.

Boekee and Lubbe (1980) defined and studied R-norm information measure of the distribution P for $R \in \mathbb{R}^+$ as given by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \quad ; \quad R > 0,\ R \neq 1. \qquad (3)$$

The measure (3) is a real function from Δ_n to $\mathbb{R}^+$ and is called R-norm information measure. The most important property of this measure is that when $R \to 1$, it approaches to Shannon's entropy and in case $R \to \infty$, $H_R(P) \to (1 - \max p_i)$; $i = 1, 2, \ldots, n$.

Analogous to measure (3), Hooda (2004) proposed and characterized the following fuzzy entropy:

$$H_R(A) = \frac{R}{R-1}\left[\sum_{i=1}^{n} 1 - \left(\mu_A^R(x_i) + (1-\mu_A(x_i))^R\right)^{\frac{1}{R}}\right]; R > 0,\ R \neq 1. \quad (4)$$

Further, Hooda and Ram (1998) gave a parametric generalization of (3) by

$$H_R^\beta(P) = \frac{R}{R+\beta-2}\left[1 - \left(\sum_{i=1}^{n} p_i^{\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}}\right]; \quad (5)$$

where $0 < \beta \leq 1$, $R > 0$ and $R + \beta \neq 2$.

The measure (5) is called the generalized R-norm entropy of degree β and it reduces to (3), when $\beta \to 1$. In case $R \to 1$, (5) reduces to

$$H_1^\beta(P) = \frac{1}{\beta-1}\left[1 - \left(\sum_{i=1}^{n} p_i^{1/(2-\beta)}\right)^{2-\beta}\right]; \quad (6)$$

where $0 < \beta \leq 1$, $R > 0$ and $R + \beta \neq 2$.

Setting $\theta = \frac{1}{2-\beta}$ in (6), we get

$$H^\theta(P) = \frac{\theta}{\theta-1}\left[1 - \left(\sum_{i=1}^{n} p_i^\theta\right)^{\frac{1}{\theta}}\right]; \quad \frac{1}{2} < \theta \leq 1. \quad (7)$$

This is an information measure which has been mentioned by Arimoto (1971) as an example of a generalized class of information measures. It may also be noted that (7) approaches to Shannon's entropy when $\theta \to 1$.

Next, suppose that $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$ be two given probability distributions belonging to Δ_n. Kullback and Leibler (1951) obtained the measure of directed divergence of P from Q as

$$D(P:Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}. \quad (8)$$

Kullback (1959) suggested the measure of symmetric divergence as

$$J(P:Q) = D(P:Q) + D(Q:P) = \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i}. \quad (9)$$

Motivated by Kullback and Leibler measure, Bhandari and Pal (1993) suggested the following fuzzy directed divergence measure of fuzzy set A from B:

$$I(A,B)=\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right], \tag{10}$$

and the corresponding symmetric divergence measure by

$$J(A,B)=I(A,B)+I(B,A),$$

which on simplification gives

$$J(A,B)=\sum_{i=1}^{n}[(\mu_A(x_i)-\mu_B(x_i)]\log\frac{\mu_A(x_i)(1-\mu_A(x_i))}{\mu_B(x_i)(1-\mu_B(x_i))}. \tag{11}$$

Further, corresponding to Harvda and Charvat's (1967) measure of directed divergence given by

$$D^{\beta}(P:Q)=\frac{1}{\beta-1}\sum_{i=1}^{n}(p_i^{\beta}q_i^{1-\beta}-1);\quad \beta>0,\ \beta\neq 1, \tag{12}$$

Hooda (2004) suggested the following measures of fuzzy directed divergence and symmetric divergence measure:

$$I^{\beta}(A,B)=\frac{1}{\beta-1}\sum_{i=1}^{n}\left[\mu_A^{\beta}(x_i)\mu_B^{1-\beta}(x_i)+(1-\mu_A(x_i))^{\beta}(1-\mu_B(x_i))^{1-\beta}-1\right]; \tag{13}$$

where $\beta>0$, $\beta\neq 1$

and

$$J^{\beta}(A,B)=I^{\beta}(A,B)+I^{\beta}(B,A) \tag{14}$$

respectively.

A new generalized measure of fuzzy information analogous to (5) is proposed and its validity to be a fuzzy information measure is proved in Section 2. In Section 3, we propose a generalized fuzzy directed divergence measure analogous to a R-norm directed divergence and prove its validity. In Section 4, we investigate the monotonic nature of the generalized measure of fuzzy information and the R-norm fuzzy directed divergence. R-norm generalized measures of total ambiguity and Fuzzy information improvement are also studied in Section 5.

2. A GENERALIZED R-NORM FUZZY INFORMATION MEASURE

Analogous to (5), we propose the following measure of fuzzy information:

$$H_R^\beta(A) = \frac{R}{R+\beta-2}\left[\sum_{i=1}^{n} 1 - \left[(\mu_A(x_i))^{\frac{R}{2-\beta}} + (1-\mu_A(x_i))^{\frac{R}{2-\beta}}\right]^{\frac{2-\beta}{R}}\right]; \qquad (15)$$

where $0 < \beta \leq 1,\ R > 0\ R+\beta \neq 2$ and prove its validity in the next theorem.

Theorem 1: The measure (15) is a valid measure of fuzzy information.

Proof: To prove that the measure (15) is a valid fuzzy information measure, we shall show that four properties (P1) to (P4) are satisfied.

The measure (15) can be written

$$H_R^\beta(A) = \lambda\left[\sum_{i=1}^{n} 1 - \left[(\mu_A(x_i))^{\nu} + (1-\mu_A(x_i))^{\nu}\right]^{\frac{1}{\nu}}\right]; \qquad (16)$$

where $\lambda = \frac{R}{R+\beta-2},\ \nu = \frac{R}{2-\beta},\ \nu > 0,\ \nu \neq 1.$

P1 (*Sharpness*):

If $H_R^\beta(A) = 0$, then

$$(\mu_A(x_i))^{\nu} + (1-\mu_A(x_i))^{\nu} = 1. \qquad (17)$$

Since $\nu(\neq 1) > 0$, therefore, (17) is satisfied in case $\mu_A(x_i)= 0$ or $1, \forall i = 1, 2, \ldots, n.$

Conversely, if A be a non-fuzzy set, then either $\mu_A(x_i)= 0$ or $\mu_A(x_i)= 1$. It implies $(\mu_A(x_i))^{\nu} + (1-\mu_A(x_i))^{\nu} = 1$ for $\nu > 0,\ \nu \neq 1$, for which $H_R^\beta(A)= 0$. Hence $H_R^\beta(A)= 0$ if and only if A is non-fuzzy set or crisp set.

P2 (*Maximality*):

Differentiating $H_R^\beta(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{\partial H_R^\beta}{\partial \mu_A(x_i)} = -\lambda[(\mu_A(x_i))^{\nu} + (1-\mu_A(x_i))^{\nu}]^{\frac{1-\nu}{\nu}}\left[(\mu_A(x_i))^{\nu-1} - (1-\mu_A(x_i))^{\nu-1}\right]. \qquad (18)$$

Let $0 \leq \mu_A(x_i) < 0.5$, then two cases arise

Case 1: $R > 2-\beta$

In this case we have $\lambda > 0,\ \nu > 1$ and $(\mu_A(x_i))^{\nu-1} - (1-\mu_A(x_i))^{\nu-1} < 0$ which implies that $\frac{\partial H_R^\beta}{\partial \mu_A(x_i)} > 0.$

Case 2: $R < 2-\beta$

In this case we have $\lambda < 0,\ \nu < 1$ and $(\mu_A(x_i))^{\nu-1} - (1-\mu_A(x_i))^{\nu-1} > 0$ which implies that $\frac{\partial H_R^\beta}{\partial \mu_A(x_i)} > 0.$

Hence $H_R^\beta(A)$ is an increasing function of $\mu_A(x_i)$ satisfying $0 \le \mu_A(x_i) < 0.5$. Similarly, it can be proved that $H_R^\beta(A)$ is a decreasing function of $\mu_A(x_i)$ satisfying $0.5 < \mu_A(x_i) \le 1$. It is evident that $\frac{\partial H_R^\beta}{\partial \mu_A(x_i)} = 0$, when $\mu_A(x_i) = 0.5$. Hence $H_R^\beta(A)$ is a concave function and it has a global maximum at $\mu_A(x_i) = 0.5$. It shows that $H_R^\beta(A)$ is maximum if and only if A is the most fuzzy set.

P3 (*Resolution*):

Since $H_R^\beta(A)$ is an increasing function of $\mu_A(x_i)$ in [0, 0.5) and decreasing function in (0.5, 1], therefore

$$\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow H_R^\beta(A^*) \le H_R^\beta(A) \text{ in } [0, 0.5) \tag{19}$$

and

$$\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow H_R^\beta(A^*) \le H_R^\beta(A) \text{ in } (0.5, 1]. \tag{20}$$

Taking (19) and (20) together, we get $H_R^\beta(A^*) \le H_R^\beta(A)$.

P4 (*Symmetry*):

Clearly from the definition of $H_R^\beta(A)$and with $\mu_{\bar{A}}(x_i) = 1 - \mu_A(x_i)$, we conclude that $H_R^\beta(\overline{A}) = H_R^\beta(A)$.

Hence $H_R^\beta(A)$ satisfies all the properties of fuzzy entropy and therefore is a valid measure of fuzzy entropy.

It may be noted that (15) reduces to (4), when $\beta = 1$ and reduces to (2) when $\beta = 1$ and $R \to 1$. In case $\beta = 1$ and $R \to \infty$, (15) reduces to $\sum_{i=1}^{n} [1 - \max\{\mu_A(x_i),\ 1 - \mu_A(x_i)\}]$.

3. R-NORM FUZZY DIRECTED DIVERGENCE MEASURES

Let $P(p_1, p_2, \ldots, p_n)$ and $Q(q_1, q_2, \ldots, q_n)$ be the posterior and prior probability distribution of a random variable respectively in an experiment. Recently, Hooda and Sharma (2007) defined the R-norm directed divergence given by

$$D_R(P : Q) = \frac{R}{R-1}\left[\left(\sum_{i=1}^{n} p_i^R q_i^{1-R}\right)^{\frac{1}{R}} - 1\right] \tag{21}$$

and R-norm measure of inaccuracy given by

$$D_R(P/Q) = D_R(P:Q) + H_R(P) = \frac{R}{R-1}\left[\left(\sum_{i=1}^{n} p_i^R q_i^{1-R}\right)^{\frac{1}{R}} - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]. \tag{22}$$

It may be seen that when $R \to 1$, (21) reduces to

$$D(P:Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i},$$

which is the measure of directed divergence due to Kullback and Leibler (1951) and (22) reduces to Kerridge inaccuracy due to Kerridge (1961)

$$D(P/Q) = -\sum_{i=1}^{n} p_i \log q_i.$$

Analogous to (21) we propose the following measure of fuzzy directed divergence of fuzzy set A from fuzzy set B:

$$I_R(A,B) = \frac{R}{R-1}\sum_{i=1}^{n}\left[\left(\mu_A^R(x_i)\mu_B^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}\right)^{\frac{1}{R}} - 1\right], \tag{23}$$

where $R > 0$, $R \neq 1$ and measure of fuzzy symmetric divergence

$$J_R(A,B) = I_R(A,B) + I_R(B,A). \tag{24}$$

Next, we show that $I_R(A,B)$ is a valid measure i.e. $I_R(A,B) \geq 0$ with equality if $\mu_A(x_i) = \mu_B(x_i)$ for each $i = 1, 2, \ldots, n$.

Let $\sum_{i=1}^{n} \mu_A(x_i) = s$, $\sum_{i=1}^{n} \mu_B(x_i) = t$, then

$$\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)}{s}\right)^R\left(\frac{\mu_B(x_i)}{t}\right)^{1-R} - 1 \geq 0$$

or

$$\sum_{i=1}^{n} \mu_A^R(x_i)\mu_B^{1-R}(x_i) \geq s^R t^{1-R}. \tag{25}$$

Similarly, we can write

$$\sum_{i=1}^{n} (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} \geq (n-s)^R(n-t)^{1-R}. \tag{26}$$

Adding (25) and (26), we get

$$\sum_{i=1}^{n} \mu_A^R(x_i)\mu_B^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} \geq s^R t^{1-R} + (n-s)^R(n-t)^{1-R}. \tag{27}$$

Case 1: $0 < R < 1$

Let $\mu_A^R(x_i)\mu_B^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} = x_i$, then $x_i < 1$ and $\frac{1}{R} > 1$ which implies $x_i - 1 > (x_i)^{1/R} - 1$.

Since $\frac{R}{R-1} < 0$, therefore $\frac{R}{R-1}\sum_{i=1}^{n}\left[(x_i)^{1/R} - 1\right] \geq \frac{R}{R-1}\sum_{i=1}^{n}(x_i - 1)$.

Thus, we have

$$I_R(A,B) \geq \frac{R}{R-1}\left[s^R t^{1-R} + (n-s)^R(n-t)^{1-R} - n\right].$$

Further let $\phi(s) = \frac{R}{R-1}\left[s^R t^{1-R} + (n-s)^R(n-t)^{1-R} - n\right]$, then

$\phi'(s) = \frac{R}{R-1}\left[R\left(\frac{s}{t}\right)^{R-1} - R\left(\frac{n-s}{n-t}\right)^{R-1}\right]$ and

$\phi''(s) = R^2\left[\frac{1}{t}\left(\frac{s}{t}\right)^{R-2} + \frac{1}{n-t}\left(\frac{n-s}{n-t}\right)^{R-2}\right] > 0.$

This shows that $\phi(s)$ is a convex function of s whose minimum value arises when $\frac{s}{t}\left(=\frac{n-s}{n-t}\right) = 1$ and is equal to zero. Hence, $\phi(s) > 0$ and vanishes only when $s = t$.

Case 2: $R > 1$

In this case (27) can be written

$$\left(\sum_{i=1}^{n}\mu_A^R(x_i)\mu_B^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} - 1\right)^{1/R}$$
$$\geq \left(s^R t^{1-R} + (n-s)^R(n-t)^{1-R} - n\right)^{1/R}. \tag{28}$$

Also, we have

$$\sum_{i=1}^{n}\left[\left(\mu_A^R(x_i)\mu_B^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}\right)^{\frac{1}{R}} - 1\right]$$
$$\geq \left(\sum_{i=1}^{n}\mu_A^R(x_i)\mu_B^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} - 1\right)^{1/R} \tag{29}$$

Now (28) and (29) together implies that

$$I_R(A,B) \geq \frac{R}{R-1}\left[s^R t^{1-R} + (n-s)^R(n-t)^{1-R} - n\right]^{\frac{1}{R}}.$$

Let $\phi(s) = \frac{1}{R-1}\left[s^R t^{1-R} + (n-s)^R(n-t)^{1-R} - n\right]$, then

$\phi'(s) = \frac{1}{R-1}\left[R\left(\frac{s}{t}\right)^{R-1} - R\left(\frac{n-s}{n-t}\right)^{R-1}\right]$ and

$\phi''(s) = \left[\frac{R}{t}\left(\frac{s}{t}\right)^{R-2} + \frac{R}{n-t}\left(\frac{n-s}{n-t}\right)^{R-2}\right] > 0.$

This shows that $\phi(s)$ is a convex function of s whose minimum value arises when $\frac{s}{t}\left(=\frac{n-s}{n-t}\right) = 1$ and is equal to zero. Hence, $\phi(s) > 0$ and vanishes only when

$s = t$ i.e. for all $R(\neq 1) > 0$, $I_R(A, B) \geq 0$ and vanishes only when $A = B$. Thus $I_R(A, B)$ is a valid measure of directed divergence of fuzzy set A from fuzzy set B and consequently, the corresponding measure of fuzzy symmetric divergence $J_R(A, B)$ is a valid measure.

It may be noted that $\lim_{R\to 1} I_R(A, B) = I(A, B)$ and $\lim_{R\to 1} J_R(A, B) = J(A, B)$, where $I(A, B)$ and $J(A, B)$ are the fuzzy directed divergence and symmetric divergence measures given by (10) and (11) respectively.

4. MONOTONICITY OF FUZZY INFORMATION AND FUZZY DIRECTED DIVERGENCE MEASURES

Two fuzzy sets A and B are said to be *fuzzy-equivalent* if $\mu_B(x_i)$ = either $\mu_A(x_i)$ or $1 - \mu_A(x_i)$ for each value of i. It is clear that fuzzy-equivalent sets have the same entropy but two sets may have the same fuzzy entropy without being fuzzy equivalent. From the fuzziness point of view there is no essential difference between fuzzy equivalent sets. A *standard fuzzy set* is that member of the class of fuzzy equivalent sets all of whose membership value are ≤ 0.5.

Let A_1 = (0.2, 0.3, 0.4, 0.2, 0.3), A_2= (0.4, 0.3, 0.2, 0.2, 0.4), A_3= (0.3, 0.2, 0.3, 0.3, 0.3) be any three fuzzy sets in standard form. Consider four different values of R, i.e., 0.6, 1, 2, 3, and $0 < \beta \leq 1$. Using (15), we have constructed Table 1 listed below. Looking at Table 1, it is clear that the fuzzy information measure given by (15) is a monotonically decreasing function of β and R. It can be shown analytically that $\frac{d}{d\beta}\left(H_R^\beta(A)\right) \leq 0;\ \forall 0 < \beta \leq 1$ and $\frac{d}{dR}\left(H_R^\beta(A)\right) \leq 0;\ \forall\ R > 0$ and this implies monotonic decreasing nature of the fuzzy information measure with respect to β and R respectively. However, it is observed that for the most fuzzy set, the maximum value of $H_R^\beta(A)$ depends on the value of β and R, but it will be less than or equal to n.

Table 1

β	$H^{\beta}_{0.6}(A)$			$H^{\beta}_{1}(A)$			$H^{\beta}_{2}(A)$			$H^{\beta}_{3}(A)$		
	A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
0.001	7.76	7.86	7.84	4.41	4.47	4.46	2.89	2.96	2.94	2.45	2.52	2.50
0.002	7.75	7.85	7.84	4.41	4.47	4.46	2.89	2.96	2.94	2.45	2.52	2.50
0.005	7.74	7.83	7.82	4.40	4.47	4.46	2.89	2.95	2.94	2.45	2.52	2.50
0.01	7.71	7.80	7.79	4.39	4.46	4.45	2.89	2.95	2.94	2.45	2.52	2.49
0.02	7.65	7.75	7.73	4.38	4.44	4.43	2.88	2.94	2.93	2.44	2.51	2.49
0.05	7.49	7.58	7.56	4.32	4.39	4.38	2.86	2.92	2.91	2.43	2.50	2.48
0.1	7.22	7.31	7.29	4.24	4.30	4.29	2.83	2.89	2.88	2.40	2.48	2.45
0.2	6.71	6.80	6.79	4.06	4.13	4.12	2.76	2.83	2.81	2.36	2.43	2.40
0.3	6.25	6.33	6.32	3.90	3.96	3.95	2.70	2.76	2.75	2.31	2.39	2.36
0.4	5.82	5.90	5.89	3.74	3.80	3.79	2.63	2.70	2.68	2.26	2.34	2.31
0.5	5.43	5.50	5.49	3.59	3.65	3.64	2.56	2.63	2.61	2.21	2.29	2.26
0.6	5.06	5.13	5.12	3.45	3.50	3.49	2.50	2.57	2.54	2.16	2.25	2.21
0.7	4.73	4.79	4.78	3.30	3.36	3.35	2.43	2.50	2.48	2.11	2.20	2.15
0.8	4.41	4.48	4.47	3.16	3.22	3.21	2.36	2.43	2.40	2.06	2.15	2.10
0.9	4.12	4.18	4.17	3.03	3.09	3.08	2.29	2.36	2.33	2.01	2.10	2.04
1	3.85	3.91	3.90	2.89	2.95	2.94	2.21	2.29	2.26	1.95	2.05	1.99

Similarly, monotonic nature of the fuzzy directed divergence measure given by (23) can be observed in the computed Table 2 for three different sample pairs of standard fuzzy sets given by

$A_1 = (0.3, 0.5, 0.3, 0.2, 0.1)$; $A_2 = (0.4, 0.3, 0.4, 0.2, 0.5)$;

$A_3 = (0.5, 0.2, 0.2, 0.3, 0.4)$; $B_1 = (0.2, 0.4, 0.4, 0.3, 0.2)$;

$B_2 = (0.2, 0.4, 0.4, 0.2, 0.2)$; and $B_3 = (0.3, 0.4, 0.4, 0.2, 0.3)$.

Table 2 shows that the fuzzy directed divergence given by (23) is monotonic increasing function of R. This monotonic nature of the fuzzy directed divergence can also be proved analytically by showing that $\frac{d}{dR}\left(I_R(A,B)\right) \geq 0;\ \forall R > 0$.

Table 2

R	$I_R(A_1, B_1)$	$I_R(A_2, B_2)$	$I_R(A_3, B_3)$
0.1	0.013827	0.029164	0.032484
0.2	0.027531	0.059652	0.065004
0.5	0.067891	0.159083	0.162301
0.8	0.107101	0.269945	0.258256
1	0.132602	0.349394	0.320959
1.2	0.157595	0.432339	0.382348
1.5	0.19415	0.561022	0.471532
2	0.252654	0.777309	0.611291
5	0.543459	1.674712	1.207667
10	0.817193	2.156136	1.640287
50	1.11155	2.563536	2.06191
75	1.135962	2.597848	2.096904
100	1.148134	2.615028	2.114372
150	1.160285	2.632223	2.131821
200	1.166353	2.640828	2.140539
300	1.172415	2.649436	2.149253
400	1.175444	2.653742	2.153608
420	1.175877	2.654357	2.15423
430	1.176078	2.654644	2.154519
440	1.17627	2.654917	2.154795

5. MEASURES OF TOTAL AMBIGUITY AND FUZZY INFORMATION IMPROVEMENT

5.1 Total ambiguity

Let A and B be two fuzzy sets. The total ambiguity of the fuzzy set A about set B is the sum of two components:

—Fuzzy entropy present in the fuzzy set A, and

—fuzzy directed divergence of A from B measured by $I(A, B)$.

Using Harvda and Charvat's (1967) measure, Kapur (1997) estimated the total fuzzy ambiguity as

$$TA = \frac{1}{1-\alpha}\left[\sum_{i=1}^{n} \mu_A^{\alpha}(x_i)(1-\mu_B^{1-\alpha}(x_i)) + \sum_{i=1}^{n} (1-\mu_A(x_i))^{\alpha}\left(1-(1-\mu_B(x_i))^{1-\alpha}\right)\right].$$

Corresponding to fuzzy information measure (4) and the proposed fuzzy directed divergence (23), total ambiguity is given by

$$TA = \frac{R}{R-1}\left[\sum_{i=1}^{n}\left(1-(\mu_A^R(x_i)+(1-\mu_A(x_i))^R\right)^{\frac{1}{R}}\right.$$
$$\left.+\sum_{i=1}^{n}\left[\left\{\mu_A^R(x_i)\mu_B^{1-R}(x_i)+(1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}\right\}^{\frac{1}{R}}-1\right]\right]$$
$$= \frac{R}{R-1}\sum_{i=1}^{n}\left[\left(\mu_B(x_i)\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^R+(1-\mu_B(x_i))\left(\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right)^R\right)^{\frac{1}{R}}\right.$$
$$\left.-\left(\mu_A^R(x_i)+(1-\mu_A(x_i))^R\right)^{\frac{1}{R}}\right].$$

Total ambiguity is a fuzzy measure of inaccuracy analogous to Kerridge (1961) inaccuracy and is related to two fuzzy sets. It is not symmetric as we get something different if we interchange the role of the fuzzy sets A and B.

5.2 R-norm fuzzy information improvement measure

Let P and Q be observed and predicted distributions of a random variable respectively. Let $R = (r_1, r_2, \ldots, r_n)$ be the revised probability distribution of Q, then

$$I(P:Q) - I(P:R) = \sum_{i=1}^{n} p_i \log \frac{r_i}{q_i}, \tag{30}$$

which is known as Theil's measure (1967) of information improvement and has found wide applications in economics, accounts and financial management. Similarly, suppose the correct fuzzy set is A and originally our estimate for it was the fuzzy set B and that was revised to set fuzzy set C. The original ambiguity was $I(A,B)$ and final ambiguity is $I(A,C)$, so the reduction in ambiguity is

$$\begin{aligned} I(A,B,C) &= I(A,B) - I(A,C), \\ &= \sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_C(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))\log\frac{(1-\mu_C(x_i))}{(1-\mu_B(x_i))}\right]. \end{aligned} \tag{31}$$

The measure $I(A,B,C)$ given by (31) can be called fuzzy information improvement measure. Corresponding to fuzzy directed divergence given by (23), the reduction in ambiguity is given by

$$\begin{aligned} I_R(A,B,C) &= I_R(A,B) - I_R(A,C) \\ &= \frac{R}{R-1}\sum_{i=1}^{n}\Big[\left\{\mu_A^R(x_i)\mu_B^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}\right\}^{\frac{1}{R}} \\ &\quad - \left\{\mu_A^R(x_i)\mu_C^{1-R}(x_i) + (1-\mu_A(x_i))^R(1-\mu_C(x_i))^{1-R}\right\}^{\frac{1}{R}}\Big], \qquad (32) \end{aligned}$$

which can be called as R-norm fuzzy information improvement measure. It can also be proved that $I_R(A,B,C) \to I(A,B,C)$ i.e. (32) reduces to (31), provided $R \to 1$.

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