ON COMPLEX EXTENSIONS AND INFORMATION MEASURES OF NEUTROSOPHIC, HESITANT & PICTURE FUZZY SETS IN DECISION MAKING

Thesis submitted in fulfillment of the requirement for the degree of

DOCTOR OF PHILOSOPHY

By

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DECLARATION BY THE SCHOLAR

I hereby declare that the work reported in the Ph.D. thesis entitled, **"On Complex Extensions and Information Measures of Neutrosophic, Hesitant & Picture Fuzzy Sets in Decision Making"** submitted at **Jaypee University of Information Technology, Waknaghat, India**, is an authentic record of my work carried out under the supervision of **Prof. Rakesh Kumar Bajaj & Prof. Karanjeet Singh**. I have not submitted this work elsewhere for any other degree or diploma. I am fully responsible for the content of my Ph.D. Thesis.

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SUPERVISOR'S CERTIFICATE

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"God created teachers to make a difference in the world".

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(Mahima Poonia)

Abstract

The objective of this thesis entitled, *"On Complex Extensions and Information Measures of Neutrosophic, Hesitant & Picture Fuzzy Sets in Decision Making"* is to study the concept of fuzzy extensions on real and complex planes along with various applications in detail. The work presented in this thesis has been carried out in order to fulfill the objective to propose the notion of cohesive fuzzy set, complex neutrosophic matrix, energy of picture fuzzy graphs with their various important operations & applications in the field of decision-making.

In literature, the notion of fuzzy sets and its generalized extensions have made a large amount of contribution in the progress of scientific and engineering research area. It has large number of applications in the areas (theoretical as well as practical) related to engineering, arts, humanities, computer science, health sciences, life sciences, physical sciences etc due to its ability of dealing with the uncertainty factor. In the current work, these concepts have been explained in detail and brief structure of the format of the presented work is presented below:

We have presented the fundamental background of hesitant, neutrosophic, picture and complex fuzzy sets with their mathematical form, definitions, operations and literature survey in Chapter 1.

In Chapter 2, a novel concept of Cohesive fuzzy set (CHFS) has been proposed as a synchronized generalization from innovative notions of complex fuzzy set and hesitant fuzzy set. We have also studied the relationship and connections between the Cohesive Fuzzy Set and Complex Intuitionistic Fuzzy Set along with the validation of the obtained results. Based on the proposed notion, various properties, operations and identities have been established with their necessary proof. The applications of CHFS in the process of filtering the signals for getting the reference signal using the necessary Fourier cosine transform (FCT)/inverse FCT and identifying maximum number of sunspots in a particular interval under a solar activity have been suitably discussed with illustrative numerical examples. Some advantages of incorporating the proposed notion have also been tabulated for the sake of better understanding.

In Chapter 3, a new concept of the complex neutrosophic matrix has been intro-

duced to solve different problems related to uncertainties. Based on the proposed matrix, we have provided various algebraic operations like addition, subtraction, union many others which will be of great help in establishing the fundamental concepts. The matrix norm convergence of the proposed matrix has also been studied for the necessary foundation of the complex neutrosophic matrix. The two different types of new similarity measure matrices for complex neutrosophic matrices have been proposed and validated the axiomatic definition of the similarity measure. In addition to this, a new similarity measure has also been proposed for complex fuzzy matrices along with detailed explanatory numerical example. The application in the area of identification of reference signal has also been described.

In Chapter 4, four new similarity measures in their exponential form have been proposed for the case of single valued neutrosophic set. Numerical examples for the classification problem and the decision-making problem have also been presented and compared the obtained results with the well established existing approaches. Later, a novel concept of single valued neutrosophic information measure based on utility distribution and probabilistic randomness has also been proposed. The proposed concept has been obtained by integrating the uncertainties caused by neutrosophic information, useful information (utility based) and probabilistic information. Further, in a similar integrating way, the divergence measure of the 'useful' information has also been proposed for the study of applicable mutual information. Consequently, the hybrid ambiguity and neutrosophic information improvement measures have been studied with the help of the proposed 'useful' information measures.

In Chapter 5, the notion of energy and Laplacian energy of Picture fuzzy graph and directed Picture fuzzy graph have been proposed with the help of adjacency matrix. and the results on lower and upper bounds. On the basis of the proposed energy of picture fuzzy graph, a methodology for the ranking in a decision-making problem of site selection has been proposed. In order to illustrate the implementation of the proposed methodology, a hydro-power plant site selection problem has been considered. The novelty of the proposed approach, comparative analysis, advantages have also been studied.

Finally, the proposed work has been concluded in Chapter 6 of the current thesis along with some possible scope of future work.

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Journal Articles

- Poonia, M., Bajaj, R. K., **On Measures of Similarity for Neutrosophic Sets with Applications in Classification and Evaluation Processes**, *Neutrosophic Sets and Systems*, vol. 39(1), pp 86-100, Jan 2021. (**Indexing:** Scopus, Web of Science)
- Poonia, M., Bajaj, R. K., **On Laplacian energy of picture fuzzy graphs in site selection problem**, *Journal of Intelligent & Fuzzy Systems*, vol. 41(1), pp. 481-498, 2021. DOI: 10.3233/JIFS-202131.(**Indexing:** SCIE, Impact Factor - 1.851)
- Poonia, M., Bajaj, R. K., **Complex Neutrosophic Matrix with Some Algebraic Operations and Matrix Norm Convergence**, *Neutrosophic Sets and Systems*, vol. 47, pp 165-178, 2021. (**Indexing:** Scopus, Web of Science)
- Poonia, M., Bajaj, R. K., **Utility Distribution Based Measures of Probabilistic Single Valued Neutrosophic Information, Hybrid Ambiguity & Information Improvement**, *Communications in Computer and Information Science, Springer*, vol. 1572, pp 78-89, 2022. DOI: 10.1007/978-3-031-05767-07. (**Indexing:** Scopus)
- Poonia, M., Bajaj, R. K., **On Cohesive Fuzzy Set, Operations and Properties with Applications in Electromagnetic Signals and Solar Activities**, *Soft Computing.* **(Communicated)**
- Poonia, M., Bajaj, R. K., **On Complex Fuzzy Matrix with Algebraic Operations, Similarity Measure and its Application in Identification of Reference Signal**, *Journal of Information Science and Engineering.* **(Communicated)**
- Poonia, M., Bajaj, R. K., **On Some New Similarity Measure Matrices for Complex Neutrosophic Matrices and its Positive Definite-**

ness with Application in Medical Diagnosis. *Neutrosophic Sets and Systems.* **(Communicated)**

Conference Paper Presented

- **International Conference of IAPS on Advances in Differential Equations and Mathematical Modeling (IC-ADE-MM-2020),** organized by Jawaharlal Nehru University, New Delhi, India, held on $18^{th} - 20^{th}$, Dec 2020.
- **3rd International Conference on Soft Computing and its Engineering Applications (icSoftComp2021)**, organized by Charotar University of Science and Technology, Changa, Gujarat, India, held on $10^{th} - 11^{th}$, Dec 2021.
- **4th International conference on Recent Advances in Mathematical Sciences and its Applications (RAMSA-2020)**, organized by JIIT, Noida, India, held on $9^{th} - 12^{th}$, Jan, 2020.
- **International Conference on Differential Equations and Control Problems: Modeling, Analysis and Computations (ICDECP19)** , organized by IIT, Mandi, India, held on $17^{th} - 19^{th}$, June 2019.

Chapter 1

Introduction

Various tools have been designed by different researchers to solve the problem of uncertainty inherited in our day today life among which the probability theory and the theory of fuzzy sets are the most popular as well as widely applicable theories. It may be noted that the information regarding the relative frequency is having a due concern with the probability theory whereas in case of imprecise and inexact information having uncertainty for the decision makers, the fuzzy set theory is utilized. Zadeh introduced the concept of fuzzy sets (FSs) [72] which is found to be more efficient decision aid techniques providing the ability to deal with the uncertainty and the vagueness present in our real-life problems. In literature, it is prominently visible that the notion of fuzzy set theory plays a vital role in the areas of medical science [100], engineering applications [45], optimization [95], decision science [138], biological characterization problems [26], econometric [103], image analysis [105] etc. Due to the increasing componential factor, Atanassov [64] introduced the concept of intuitionistic fuzzy set (IFS) which includes the membership, non-membership and the hesitant function.

In an another extensional way, Torra [141] first introduced the notion of hesitant fuzzy set (HFS) along with various operations (complement, union and intersection etc.) which provided new dimensions to the research especially in the field of group decision making where the problem of multi-favorable situation can be better handled. For the sake of better understanding on some of the existing extensions and generalizations available in literature, we present an explanatory diagram given in the following Figure 1.1:

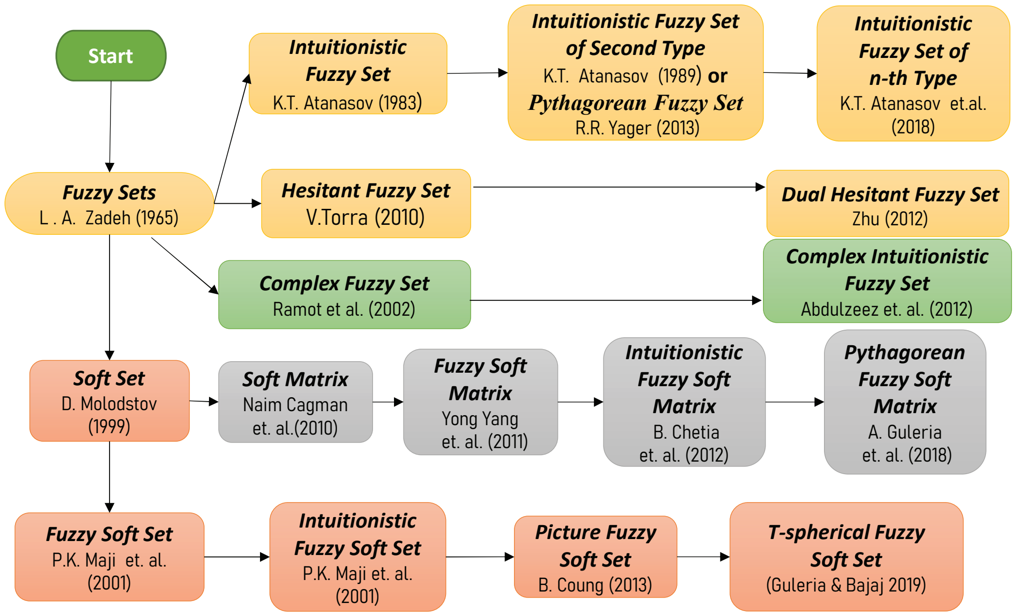

Figure 1.1: Generalizations and Extensions of Fuzzy Sets

In the hesitant fuzzy set, the decision makers provide a set of various favorable (multi-favorable situations) membership values for expressing their preferences/ assessments at the same time. On the other hand, the complex fuzzy set provides freedom to add a phase component which enables us to gain for information regarding a particular higher dimensional periodic problem. Further, Rezaei et al. [14] proposed the concept of hesitant fuzzy filters with few results on BE-algebra. In addition to this the connection between the γ- inclusive sets and hesitant fuzzy filters are also presented in detail. Further, the authors [15] have extended the concept to neutrosophic set and proposed a concept of neutrosophic filters in BE-algebra.

Further, Smarandache [32] contributed the unique concept of indeterminacy to the above-mentioned theories, which plays a vital role in obtaining solutions to various uncertain situations. This novel concept is known as the neutrosophic set, this concept of the neutrosophic set not only increase the clarity but also increase the basic information related to neutrality. "*Neutrosophic set is the branch of philosophy that deals with neutrality and its interaction with the different philosophical spectra.[32]*" Different generalized extensions of the theory of neutrosophic sets are available in literature. Some of them have been listed through the following Figure 1.2.

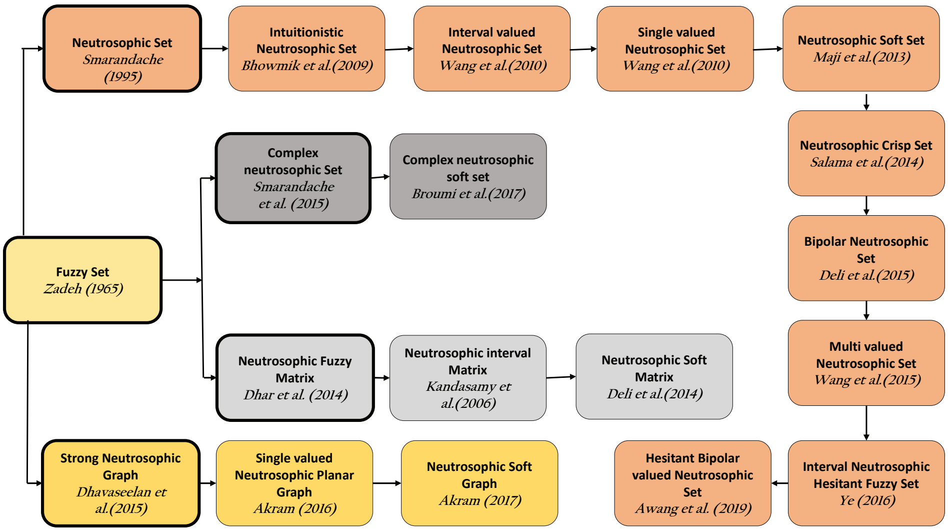

Figure 1.2: Literature Survey of Neutrosophic Theory

It may be noted that the application of neutrosophic set enhances the capability to study different types of information-based decision making problems more effectively. The generalization of fuzzy set to neutrosophic set with their information span may be understood by the geometric presentation given by the following Figure 1.3:

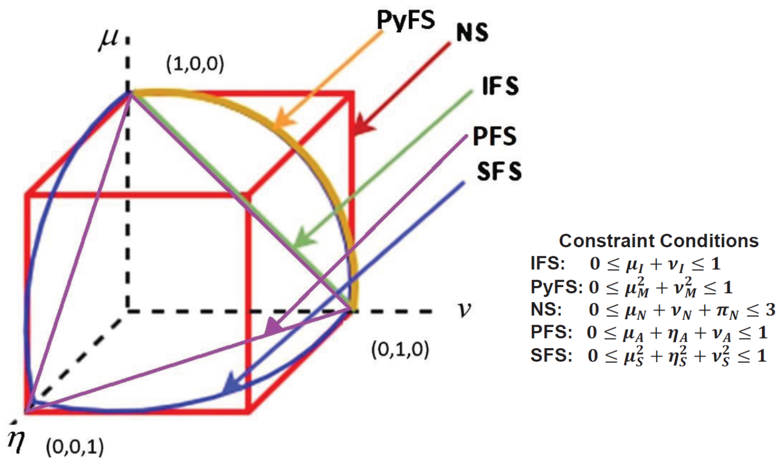

Figure 1.3: Extension of Fuzzy Set to Neutrosophic Set

Further, Cuong [16] proposed the concept of picture fuzzy set (PFS), where in all the four components, i.e., “degree of membership, degree of indeterminacy (neutral), degree of nonmembership and the degree of refusal have been taken into account". For the sake of better understanding the implementation of picture fuzzy information, we narrate an example of a voting system [16] - "*Suppose the voters have been categorized into four different classes: one who votes for (yes), one who votes against (no), one who neither vote for nor against (abstain) and one who refused for voting (refusal). It may be noted that the concept of 'refusal' is found to be an additional component which was not being taken into account by any of the sets or by their generalizations (fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set, neutrosophic set) stated above*".

Kifayat et al. [140] presented the geometrical aspects and features of these generalizations - "fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and picture fuzzy sets". It may be noted that the phenomenon of the voting system stated above can not be represented closely and sufficiently by utilizing the Pythagorean fuzzy grap -hs/sets. For capturing the information content and utilizing the flexibility in a more broader sense, the graph-theoretic literature of picture fuzzy set, picture fuzzy graph and its applications have been introduced.

1.1 Fundamental Notions and Preliminaries

In this section, some fundamental notion related to the hesitant fuzzy set, neutrosophic set, complex fuzzy set, fuzzy matrix and picture fuzzy sets/graphs along with their binary operations have been explained in detail.

1.1.1 Hesitant Fuzzy Set

Consider a fuzzy set M over a universe of discourse X characterized by the degree of membership function $\mu_M(x)(\mu_M : X \to [0,1])$. The value of μ_M represents the grade of degree to which the given element of the set X belongs to the set M. A new extension to fuzzy set has been firstly presented by Torra [141] in 2010 and named as Hesitant fuzzy set, which has various advantageous properties over the other extensions present in literature. This motivated a large amount of mathematicians to study the hesitant concept in detail, which yields various meaningful studies for extracting solutions of the problems related to uncertainty in decision making process.

The mathematical form of the hesitant fuzzy set in form of hesitant fuzzy element is given below:

Definition 1 *[141] "Suppose a universe of discourse X, then the hesitant fuzzy set over X is defined by a hesitant fuzzy element h such that when it is applied to X it again returns a subset in interval $[0,1]$. The hesitant fuzzy set M is represented as:*

$$M = \{< x, h_M(x) > | x \in X\}; \text{ where } h_M \text{ is the hesitant element."}$$

Later, Qian et al. [35] presented the concept of the generalized hesitant fuzzy set which defines its connection with the intuitionistic fuzzy set and also explained its application in case of decision support system. Some of the properties related to the hesitant fuzzy set are explained below:

Definition 2 *[141]"Some of the hesitant fuzzy set are given below:*

- ***Empty Set:*** $h(x) = 0, \forall x \in X$.
- ***Full Set:*** $h(x) = 1, \forall x \in X$.
- ***Complement Ignorance Set:*** $h(x) = [0,1]$.
- ***Nonsense Set for element x:*** $h(x) = \phi$."

Definition 3 *[141]"Suppose the lower and the upper bound in case of hesitant fuzzy set $h(x)$ is defined as*

- ***Lower Bound:*** $h^-(x) = \min h(x)$ *and*
- ***Upper Bound:*** $h^+(x) = \max h(x)$."

Definition 4 *[141] "Consider two hesitant fuzzy sets h_A and h_B over the universe of discourse set X. The binary operations defined over the hesitant fuzzy sets are given as:*

- ***Intersection:*** $h_A(x) \cap h_B(x) = \{h \in (h_A(x) \cup h_B(x)) \,| h \leq \min\left(h_A^+, h_B^+\right)\}$.
- ***Union:*** $h_A(x) \cup h_B(x) = \{h \in (h_A(x) \cap h_B(x)) \,| h \geq \min\left(h_A^-, h_B^-\right)\}$.
- ***Complement:*** $h^c(x) = \cup_{\gamma \in h(x)}\{1 - \gamma\}$."

1.1.2 Neutrosophic Set

Smarandache stated that the term neurtosophic obtained from the combination of two Latin words *Neuter* and *Sophia* which means neutral skill which definitely related to the third independent component of neutrosophic set which is the important contribution of the presented concept.

Definition 5 *[32] "Suppose the neutrosophic set A in universe X is characterized by three independent membership functions of truth* (T_A)*, neutrality* (I_A) *and falsity* (F_A)*, where* $T_A, I_A \,\&\, F_A$ *lie in interval* $]0^-, 1^+[$*. The mathematical form of the definition is given as*

$$A = \{< x, (T_A, I_A, F_A) > | x \in X\};$$

and it must also satisfy the condition: $0^- \leq sup(T_A) + sup(I_A) + sup(F_A) \leq 3^+$*."*

Definition 6 *[47] "Suppose A and B be two neutrosophic sets over the universe X. Then, the binary operation defined over the two NSs are given as;*

- ***Intersection:*** *For truth function* $T_{A\cap B} = T_A(x) \times T_B(x)$*,*
  - *For neutrality function* $I_{A\cap B} = I_A(x) \times T_B(x)$*,*
  - *For falsity function* $F_{A\cap B} = F_A(x) \times F_B(x)$*.*
- ***Union:*** *For truth function* $T_{A\cup B} = T_A(x) + T_B - T_A \times T_B$*,*
  - *For neutrality function* $I_{A\cup B} = I_A(x) + I_B - I_A \times I_B$*,*
  - *For falsity function* $F_{A\cup B} = F_A(x) + F_B - F_A \times F_B$*.*
- ***Complement:*** *For truth function* $T_A^c = \{1^+\} - T_A(x)$*,*
  - *For neutrality function* $I_A^c = \{1^+\} - I_A(x)$*,*
  - *For falsity function* $F_A^c = \{1^+\} - F_A(x)$*."*

However, it is very difficult to apply the concept of non-standard NS to practical life problems. Therefore, the concept of three independent functions in NS is bounded in the range of unit interval $[0, 1]$ to extract the solution of real life problems. This leads to the concepts of a single valued neutrosophic set (SVNS) [47] and an interval neutrosophic set (INS) [48] which are known as the branches of a neutrosophic theory/set. The formal definition for the case of SVNS is given as:

Definition 7 *[47] "Consider a single valued neutrosophic set A in X (universe of discourse) and is characterized by three membership function truth (T_A), indeterminacy (I_A) and falsity (F_A). The mathematical form is represented as*

$$A = \{\int_X < T(x), I(x), F(x) > /x | x \in X\};$$

when the universe X is continuous.

$$A = \{\sum_{i=1}^{n} < T(xi), I(xi), F(xi) > /xi | xi \in X\};$$

when the universe X is discrete. All the functions must satisfy, i.e., $T_A, I_A, F_A \in [0, 1]$."

Definition 8 *[47] "Suppose A and B be two single valued neutrosophic sets over the universe X. The, the binary operation defined over the two NSs are given as;*

- ***Intersection:*** *For truth function $T_{A\cap B} = \min(T_A(x), T_B(x))$,*
  - *For neutrality function $I_{A\cap B} = \min(I_A(x), T_B(x))$,*
  - *For falsity function $F_{A\cap B} = \max(F_A(x), F_B(x))$.*
- ***Union:*** *For truth function $T_{A\cup B} = \max(T_A(x), T_B(x))$,*
  - *For neutrality function $I_{A\cup B} = \max(I_A(x), T_B(x))$,*
  - *For falsity function $F_{A\cup B} = \min(F_A(x), F_B(x))$.*
- ***Complement:*** *For truth function $T_A^c = F_A(x)$,*
  - *For neutrality function $I_A^c = \{1\} - I_A(x)$,*
  - *For falsity function $F_A^c = T_A(x)$."*

1.1.3 Complex Fuzzy Set

Later, the fuzzy set is extended from the unit interval of [0, 1] on real plane to unit disc in complex plane by Ramot[28] in 2002. The complex fuzzy set added the phase term to the amplitude term present in the Zadeh's theory of fuzzy set and this prove to be very useful in tracking the cycle or the pattern of occurrence of the uncertainty events. The mathematical form of the complex fuzzy set is explained below:

Definition 9 *[28] "A complex fuzzy set M in the universe of discourse X, is characterized by the complex valued form of membership function $\mu_M(x)$, i.e., $\mu_M(x) = a_M(x)e^{ib_M(x)}$ where $a_M(x)\,\&\,b_M(x)$ are real valued functions and $a_M(x) \in [0,1]$. The mathematical representation of the complex fuzzy set M is given as:*

$$M = \{< x, \mu_M(x) > |x \in X.\}\text{"}$$

Definition 10 *[28] "Some of the binary operations between two CFSs M and N are given below:*

- ***Intersection:*** $\mu_{M\cup N}(x) = [a_M(x) \oplus a_N(x)]e^{ib_{M\cup N}(x)}$.
- ***Union:*** $\mu_{M\cap N}(x) = [a_M(x) * a_N(x)]e^{ib_{M\cap N}(x)}$.
- ***Complement:*** $\mu_{M}c(x) = (1 - a_M(x))\, e^{i(2\pi - b_M(x))}$."

Thomason [91] proposed the concept of matrix in case of fuzzy set and explained it with the help of the various properties of convergence.

Definition 11 *[91] "Let fuzzy matrix is denoted by P, defined on a universe X consists of fuzzy element a_{ij} and is represented by*

$$P = [a_{ij}]_{m\times n}\,;\ \ where\ \ a_{ij} \in [0,1]\,,\ (1 \leq i \leq m,\ 1 \leq j \leq n)\,.\text{"}$$

Definition 12 *[91] "Some of the algebraic operations between two fuzzy matrices A and B are given below:*

- ***Subtraction:*** $A - B = \begin{cases} a_{ij}, & iff\ A > B \\ 0, & otherwise \end{cases}$.

- ***Addition:*** $A + B = \max(a_{ij}, b_{ij})$.
- ***Multiplication:*** $A.B = \min(a_{ij}, b_{ij})$."

Definition 13 *[158] "The complex fuzzy matrix denoted by P, is defined as*

$$P = [a_{ij}(x) + ib_{ij}(x)]_{m \times n};$$

where $(a_{ij}, b_{ij}) \in [0, 1]$, $(1 \leq i \leq m,\ 1 \leq j \leq n)$."

1.1.4 Picture Fuzzy Set and Picture Fuzzy Graph

Picture fuzzy set [16] are direct extensions of fuzzy sets and intuitionistic fuzzy sets which additionally incorporates the concept of positive, negative, neutral membership and degree of refusal in the decision making problems related to human opinions. One of the most common example to understand the concept more clearly has been presented in case of voting where we have to deal with four categories (vote for/ abstain/ vote against/ refusal of the voting). This theory of picture fuzzy set has been further extended by Zuo[24] for the case of fuzzy graph and the author presented the concept of picture fuzzy graph with several types. The researchers utilized the concepts of picture fuzzy set/ graph to solve various complex situations created in real life under uncertain environment. The detailed definition for both the concepts are presented below:

Definition 14 *[16]"A picture fuzzy set A in U (universe of discourse) is given by*

$$A = \{< x, (\mu_A(x), \eta_A(x), \nu_A(x)) >| \ x \in U\};$$

where $\mu_A : U \rightarrow [0, 1]$, $\eta_A : U \rightarrow [0, 1]$ and $\nu_A : U \rightarrow [0, 1]$ denote the degree of membership, degree of neutral membership (abstain) and degree of non-membership respectively and for every $\alpha \in U$ satisfy the condition

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1.$$

The degree of refusal for any picture fuzzy set A and $x \in U$ is given by $r_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$."

Definition 15 *[16] "Suppose A and B be two picture fuzzy sets over the universe X. The, the binary operation defined over the two PFSs are given as;*

- ***Intersection:*** *For truth function* $\mu_{A\cap B} = \min(\mu_A(x), \mu_B(x))$,
  - *For neutrality function* $\eta_{A\cap B} = \min(\eta_A(x), \eta_B(x))$,
  - *For falsity function* $\nu_{A\cap B} = \max(\nu_A(x), \nu_B(x))$.
- ***Union:*** *For truth function* $\mu_{A\cup B} = \max(\mu_A(x), \mu_B(x))$,
  - *For neutrality function* $\eta_{A\cup B} = \min(\eta_A(x), \eta_B(x))$,
  - *For falsity function* $\nu_{A\cup B} = \min(\nu_A(x), \nu_B(x))$.
- ***Complement:*** *For truth function* $\mu^c_A = \nu_A(x)$,
  - *For neutrality function* $\eta^c_A = \eta_A(x)$,
  - *For falsity function* $\nu^c_A = \nu_A(x)$."

Definition 16 *[24] "A picture fuzzy graph on* (S, R)*, denoted by* $G = (M, N)$*, where* M *is a picture fuzzy set on* S *and* N *is a picture fuzzy relation in* $R = S \times S$ *such that*

$$\mu_N(x, y) \leq \min\{\mu_M(x), \mu_M(y)\},$$

$$\eta_N(x, y) \geq \max\{\eta_M(x), \eta_M(y)\},$$

$$\nu_N(x, y) \geq \min\{\nu_M(x), \nu_M(y)\};$$

satisfying the constraint condition $0 \leq \mu_N(x, y) + \eta_N(x, y) + \nu_N(x, y) \leq 1,\ \forall x, y \in S$. *The set* M *is called the picture fuzzy vertex set of the graph* G *and* N *is called the picture fuzzy edge set of the graph* G."

1.2 Literature Survey

A brief literature survey related to our present work has been summarized below:

1.2.1 Complex Extensions of Fuzzy Sets & Neutrosophic Sets

In due course of time, several types of complexities got added upon and researchers proposed various other generalizations of fuzzy sets and intuitionistic fuzzy sets. One of the major limitation of the application of FSs and IFSs that these sets are not capable to address the periodicity occurring in some uncertain and incomplete/inexact information. In addition to this, various other problems having two dimensional framework can not be modeled with FSs and IFSs. In order to encounter this deficiency, Ramot et al.[28] extended the existing structure of fuzzy set to complex fuzzy set (CFS) which added the phase variable and also extended the range from [0,1] to the unit circle in the complex plane which spans the information in a wider sense. The membership function $\mu_S(x) = r_S(x)\, e^{iw_S(x)}$ in the complex fuzzy set implies that all the membership values must lie inside the unit circle on the complex plane. There is a kind of specific mapping between a CFS and Fourier transform which can be observed by restricting the range to a complex unit disk and henceforth having various applications in the field of communication system, geological phenomena, optical systems etc. The CFS has been extended to complex intuitionistic fuzzy set (CIFS) by Abdulzeez et. al. [7] which added the complex membership and non-membership function. Garg & Rani [41] [42] contributed two studies in the field of CIFS. First, they developed correlation/weighted correlation coefficients under the CIFS setup where the membership degrees were utilized to represent the two-dimensional information. Secondly, they introduced and discussed the transformation relationships among the similarity, distance, entropies, and inclusion measures. Yaqoob et.al [102] introduced the notion of complex intuitionistic fuzzy graphs by combining two efficient theories (CIFS and graph theory) and also explained their advantage with the help of examples in the field of cellular network.

Besides various generalizations of fuzzy sets and their respective measures available in literature, Xu & Xia [160] presented various distance measures, similarity measures and correlation coefficients for hesitant fuzzy sets. Also, Torra [141] established a relation between HFS and IFS stating the enveloping procedure of IFS over HFS. Xu et.al [161] elaborated the hesitant fuzzy sets theoretically with different support system and methodologies which have some kind of special advantageous features in the group decision making processes. They also described the consensus process in hesitant fuzzy setup to complete the decision making process. Ren et.al. [159] extended the concept of HFS to normal wiggly hesitant fuzzy sets to improve the rationality of decision making process and also proposed two introductory aggregation operators. The another important contribution made in the study of HFS is dual hesitant fuzzy set (DHFS) which was proposed by Xu et.al [19] in which the membership hesitancy function and non-membership hesitancy function are used to support a more flexible access to assign the values to each element in the domain. It may be noted that FS, IFS & HFS can be treated as the special cases of DHFS. Further, Garg et.al. [40] added the probability factor to DHFS and proposed the coefficients along with the weighted correlation coefficients for probabilistic dual hesitant fuzzy sets (PDHFSs).

Further, the application of the cosine similarity measure to show the connectivity between the CFSs is also displayed by Guo [145]. Then, Mahmood and Rehman [136] extended the concept of CFS towards a bipolar complex fuzzy set and presented some basic operators with two real-life applications in pattern recognition and medicine. Further, Qudah and Hasan [150] presented a hybrid model known as a complex multi-fuzzy set by adding the properties of both complex fuzzy set and multi fuzzy set. This concept is further extended by Alkouri and Salleh [7] which contains the properties of complex-valued membership and non-membership functions and the basic operators (Union, Intersection and Complement), which have been explained in detail for the basic understanding of the concept. Later, Rani and Garg [29] introduced a distance measure under the environment of a series of distance measures (Hamming, Euclidean, and Hausdorff metrics) present in literature and validated this theory with the help of decision-making problems.

Ali and Smarandache [87] extended this concept of complex value to neutrosophic and named it as Complex Neutrosophic Fuzzy Set (CNFS) which contains the properties of both the complex fuzzy set and neutrosophic set. The complex neutrosophic fuzzy set has been studied in detail with the basic operations (Union, Intersection, complement etc) and the concept has been validated with the help of the suitable application. Further, Ali and Mahmood [155] proposed a dice similarity measure for two complex neutrosophic sets and explained the proposed concept by using it to solve the pattern recognition problem.

1.2.2 Fuzzy Matrix, Neutrosophic & Complex Fuzzy Matrix

Thomason [91] introduced the concept of fuzzy matrix and studied its convergence with respect to matrix norms. Later, the determinant of the intuitionistic fuzzy matrix was proposed by Pal [93] and then the notion of interval-valued intuitionistic fuzzy matrix was introduced by Khan and Pal [123]. Dhar et al. [88] introduced the matrix form of the neutrosophic set which plays a significant role in dealing with a big database of information which was extended by Kandasamy et al. [148] who proposed the concept of neutrosophic interval matrices with its application. Various researchers have extended their study in the direction of extension of fuzzy theories, which later turned to complex fuzzy matrices by Zhao and Ma [158] in 2016. They defined the complex fuzzy matrices in the form of $C = (A_{ij}(x) + iB_{ij}(x))$ and also explained the norm convergence. Khan et al. [147] extended the concept of complex fuzzy matrices to complex fuzzy soft matrices in 2020 and also proposed some theorems, which have been explained with the help of its application in Decision-Making Problems.

Various other mathematicians like Abobala [81] in 2021, presented the refined neutrosophic matrices and their algebraic operations with their application in the refined algebraic equation. Deli et al. [52] contributed the neutrosophic soft matrices and gave a methodology for storing the concept of the neutrosophic soft set to the memory of the computer. Further, the concept of convergence was discussed by many researchers [[146]-[69]] for a better understanding of the concept. This also proves the advantage of matrix form over the set form for the uncertainty parameter where one event can be taken at an interval of time instead of one problem. Ragab et al. [97]discussed the determinant and adjoint of the square fuzzy matrix in detail. Various other properties like the canonical form of transitive and strongly transitive matrices are studied [[43],[146]]. In Kamaci [44] proposed various similarity measures in case of soft matrices and also explained the advantages of soft matrices in many computational process. Das [118] introduced the novel concept of intuitionistic fuzzy matrix and studied various operators. The author also presented several similarity measure and validated the theory with the help of application in case of proposed similarity measure. The adjoint and determinant of square intuitionistic fuzzy matrices were discussed by Le and Park [[151]-[152]]. Further, Muthuraji and Lalitha [137] discussed the unitary and binary operators for intuitionistic fuzzy matrices.

1.2.3 Information Measures of Neutrosophic Sets

The concept of similarity plays a fundamental role in solving various complex indeterminate matters in human life. The theory of similarity is very efficient in the fields of taxonomy, recognition, case-based reasoning and many others. Pramanik and Mondal [63] introduced weighted fuzzy similarity measures based on tangent function and explained its importance through its application to medical diagnosis. Peng and Smarandache [149] in 2019 generated multi-similarity model for the neutrosophic set in order to brief the detailed process in decision-making problems in the economic sector.

Wang [47] restricted the benefits of the neutrosophic set to a Single Value Neutrosophic Set (SVNS) to increase its applicability in solving the problems. The similarity and entropy measures play a critical role in the study of measuring uncertain information related to the data available for fuzzy sets and their hybrid structures. The necessary axioms for fuzzy entropy were introduced and explained by De Luca and Termini [1]. On the other hand, the similarity measure is considered an important tool in comparison with the entropy measure due to its ability to calculate the similarity between the sets according to the data present in the literature.

Mehmet et al.[125] have proposed the transformations between single-valued neutrosophic values based on the centroid points and the values are according to the truth, indeterminacy and falsity values of SVNS. The authors have also proposed a new similarity measure based on the falsity function and presented its applicability in the case of pattern recognition. Ulucay et al.[143] proposed some new similarity measures (Dice similarity measure, weighted dice similarity measure, Hybrid vector) for the case of bipolar neutrosophic sets. Later, Ulucay et al. [144] proposed some distance, similarity and entropy measures between the two bipolar neutrosophic sets after careful consideration of positive and negative membership functions. Finally, the results obtained have been validated in accordance with the proposed methodology. Also, Shahzadi et al. [37] have used the distance and similarity measures in the case of single-valued neutrosophic sets to propose two algorithms in the field of medical diagnosis and validated the algorithms with help of numerical examples. Further, Pamucar et al. [30] have proposed a fuzzy decision-making approach in the case of a construction company by using a new weight aggregation operator that uses pair-wise comparison. In addition, a novel fuzzy neutrosophic based approach for resilient supplier selection has also been proposed, which mainly contributes to the design, implementation and analysis of a multi-attribute evaluation system concerning fuzzy neutrosophic values.

Majumdar and Samanta [108] presented the similarity and entropy measure between the two single-valued neutrosophic sets. Later, Pappis et al. [21], [22] presented an axiomatic view of similarity measure for a better understanding of the similarity measure concept. Various researchers have published many research articles on similarity and entropy measures and utilized it in many applications like in the case of fuzzy soft sets [157], intuitionistic soft sets [109] and interval-valued fuzzy sets [51] so on.

Various authors compared their algorithms with the similarity measures present in the literature. The advancement in the case of similarity measures has been explained through the following sequential development given in Figure 1.4:

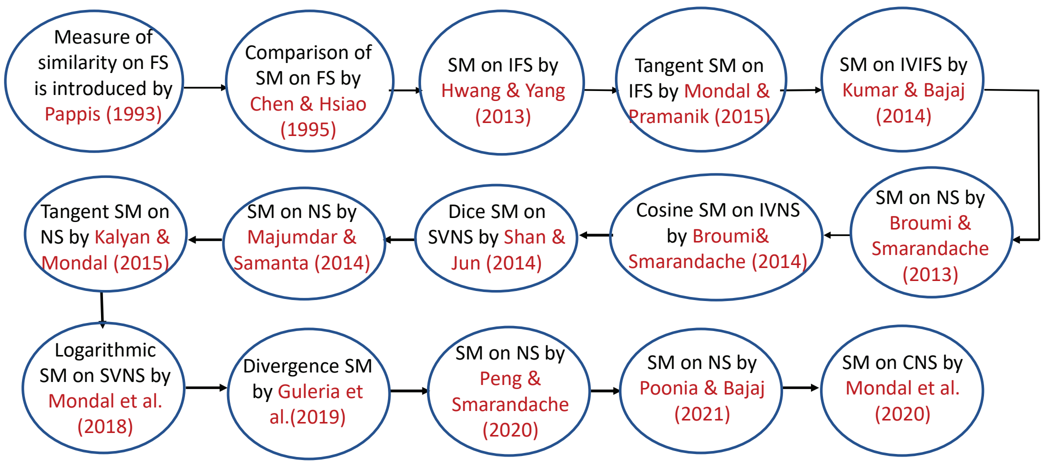

Figure 1.4: Methodologies using Similarity Measures

Different kinds of similarity/distance measures of NSs have been well studied by Broumi & Smarandache [117]. Utilizing the distance measure between two SVNSs, Majumdar and Samanta [108] defined some important measures of similarity along with their characteristics. Ye [59] presented the three different similarity measures between SVNSs as an extension of the Jaccard, Dice, and cosine similarity measures in vector space and utilized then to solve the MCDM problem under simplified neutrosophic information. Mondal and Pramanik [63] proposed a new trigonometric measure called tangent similarity measure as an improvement of cosine similarity and used this to solve the applications problem of selection of educational stream and medical diagnosis. Ye [60] has given different similarity measures for the interval neutrosop -hic sets based on distance measures with application in decision processes. Next, Ye et al. [61] [134] and Wu et al. [49] discussed the problem of diagnosis based on the similarity measures for SVNSs.

Thao and Smarandache [101] proposed new divergence measure for neutrosophic set with some properties and utilized to solve the medical diagnosis problem and the classification problem. Abdel-Basset et al. [73] developed a new model to handle the hospital medical care evaluation system based on plithogenic sets and also studied intelligent medical decision support model [74] based on soft computing and intern -et of things. In addition to this, a hybrid plithogenic approach [75] by utilizing the quality function in the supply chain management has also been developed. Further, a new systematic framework for providing aid and support to the cancer patients by using neutrosophic sets has been successfully suggested by Abdel-Basset et al. [76]. Based on neutrosophic sets, some new decision-making models have also been successfully presented for project selection [77] and heart disease diagnosis [78] with advantages and defined limitations. In subsequent research, Abdel-Basset et al. [79] have proposed a modified forecasting model based on neutrosophic time series analysis and a new model for linear fractional programming based on triangular neutrosophic numb -ers [80]. Also, Yang et al. [50] have studied some new similarity and entropy measures of the interval neutrosophic sets on the basis of new axiomatic definition along with its application in MCDM problem.

A new integrated method based on the Weighted Aggregated Sum Product Assessment (WASPAS) approach has also been proposed by Mishra et al. [8] to solve a decision-making problem with hesitant fuzzy information. The applicability of the proposed technique has been presented in the case of the green supplier selection problem and the results obtained have been duly compared with the result that exists in the literature. Further, Mishra et al.[9] studied IVIF-divergence and entropy measures and proposed a new technique for solving the classical interactive multi-criteria decision making by calculating the dominance degrees along with its applicability with vehicle company example.

In 1972, Luca and Termini [1] introduced a new measure of fuzzy entropy based on the Shannon function [23] and with the help of these two theories a new set of properties have been designed. These properties play a significant role in describing the fuzzy entropy. The fuzzy entropy by Luca et al. [1] is one of the simplest forms of entropy present in the literature and is defined as,

$$M'(A) = -\sum_{i=1}^{n}[\Gamma_A(y_i)\log\Gamma_A(y_i) + (1-\Gamma_A(y_i))\log(1-\Gamma_A(y_i))]; \tag{1.2.1}$$

where $\Gamma_A(y_i)$ denotes the degree of membership function and the properties of entropy measure are given below:

- $M'(A) = 0$ iff $\Gamma_A(y) = 0$ or 1.
- $M'(A)$ is maximum when $\Gamma_A(y) = 0.5$.
- $M'(A) = M'(A')$, where the sharpen version of A is A', i.e.
  $\Gamma_{A'}(x_i) \leq \Gamma_A(y_i)$ for $\Gamma_A(y_i) \leq 0.5$ & $\Gamma_A(y_i) \leq \Gamma_{A'}(y_i)$ for $\Gamma_A(y_i) \geq 0.5$
- $M'(A) = M'(\tilde{A})$ where $\tilde{A}$ is complement of A.

These properties are the necessary and sufficient conditions to form the fuzzy entropy measure. The Luca and Termini measure given by equation 1.2.1 also fulfills these conditions. In 1980, Kaufmann [5] proposed an entropy measure which played the basic for various new entropies in literature and is of form,

$$M'(A) = -\frac{1}{\log n}\sum_{i=1}^{n}\Gamma_A(y_i)\log\Gamma_A(y_i). \tag{1.2.2}$$

In 1967, Havrda and Charvat [56] extended the concept of Kaufmann [5] and defined the following entropy measure;

$$M'(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}[(\Gamma_A^{\alpha}(y_i) + (1-(\Gamma_A(y_i))^{\alpha} - 1]. \tag{1.2.3}$$

Thus, with the help of fuzzy entropy, the quantity of information is obtained from the systems of fuzzy theory and the measure of information collected from this fuzziness is known as fuzzy information measure. This concept of information measure was further used by Joshi [111] in 2019 to generate a new measure based on Tsallis-Havrda-Charvat entropy. Later, in 2020, Li et al. [156] and Mahmood et al. [136] studied this concept using the structures of Gaussian kernels and Complex q-rung orthopair fuzzy set respectively. The entropy was first proposed by Shannon[23] in 1948, which is a probabilistic theory and contributes majorly to the communication sector. As per deliberation given by Robert et al. [13], the defined quantity of information conveyed is directly proportional to the probability of the probabilistic task. This implies that the information quantity defines the log of the event probability of A, i.e., $M'(A) = -\log p(A)$; where $p(A)$ denotes the probability and the average information over all the events is known as Shannon entropy.

Further, the concept of information entropy measure was extended to 'useful' information measure by Bhaker and Hooda [139] in 1993, who defined the generalized mean value characteristic measure for incomplete probability measure. This theory was later used by Hooda and Bajaj [31] in 2010 and they introduced a new 'useful' information measure for directed divergence of Zadeh's theory. Then, in 2016 the briefly description related to the overview of fuzzy information measure and generalized form of fuzzy entropy was presented by Ohlan [11] and in the same year Arora and Dhiman [38] contributed a new measure of fuzzy directed divergence and its applications in decision making problems in the literature. Sharma et al. [114] established the primary decomposition of k-ideals of semirings with its uniqueness and also generalized it for fuzzy k-ideals of the semirings. In 2018 & 2019, Sofi et al. [130]-[131] used the concept of parametric 'useful' fuzzy information measure for R -norm and obtained new properties with numerical examples respectively.

1.2.4 Various Fuzzy Graphs and Notion of Energy

Meenakshi et al. [124] explained that energy of graph connects the graph more closely to the chemical quantity known as π- electron energy of conjugated hydro carbon mole -cule. The concept of energy of a graph [53] [55] [110] has been utilized in chemical engineering applications- the molecular orbital theory of conjugated molecules [4] [54] [46]. Sridhara and Khanna [36] studied the bounds on energy and Laplacian energy of graphs with various important observations and results. The concept of energy to fuzzy graphs has been extended by Narayanan and Mathew [10] with some bounds on the energy of fuzzy graphs. Further, Praba et al. [18] extended the energy concept for intuitionistic fuzzy graph with important results. Recently, the application of energy of Pythagorean fuzzy graphs has been studied in the decision making example of a satellite communication system and in the evaluation of the schemes of reservoir operation [86].

Based on the fuzzy relation [71], Kaufmann [6] proposed the concept of fuzzy graphs and Rosenfeld [12] subsequently developed the concept of fuzzy vertex and fuzzy edge. Some standard operations on the fuzzy graphs were studied by the Mordeson and Peng [57] along with their properties. Further, Parvathi et al. [112, 113] extended the notion of a fuzzy graph to intuitionistic fuzzy graph and analyzed various properties related to minmax intuitionistic fuzzy graph. Karunambigai et al. [90] proposed a category of constant and totally constant intuitionistic fuzzy graphs and subsequently Akram et al. [84] presented the concept of strong intuitionistic fuzzy graphs along with their properties. Also, Akram et al. [85] presented intuitionistic fuzzy hypergraphs with their applications and Alshehri et al. [98] defined the planarity, duality and multigraphs in context with intuitionistic fuzzy graphs. Sahoo and Pal [126] [127] proposed various types of product operations for intuitionistic fuzzy graphs, intuitionistic fuzzy tolerance graph with their applications.

Various researchers [82] [89] [94] [129] utilized the flexibility and its applicability to set forward some new ideas concerning the extended structures of intuitionistic fuzzy graphs and provided many interesting applications in clustering and decision-making problems and support systems. Naz et al. [133] proposed a generalization of the intuitionistic fuzzy graph, termed as the Pythagorean fuzzy graphs, and studied their applications in various decision making problems. Some graph-theoretic operations related with Pythagorean fuzzy graphs have been well studied by Verma et al. [115]. Zuo et al. [24] introduced the new concept of picture fuzzy graph and its various types with different properties. Earlier classical graphs were used to represent the social network but the main drawback of using classical graphs was that in its case all the individuals are of same value which is not the case in real world. Therefore, the relationship between these individuals will also be same in all the cases which is again not true in reality. This problem was solved by Samanta and Pal [132] by using type 1 fuzzy graph in place of classical fuzzy graph. But it is also not able to deal with the complexity of the real world problem. Das et al. did a lot of research in the field of picture fuzzy graph in references [121] and [118] in which the authors proposed the picture fuzzy planar graphs and explained it with the help of its application in road map whereas m-step picture fuzzy competition graphs, picture fuzzy economic competition graphs and picture fuzzy competition hypergraphs are introduced and its applications are shown in the field of academics, environment and companies etc respectively. With the above application Das et al. also gave the novel concept of the m-step picture fuzzy competition graphs, picture fuzzy economic competition graphs and picture fuzzy competition hypergraphs in [120] and its plays a great role in the field of medicine and Pal et al. [92] explained the modern trends in study of fuzzy theory.

1.3 Motivation

Uncertainty is one of the root cause of real life problems which made it difficult to extract solutions to various situations. The theory of fuzziness has been dealt by various extensions but tracking the pattern of happening of an event is an important contribution.

The following structural figure explains the motivation behind the present work:

It may be noted that the notions presented in the grey boxes of the above figure already exist in the literature whereas the notions in green boxes have been proposed and explained in detail in the present work. The outline of the research gap & rationale of the present work have been discussed as follows.

- Extension of the fuzzy set to a complex fuzzy set increases the range of solving the problem from the interval [0, 1] on the real plane to the unit disc in the complex plane. This lays the foundation of two novel concepts of the complex neutrosophic matrix and cohesive fuzzy set forvsolving the problems related

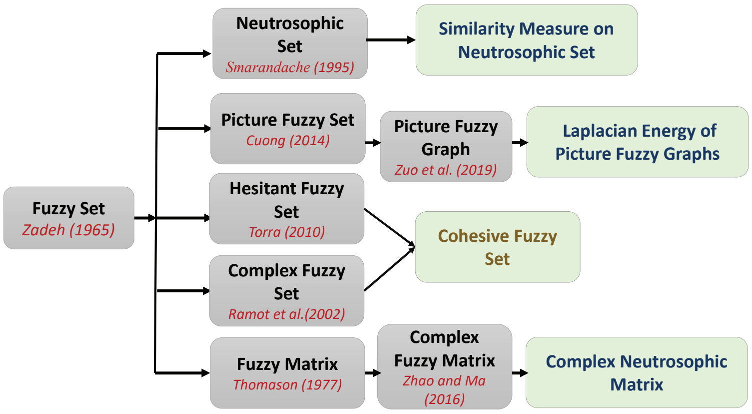

Figure 1.5: Flow of Proposed Work & Motivation

to uncertainties more efficiently. We present a natural extension of the existing set to a novel concept of cohesive fuzzy set (CHFS) which has the capability to explicitly focus on the set of the favorable situations for a particular uncertain higher dimensional problem with the possible extended range of unit disk having a phase component. The phase component gives the advantage of addressing the impreciseness which occurs in a periodic fashion. The objective behind introducing the concept of CHFS is that it not only deals with the situation in which we are facing difficulty in choosing the best among the various favorable options, but also helps in neglecting the unfavorable situations among the wide range of situations which would certainly save our time and energy both.

- Considering the importance of matrix form in solving a large number as well as higher dimension problems in a single interval of time motivated us to extend these advantages of the matrix form from the real plane to the complex pane of unit range. Thus, we extended the theory of neutrosophic matrices to the complex plane and introduced the novel concept of a complex neutrosophic matrix. In this article, we have defined some new types of similarity measure matrices for the Complex Neutrosophic Matrices (CNM) have been proposed. The positive definiteness and importance of these measures have been explained with the help of various properties. The concept of complex fuzzy matrix plays a significant role in the complex plane. Various new operators for the proposed matrix have been presented for the detailed understanding of the concept. A similarity measure for the proposed matrices has also been designed and subsequently its applicability in the field of identification of reference signal has been presented.

- Similarity measure plays a significant role in defining the similarity among the fuzzy sets. Therefore, there is a wide area of research in defining the similarity in the case of neutrosophic set and this plays a vital role in explaining the similarity among the uncertainties of events present. We have incorporated the exponential function for framing the new similarity measures for the neutrosophic sets along with their weighted form and utilized them for the solving a standard classification problem of pattern recognition and the decision-making problem. The concept of probabilistic occurrence and neutrosophic entropy have been put together along with utility distribution of the events to obtain a novel concept of 'useful' single valued neutrosophic probabilistic information measure. The proposed measure is a new kind of measure that will be helpful for the study of decision problems under utility distribution. In addition to this, some extended measures like hybrid ambiguity measure, analogous divergence measure and information improvement measure have also been discussed.

- In literature, no study was presented using the Laplacian energy of picture fuzzy graph to construct a methodology to identify the location in case of a site selection problem. We introduce the novel concept of adjacency matrix, energy and Laplacian energy for picture fuzzy graphs with applications. A new methodology for solving a selection problem based on the proposed notions of picture fuzzy graph has also been provided with an example. In whole, the purpose of the proposed work is to further expand the fuzzy graph related concepts under picture fuzzy environment. Such extensions and enrichment will certainly help in widening the span and coverage of the information significantly.

Chapter 2

Cohesive Fuzzy Set, Operations & Applications

In this chapter, we have introduced the notion of cohesive fuzzy set (CHFS) with various operations, properties and standard identities. This extension of fuzzy set is capable to deal with the situation in which there are multi-favorable situations in the complex plane. We have also presented the application of cohesive fuzzy sets in the process of filtering the signals using the Fourier Cosine Transformation (FCT) and Inverse Discrete Fourier Cosine Transformation (IDFCT/DFCT). In addition to this, another application of identifying maximum number of sunspots in a particular interval under a solar activity has been presented with an illustrative example. The advantages and the limitations of the proposed methodology have also been studied.

2.1 Notion of Cohesive Fuzzy Set

In this section, we introduce the concept of cohesive fuzzy set and provide its formal definition along with various operations and related important properties.

The complex fuzzy set captures the phase component to process the information of a higher dimensional periodic problem while in the theory of hesitant fuzzy set theory, experts provide a set of various multi-favorable situations for presenting their assessments. In order to merge both the requirements in a synchronized way, a natural extension to a set called cohesive fuzzy set is being introduced for explicitly focussing on the set of the favorable situations for a particular uncertain higher dimensional problem with the possible extended range of unit disk having a phase component.

Definition 17 (**Cohesive fuzzy set**) *Consider a fuzzy set T defined on a fixed universe of discourse S, a cohesive fuzzy set (CHFS) on T is in terms of function h when applied on S returns a subset of unit circle, i.e.,*

$$S_1 = \{< x, h_T(x) > | x \in S\};$$

where h_T is a complex set of values in a unit circle of the complex plane, denoting the possible membership degrees of elements $x \in S$ to the set $T \subset S$. Here, h_T is of the form $r_T(x) \exp(iw_T(x))$, where $i = \sqrt{-1}$, $r_T(x)$ and $w_T(x)$ both are real values and $r_T(x) \in [0, 1]$.

Example: For understanding the basic structure of CHFS, let $S = \{x_1, x_2, x_3\}$ be the reference set. Suppose

$$h_{T_1}(x_1) = \{0.5 \exp \pi, 0.8 \exp \frac{\pi}{2}, 0.7 \exp \frac{\pi}{2}\},$$

$$h_{T_2}(x_2) = \{0.6 \exp \pi, 0.9 \exp \pi, 0.7 \exp \frac{\pi}{4}\},$$

and

$$h_{T_3}(x_3) = \{0.5 \exp \pi, 0.7 \exp \frac{\pi}{2}, 0.7 \exp \pi\}$$

denote the membership set of $x_i\,(i = 1, 2, 3)$ to the set T respectively. Then the cohesive fuzzy set can be represented as

$$T = \{< x_1, \{0.5 \exp \pi, 0.8 \exp \frac{\pi}{2}, 0.7 \exp \frac{\pi}{2}\} >, < x_2, \{0.6 \exp \pi, 0.9 \exp \pi, 0.7 \exp \frac{\pi}{4}\} >, < x_3, \{0.5 \exp \pi, 0.7 \exp \frac{\pi}{2}, 0.7 \exp \pi\} >\}.$$

Various Basic Operations/Results on Cohesive Fuzzy Sets

Given a cohesive fuzzy set T whose membership function is given by h_T, we suitably propose its lower and upper bound as given below:

- <u>lower bound:</u> $h_T^- = min\,(h_T)$ and
- <u>upper bound:</u> $h_T^+ = max\,(h_T)$.

It may be noted that the pair of complex hesitant functions h_T^- and $1 - h_T^+$ define the complex intuitionistic fuzzy set. Next, we first propose the definition of the complement of cohesive fuzzy set as follows:

Definition 18 ***(Complement)*** *Given a cohesive fuzzy set represented by membership function h_T, its complement set is defined as follows:*

$$h_T^c = \cup_{\mu_T \in h_T} \{\mu_T\}^c; \tag{2.1.1}$$

where $\mu_T = r_T e^{iw_T}$, i.e.,

$$h_T^c = \cup_{\mu_T \in h_T} \{\mu_T\}^c = \cup_{r_T \in h_T, w_T \in h_T} \{(1 - r_T)\, e^{i(-w_T)}\}.$$

Proposition 1. The operation of complement is involution, i.e.,

$$(h_T^c)^c = h_T \tag{2.1.2}$$

Proof: It is easy to observe that $(1 - (1 - r_T))\, e^{i(-(-w_T))}$ for all $r_T, w_T \in h_T$. Hence the result.

Definition 19 ***(Union)*** *Suppose there are two cohesive fuzzy sets represented by their hesitant membership functions h_{T_1} and h_{T_2} respectively. The union of these CHFSs, denoted by $h_{T_1} \cup h_{T_2}$, can be defined as*

$$(h_{T_1} \cup h_{T_2})\,(x) = \{h_T \in (h_{T_1}(x) \cup h_{T_2}(x))\,| h_T \geq \max\left(h_{T_1}^-, h_{T_2}^-\right)\}.$$

Definition 20 ***(Intersection)*** *Suppose there are two cohesive fuzzy sets represented by their hesitant membership functions h_{T_1} and h_{T_2} respectively. The intersection of these CHFSs, denoted by $h_{T_1} \cap h_{T_2}$, can be defined as*

$$(h_{T_1} \cap h_{T_2})\,(x) = \{h_T \in (h_{T_1}(x) \cap h_{T_2}(x))\,| h_T \leq \min\left(h_{T_1}^+, h_{T_2}^+\right)\}.$$

Hence, from the Definitions 18, 19 and 20 given above, we write the following equations:

$$\begin{aligned} h_T^c &= \cup_{\mu_T \in h_T} \{\mu_T\}^c = \cup_{r_T, w_T \in h_T} \{(1 - r_T)\, e^{-iw_T}\}; \\ h_{T_1} \cup h_{T_2} &= \cup_{\mu_{T_1} \in h_{T_1},\, \mu_{T_2} \in h_{T_2}} \max\{\mu_{T_1}, \mu_{T_2}\} = \cup_{r_T, w_T \in h_T} \{\max(r_{T_1}, r_{T_2})\, e^{i\max\left(w_{T_1}, w_{T_2}\right)}\}; \\ h_{T_1} \cap h_{T_2} &= \cup_{\mu_{T_1} \in h_{T_1},\, \mu_{T_2} \in h_{T_2}} \min\{\mu_{T_1}, \mu_{T_2}\} = \cup_{r_T, w_T \in h_T} \{\min(r_{T_1}, r_{T_2})\, e^{i\min\left(w_{T_1}, w_{T_2}\right)}\}. \end{aligned} \tag{2.1.3}$$

where μ_T, μ_{T_1} and μ_{T_2} are of form $r_T e^{iw_T}, r_{T_1} e^{iw_{T_1}}$ and $r_{T_2} e^{iw_{T_2}}$ respectively.

Remark: The Complex Intuitionistic Fuzzy Set (CIFS) contains complex membership and non-membership functions. However, in case of CHFS, only the complex membership function is considered. Therefore, we can say that every CHFS is contained in CIFS whereas the reverse is not true.

Definition 21 *Suppose there is a cohesive fuzzy set given by h_T, we define CIFS $A_{env}(h_T)$ as the envelope of h_T. Now the set $A_{env}(h_T)$ is represented by $< x, \mu_S(x), \gamma_S(x) >$ with*

$$\begin{aligned} \mu_S(x) &= \min(h_T) = \min(\mu_T) \\ \gamma_S(x) &= 1 - \max(h_T) = 1 - \max(\mu_T) \end{aligned} \tag{2.1.4}$$

where $\mu_T = r_T e^{iw_T}$.

Proposition 2. Now the relationship between the cohesive fuzzy set and Complex intuitionistic fuzzy set is given by:

- $A_{env}(h_T^c) = (A_{env}(h_T))^c$;
- $A_{env}(h_{T_1} \cup h_{T_2}) = A_{env}(h_{T_1}) \cup A_{env}(h_{T_2})$;
- $A_{env}(h_{T_1} \cap h_{T_2}) = A_{env}(h_{T_1}) \cap A_{env}(h_{T_2})$.

Proof: We know that

$$\begin{aligned} A_{env}(h_T) &=< \min h(x), 1 - \max h(x) >=< h_T^-(x), 1 - h_T^+(x) > \\ (A_{env}(h_T))^c &=< 1 - h_T^+(x), h_T^-(x) > \end{aligned}$$

and that

$$
\begin{aligned}
A_{env}\left(h_T^c\right) &=< \min h^c\left(x\right), 1-\max h^c\left(x\right) > \\
&=< \min\left(\left(1-r_T\left(x\right)\right)e^{-iw_T(x)}\right), 1{-}\max\left(\left(1-r_T\left(x\right)\right)e^{-iw_T(x)}\right) > \\
&=< 1-\max\left(r_T\left(x\right)e^{iw_T(x)}\right), 1-1+\min\left(r_T\left(x\right)e^{iw_T(x)}\right) > \\
&=< 1-h_T^+\left(x\right), h_T^-\left(x\right) >
\end{aligned}
$$

So, it proves the first inequality.
Then,

$$A_{env}\left(h_{T_1}\cup h_{T_2}\right) = A_{env}\left(\left\{h_T\in\left(h_{T_1}\left(x\right)\cup h_{T_2}\left(x\right)\right)|h_T\geq\max\left(h_{T_1}^-,h_{T_2}^-\right)\right\}\right)$$

Thus, it implies that x lie in interval $[\max\left(h_{T_1}^-\left(x\right),h_{T_2}^-\left(x\right)\right), \max(h_{T_1}^+\left(x\right),h_{T_2}^+\left(x\right)]$. This implies that

$$A_{env}\left(h_{T_1}\cup h_{T_2}\right) =< \max\left(h_{T_1}^-,h_{T_2}^-\right), \min\left(1-h_{T_1}^+, 1-h_{T_2}^+\right) >$$

This proves the second inequality.
Similarly, we can prove the third inequality. Finally, all the equalities are proved.
Next, for the sake of relative ordering over the cohesive fuzzy elements, some necessary comparing laws are being provided as follows:

Definition 22 *For a given cohesive fuzzy element h_T,*

$$f\left(h_T\right) = \frac{1}{\#h_T}\sum_{r_T,w_T\in h_T} r_T e^{iw_T};$$

is called the score function of h_T, where $\#h_T$ is the number of the elements in h_T. For two cohesive fuzzy elements h_{T_1} and $h_{T\ \ 2}$,

$$\textit{if } f\left(h_{T_1}\right) > f\left(h_{T_2}\right) \textit{ then } h_{T_1} > h_{T_2};\ \textit{iff}\left(h_{T_1}\right) = f\left(h_{T_2}\right), \textit{ then } h_{T_1} = h_{T_2}.$$

Next, we have defined some new operations on the cohesive fuzzy elements h_T, h_{T_1} and h_{T_2} on the basis of the relations proposed in proposition 2 are given below:

- $\left(h_T\right)^\lambda = \cup_{r_T,w_T\in h_T}\left(r_T e^{iw_T}\right)^\lambda$; where $\lambda\in\mathbb{R}, \lambda>0$
- $\lambda h_T = \cup_{r_T,w_T\in h_T}\left(1-\left(1-r_T\right)^\lambda\right)e^{i\lambda w_T}$

- **(Direct Sum)** $h_{T_1} \oplus h_{T_2} = \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{(r_{T_1} + r_{T_2} - r_{T_1} r_{T_2}) \, e^{i\left(w_{T_1} + w_{T_2}\right)}\}$
- **(Direct Product)** $h_{T_1} \otimes h_{T_2} = \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{(r_{T_1} r_{T_2}) \, e^{i\left(w_{T_1} + w_{T_2}\right)}\}$

Some more important operations have been established using the above operations on cohesive fuzzy elements as follows:

Theorem 1 *For given three cohesive fuzzy elements h_T, h_{T_1} and h_{T_2}, the following identities hold:*

(a) $h_{T_1}{}^c \cup h_{T_2}{}^c = (h_{T_1} \cap h_{T_2})^c$.

(b) $h_{T_1}{}^c \cap h_{T_2}{}^c = (h_{T_1} \cup h_{T_2})^c$.

(c) $(h_T^c)^\lambda = (\lambda h_T)^c$.

(d) $\lambda \, (h_T^c) = \left(h_T^\lambda\right)^c$.

(e) $h_{T_1}^c \oplus h_{T_2}^c = (h_{T_1} \otimes h_{T_2})^c$.

(f) $h_{T_1}^c \otimes h_{T_2}^c = (h_{T_1} \oplus h_{T_2})^c$.

Proof. The proof for the above stated identities have been outlined below:

(a)
$$\begin{aligned} h_{T_1}{}^c \cup h_{T_2}{}^c &= \cup_{r_{T_1}, w_{T_1} \in h_{T_1} r_{T_1}, w_{T_1} \in h_{T_1}} \max\left((1 - r_{T_1}) \, e^{-iw_{T_1}}, (1 - r_{T_2}) \, e^{-iw_{T_2}}\right) \\ &= (1 - \min(r_{T_1}, r_{T_2})) \, e^{-i\left(1 - \min\left(w_{T_1}, w_{T_2}\right)\right)} \\ &= (h_{T_1} \cap h_{T_2})^c. \end{aligned}$$

(b)
$$\begin{aligned} h_{T_1}{}^c \cap h_{T_2}{}^c &= \cup_{r_{T_1}, w_{T_1} \in h_{T_1} r_{T_1}, w_{T_1} \in h_{T_1}} \min\left((1 - r_{T_1}) \, e^{-iw_{T_1}}, (1 - r_{T_2}) \, e^{-iw_{T_2}}\right) \\ &= (1 - \max(r_{T_1}, r_{T_2})) \, e^{-i\left(1 - \max\left(w_{T_1}, w_{T_2}\right)\right)} \\ &= (h_{T_1} \cup h_{T_2})^c. \end{aligned}$$

(c)
$$\begin{aligned} (h_T^c)^\lambda &= \cup_{r_T, w_T \in h_T} \left((1 - r_T) \, e^{-iw_T}\right)^\lambda = \cup_{r_T, w_T \in h_T} (1 - r_T)^\lambda \, e^{-i\lambda w_T} \\ &= \cup_{r_T, w_T \in h_T} \left(\left(1 - (1 - r_T)^\lambda\right) e^{i\lambda w_T}\right)^c \\ &= (\lambda h_T)^c. \end{aligned}$$

(d)
$$\lambda \, (h_T^c) = \cup_{r_T, w_T \in h_T} \left(1 - \left(1 - (1 - r_T)^\lambda\right)\right) e^{-i\lambda w_T} = \left(h_T^\lambda\right)^c.$$

(e) $h^c_{T_1} \oplus h^c_{T_2} = \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{((1 - r_{T_1}) + (1 - r_{T_2}) - (1 - r_{T_1})(1 - r_{T_2}))\, e^{-i\left(w_{T_1} + w_{T_2}\right)}\}$
$= \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{(1 - r_{T_1} r_{T_2})\, e^{-i\left(w_{T_1} + w_{T_2}\right)}\}$
$= (h_{T_1} \otimes h_{T_2})^c.$

(f) $h^c_{T_1} \otimes h^c_{T_2} = \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{(1 - r_{T_1})(1 - r_{T_2})\, e^{-i\left(w_{T_1} + w_{T_2}\right)}\}$
$= \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{(1 - (r_{T_1} + r_{T_2} - r_{T_1} r_{T_2}))\, e^{-i\left(w_{T_1} + w_{T_2}\right)}\}$
$= (h_{T_1} \oplus h_{T_2})^c.$

Definition 23 *Let h_{T_1} and h_{T_2} are two cohesive fuzzy elements, we propose the operators given below:*

(a) $h_{T_1} o_1 h_{T_2} = \cup_{\mu_{T_1} \in h_{T_1}} \{\frac{|\mu_{T_1} - \mu_{T_2}|}{1 + |\mu_{T_1} - \mu_{T_2}|}\}$

(b) $h_{T_1} o_2 h_{T_2} = \cup_{\mu_{T_1} \in h_{T_1}} \{\frac{|\mu_{T_1} - \mu_{T_2}|}{1 + 2|\mu_{T_1} - \mu_{T_2}|}\}$

(c) $h_{T_1} o_3 h_{T_2} = \cup_{\mu_{T_1} \in h_{T_1}} \{\frac{|\mu_{T_1} - \mu_{T_2}|}{2}\}$

(d) $h_{T_1} o_4 h_{T_2} = \cup_{\mu_{T_1} \in h_{T_1}} \{\frac{|\mu_{T_1} . \mu_{T_2}|}{2}\}$

where μ_{T_1} & μ_{T_2} are in the form of $r_{T_1} e^{i w_{T_1}}$ & $r_{T_2} e^{i w_{T_2}}$ respectively.

Remarks:

It may be observed from the above definition that

- $h_{T_1} \oplus h_{T_2} = \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{(r_{T_1} + r_{T_2} - r_{T_1} r_{T_2})\, e^{i\left(w_{T_1} + w_{T_2}\right)}\}$
  $= \cup_{\mu_{T_1}, w_{T_1} \in h_{T_1}, \mu_{T_2}, w_{T_2} \in h_{T_2}} \{\mu_{T_1} e^{i w_{T_2}} + \mu_{T_2} e^{i w_{T_1}} - \mu_{T_1} \mu_{T_2}\}.$
- $h_{T_1} \otimes h_{T_2} = \cup_{r_{T_1}, w_{T_1} \in h_{T_1}, r_{T_2}, w_{T_2} \in h_{T_2}} \{(r_{T_1} r_{T_2})\, e^{i\left(w_{T_1} + w_{T_2}\right)}\}$
  $= \cup_{\mu_{T_1} \in h_{T_1}, \mu_{T_2} \in h_{T_2}} \{\mu_{T_1} \mu_{T_2}\}.$

Theorem 2 *For h_{T_1} and h_{T_2} be the two cohesive fuzzy elements. Then, we have the following identities*

(a) $(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1} o_1 h_{T_2}) = (h_{T_1} o_1 h_{T_2})$

(b) $(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1} o_1 h_{T_2}) = (h_{T_1} \oplus h_{T_2})$

(c) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1} o_1 h_{T_2}) = (h_{T_1} o_1 h_{T_2})$

(d) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1} o_1 h_{T_2}) = (h_{T_1} \otimes h_{T_2})$

(e) $(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1} o_2 h_{T_2}) = (h_{T_1} o_2 h_{T_2})$

(f) $(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1} o_2 h_{T_2}) = (h_{T_1} \oplus h_{T_2})$

(g) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1} o_2 h_{T_2}) = (h_{T_1} o_2 h_{T_2})$

(h) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1} o_2 h_{T_2}) = (h_{T_1} \otimes h_{T_2})$

(i) $(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1} o_3 h_{T_2}) = (h_{T_1} o_3 h_{T_2})$

(j) $(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1} o_3 h_{T_2}) = (h_{T_1} \oplus h_{T_2})$

(k) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1} o_3 h_{T_2}) = (h_{T_1} o_3 h_{T_2})$

(l) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1} o_3 h_{T_2}) = (h_{T_1} \otimes h_{T_2})$

(m) $(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1} o_4 h_{T_2}) = (h_{T_1} o_4 h_{T_2})$

(n) $(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1} o_4 h_{T_2}) = (h_{T_1} \oplus h_{T_2})$

(o) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1} o_4 h_{T_2}) = (h_{T_1} o_4 h_{T_2})$

(p) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_{11}} o_4 h_{T_2}) = (h_{T_1} \otimes h_{T_2})$.

Proof. All the above listed properties have been proved one by one. In view of the Definition 12 stated above, we have

(a) $$\begin{aligned}
&(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1}\ o_1\ h_{T_2}) \\
&= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \right) \\
&= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} min\{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \\
&= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} = (h_{T_1}\ o_1\ h_{T_2}).
\end{aligned}$$

(b) $$\begin{aligned}
&(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1}\ o_1\ h_{T_2}) \\
&= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \right) \\
&= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} max\{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \\
&= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} = (h_{T_1} \oplus h_{T_2}).
\end{aligned}$$

(c) $$\begin{aligned}
&(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1}\ o_1\ h_{T_2}) \\
&= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \right) \\
&= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \min\{\mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \\
&= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} = (h_{T_1}\ o_1\ h_{T_2}).
\end{aligned}$$

(d) $$\begin{aligned}
&(h_{T_1} \otimes h_{T_2}) \cup (h_{T_1}\ o_1\ h_{T_2}) \\
&= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \right) \\
&= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \max\{\mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+|\mu_{T_1} - \mu_{T_2}|}\} \\
&= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} = (h_{T_1} \otimes h_{T_2}).
\end{aligned}$$

(e) $(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1}\ o_2\ h_{T_2})$

$$= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\} \right)$$

$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \min\{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\}$$

$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\} = (h_{T_1}\ o_2\ h_{T_2}).$$

(f) $(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1}\ o_2\ h_{T_2})$

$$= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\} \right)$$

$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \max\{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\}$$

$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} = (h_{T_1} \oplus h_{T_2}).$$

(g) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1}\ o_2\ h_{T_2})$

$$= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\} \right)$$

$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \min\{\mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\}$$

$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\} = (h_{T_1}\ o_2\ h_{T_2}).$$

(h) $(h_{T_1} \otimes h_{T_2}) \cup (h_{T_1}\ o_2\ h_{T_2})$

$$= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\} \right)$$

$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \max\{\mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{1+2|\mu_{T_1} - \mu_{T_2}|}\}$$

$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} = (h_{T_1} \otimes h_{T_2}).$$

(i) $(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1}\ o_3\ h_{T_2})$

$$= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\} \right)$$

$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \min\{\mu_{T_{11}} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\}$$

$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\} = (h_{T_1}\ o_3\ h_{T_2}).$$

(j) $(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1}\ o_3\ h_{T_2})$

$$= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\} \right)$$

$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \max\{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\}$$
$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} = (h_{T_1} \oplus h_{T_2}).$$

(k) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1}\ o_3\ h_{T_2})$
$$= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\} \right)$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \min\{\mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\}$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\} = (h_{T_1}\ o_3\ h_{T_2}).$$

(l) $(h_{T_1} \otimes h_{T_2}) \cup (h_{T_1}\ o_3\ h_{T_2})$
$$= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\} \right)$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \max\{\mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1} - \mu_{T_2}|}{2}\}$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} = (h_{T_1} \otimes h_{T_2}).$$

(m) $(h_{T_1} \oplus h_{T_2}) \cap (h_{T_1}\ o_4\ h_{T_2})$
$$= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\} \right)$$
$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \min\{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\}$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\} = (h_{T_1}\ o_4\ h_{T_2}).$$

(n) $(h_{T_1} \oplus h_{T_2}) \cup (h_{T_1}\ o_4\ h_{T_2})$
$$= \left(\cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\} \right)$$
$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \max\{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\}$$
$$= \cup_{\substack{\mu_{T_1}, w_{T_1} \in h_{T_1} \\ \mu_{T_2}, w_{T_2} \in h_{T_2}}} \{\mu_{T_1} e^{iw_{T_2}} + \mu_{T_2} e^{iw_{T_1}} - \mu_{T_1}\mu_{T_2}\} = (h_{T_1} \oplus h_{T_2}).$$

(o) $(h_{T_1} \otimes h_{T_2}) \cap (h_{T_1}\ o_4\ h_{T_2})$
$$= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cap \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\} \right)$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} min\{\mu_{T_1}\mu_{T_2}, \tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\}$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\tfrac{|\mu_{T_1}\mu_{T_2}|}{2}\} = (h_{T_1}\ o_4\ h_{T_2}).$$

(p) $(h_{T_1} \otimes h_{T_2}) \cup (h_{T_1}\ o_4\ h_{T_2})$

$$= \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} \right) \cup \left(\cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\frac{|\mu_{T_1}\mu_{T_2}|}{2}\} \right)$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \max\{\mu_{T_1}\mu_{T_2}, \frac{|\mu_{T_1}\mu_{T_2}|}{2}\}$$
$$= \cup_{\substack{\mu_{T_1} \in h_{T_1} \\ \mu_{T_2} \in h_{T_2}}} \{\mu_{T_1}\mu_{T_2}\} = (h_{T_1} \otimes h_{T_2}).$$

Hence, this proves all the above stated identities from $((a) - (p))$. Similarly, various other operations and relations can further be established for cohesive fuzzy set.

2.2 Application of Cohesive Fuzzy Sets in Reference Signal

In this section, we incorporate the proposed notion of cohesive fuzzy set in the application field of filtering the electromagnetic signals for obtaining the reference signal from number of signals obtained. The propagation and parameter of an electromagnetic signal can be understood through the following diagram given in Figure 2.1:

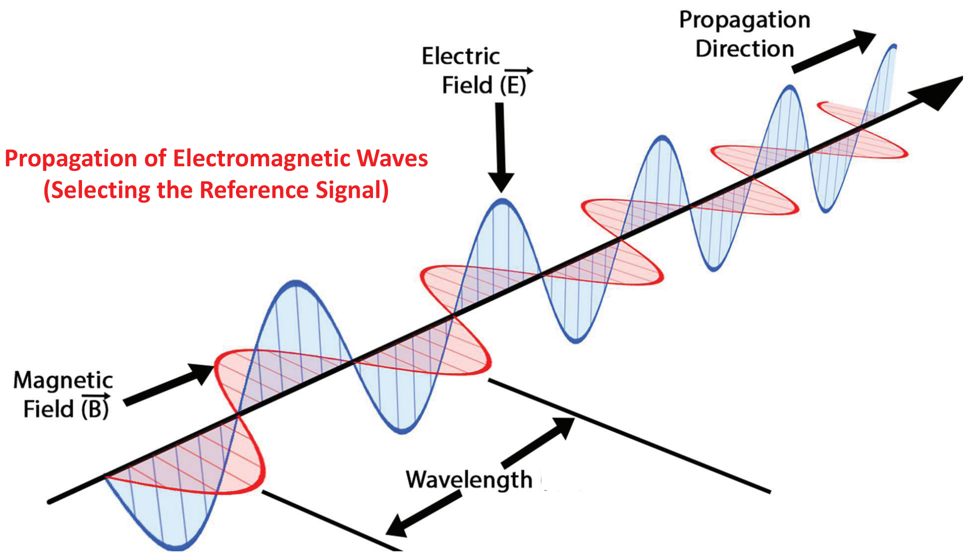

Figure 2.1: Components of Electromagnetic signal

In the subsequent sections, we first present new methodology by incorporating Fourier

Cosine Transformation in order to identify reference electromagnetic signal and secondly by using Inverse Discrete Fourier Cosine Transformation we present another methodology for identifying reference electromagnetic signal. To increase the clarity the flowchart explaining the procedure is given in Figure 2.2.

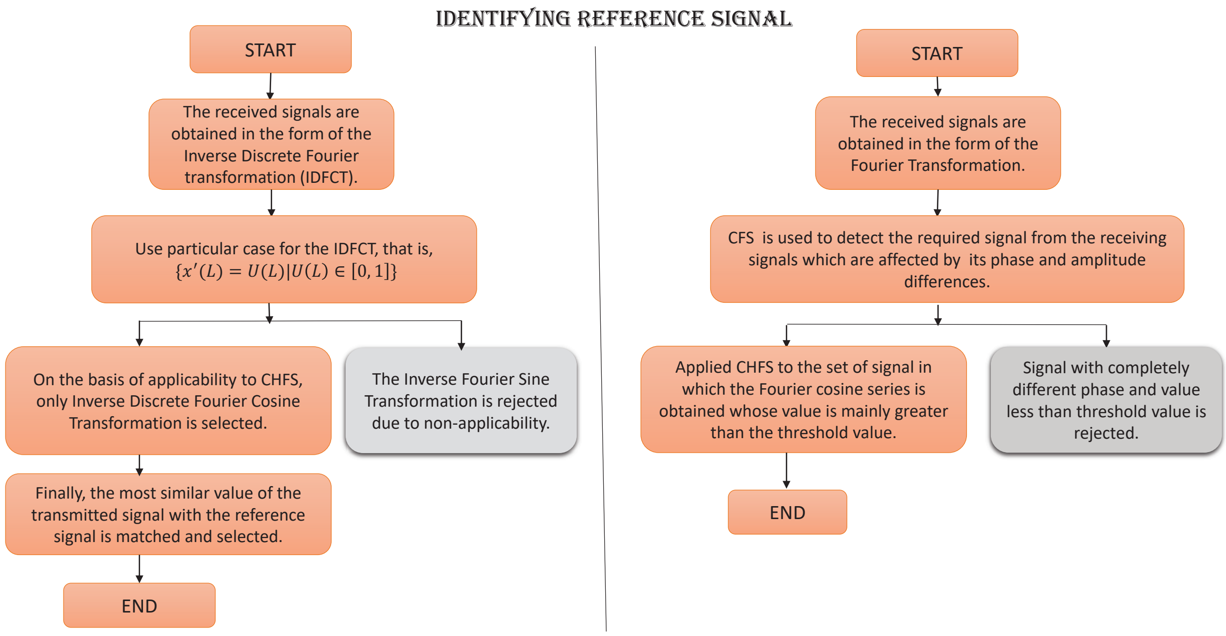

Figure 2.2: Methodology for Electromagnetic signal

2.2.1 Identifying Reference Electromagnetic Signal using Fourier Cosine Transformation (FCT)

Here, the processing of electromagnetic signal has been carried over by implementing the introduced concept of cohesive fuzzy set in identifying the signal of interest among the large number of signals received by the receiver. Ramot et al. [28] demonstrated the use of complex fuzzy set in signal processing where L different speech signals and electromagnetic signals, viz., $T_1(t), T_2(t), ..., T_L(t)$, have been detected & sampled by the digital receiver. Each received signal is sampled N times. Let $T_l(k)$ denotes the k^{th} $(1 \leq k \leq N)$ sample and the l^{th} signal $(1 \leq l \leq L)$.
Further, the Fourier transformation of the received signals have been obtained and

each being represented as the sum of Fourier components given below:

$$T_l\left(k\right) = \frac{1}{N}\sum_{n=1}^{N} C_{l,n} e^{\frac{i2\pi(n-1)(k-1)}{N}}; \tag{2.2.1}$$

where $C_{l,n}\left(1 \leq n \leq N\right)$ are complex Fourier coefficients of T_l.

It may be noted that in case of cohesive fuzzy set, we take Fourier cosine transformation of the received signals and each of which is represented as the sum of Fourier cosine components

$$T_l\left(k\right) = \frac{1}{N}\sum_{n=1}^{N} C_{l,n} \cos\left(\frac{i2\pi\left(n-1\right)\left(k-1\right)}{N}\right); \tag{2.2.2}$$

where $C_{l,n}\left(1 \leq n \leq N\right)$ are complex Fourier coefficients of T_l.
Therefore, the above mentioned sum may be rewritten as

$$T_l\left(k\right) = \frac{1}{N}\sum_{n=1}^{N} P_{l,n} e^{i\alpha_{l,n}} \cos\left(\frac{i2\pi\left(n-1\right)\left(k-1\right)}{N}\right); \tag{2.2.3}$$

where $C_{l,n} = A_{l,n}e^{i\alpha_{l,n}}$, with $P_{l,n}$ and $\alpha_{l,n}$ to be real valued & $P_{l,n} \geq 0\ \forall n$.
The aim of the above proposed application is to determine whether any signal among received L signals can be identified as the reference signal R. Therefore, in similar manner the reference signal R is also sampled N times and its corresponding Fourier cosine series may be written as:

$$R\left(k\right) = \frac{1}{N}\sum_{n=1}^{N} P_{R,n} e^{i\alpha_{R,n}} \cos\left(\frac{i2\pi\left(n-1\right)\left(k-1\right)}{N}\right); \tag{2.2.4}$$

where

$$C_{R,n} = P_{R,n}e^{i\alpha_{R,n}}, \text{with } 1 \leq n \leq N,$$

$$P_{R,n} \text{ and } \alpha_{R,n} \text{ to be real valued \& } P_{R,n} \geq 0\ \forall\ n.$$

Next, we formally list the steps of the proposed methodology for identifying the reference signal with the help of the similarity measures between the signals $T_1, T_2, ..., T_L$ to R as follows:
Step 1: We first normalize the amplitudes of all Fourier cosine coefficients for any candidate signal $T_l\left(1 \neq l \neq L\right)$. Suppose P_l denotes the N-dimensional vector of amplitudes of the candidate signal's (T) Fourier coefficients:

$$P_l = \left(P_{l,1}, P_{l,2}, ..., P_{l,N}\right),$$

and P_R denotes the N-dimensional vector of amplitudes of the reference signal's (R) Fourier coefficients:

$$P_R = (P_{R,1}, P_{R,2}, ..., P_{R,N}).$$

We consider the normalized vector Q_l in the form as given below:

$$Q_l = \frac{1}{P_l.\|P_l\|}, \text{ where } \|P_l\| = \sqrt{\sum_{n=1}^{N} (P_{l,n})^2},$$

and the normalized vector Q_R in the form as given below:

$$Q_R = \frac{1}{P_R.\|P_R\|}, \text{ where } \|P_R\| = \sqrt{\sum_{n=1}^{N} (P_{R,n})^2}.$$

Thus, the vector $Q_l = (Q_{l,1}, Q_{l,2}, ..., Q_{l,N})$ represents the normalized amplitudes of $T_l's$ Fourier cosine coefficients. Similarly, $Q_R = (Q_{R,1}, Q_{R,2}, ..., Q_{R,N})$ is the normalized amplitude of $R's$ Fourier cosine coefficients.

Step 2: Next, we calculate the complex grade similarity for every Fourier cosine coefficient of T_l in relevance with the reference signal R. Then, the grade of similarity between $C_{l,n}$ *to* $C_{R,n}$ may be denoted by $\nu_{R,T_l}(n)$ and given by:

$$\nu_{R,T_l}(n) = r_{R,T_l}(n)\, e^{iw_{R,T_l}(n)}; \tag{2.2.5}$$

where

$$r_{R,T_l}(n) = e^{\frac{-\left(Q_{R,n}-Q_{l,n}\right)^2}{Q_{R,n}Q_{l,n}}} \; \& \; w_{R,T_l} = (\alpha_{R,n} - \alpha_{l,n}).$$

Here, $\nu_{R,T_l(n)}$ represents the complex grade of membership which includes a phase and amplitude terms. The phase term contains the information of the relative phase between the $C_{l,n}$ and $C_{R,n}$. The amplitude term r_{R,T_l} in range [0,1] is normalized and used to measure the distance exponentially between the $C_{l,n}$ and $C_{R,n}$. The effect of outside factors such as path loss, distance of transmission source from digital receiver etc are reduced by using normalized amplitudes $Q_{l,n}$ and $Q_{R,n}$. In case of the relative amplitude of $C_{l,n}$ in T_l is compared to $C_{R,n}$ in R, so that synchronized results may be obtained in either cases of strong and weak signals.

Step 3: Further, the complex grade similarity ν_{R,T_l}, is obtained by summing the grade similarity of each of the Fourier cosine coefficient $\nu_{R,T_l}(n)\,\forall n\,(1 \leq n \leq N)$, in which either $Q_{l,n}$ or $Q_{R,n}$ must be larger than the $Q_{Threshold}$. This $Q_{Threshold}$ is used to prevent ν_{R,T_l} from the Fourier cosine coefficients with small amplitudes in T_l and R. Next, the sum of the complex grade similarity is divided by the number of coefficients (m). The considered coefficients of $Q_{l,n}$ and $Q_{R,n}$ must have greater amplitudes compare to the $Q_{Threshold}$ and consequently mapping the amplitude of ν_{R,T_l} in the range of [0,1], subject to

$$\nu_{R,T_l} = \frac{\sum_M \nu_{R,T_l}(n)}{m}; \tag{2.2.6}$$

where

$M = \{n|Q_{l,n}\ or\ Q_{R,n}\ >\ Q_{Threshold}\}$ and the number of elements in M is denoted by$m$.

Hence, the sum of ν_{R,T_l} given in equation (2.2.6) is totally dependent on the phase term of ν_{R,T_l}. The phase term is an important factor to determine whether the grade of similarity increase or decreases among $C'_{l,n}s$ and $C'_{R,n}s$. This issue of phase has been reduced in our proposed methodology as we are taking Fourier cosine transformation due to which only one factor will affect the phase term.

Thus, the amplitude of ν_{R,T_l}, which is used to determine T_l to R, subject to the following conditions:

- The identified signal T_l w.r.t R must be close to 1.
- The normalized amplitudes of the Fourier coefficients of T_l and R are similar.
- The relative phases of the Fourier coefficients of candidate and reference signals i.e. T_l and R are similar.

Step 4: Finally, the electromagnetic signal T_l may be identified as R, by comparing the values of $|\nu_{(R,T_l)}|$ to $\nu_{Threshold}$. If the obtained value of $|\nu_{(R,T_l)}|$ exceeds the threshold, then the identified signals T_l may be considered as R.

The above proposed methodology, which utilizes the Fourier Cosine Transformation in calculating the similarity between two signals, is suppose to play a significant role in signal analysis applications where the relative phase between the Fourier component of the signals is considered to be important factor.

2.2.2 Identifying Reference Electromagnetic Signal using Inverse Discrete Fourier Cosine Transformation (IDFCT)

In this subsection, we have used the Inverse Discrete Fourier Cosine Transformation to develop a methodology to find the reference signal among the transmitted signals received by the receiver.

Xueling et al. [96] used the L^{th} Inverse Discrete Fourier Transform (IDFT) coefficient of a length L sequence $x(L)$ and have defined it as:

$$x\left(p\right)=\frac{1}{L}\sum_{p=0}^{L-1}x'\left(L\right)e^{i\frac{2\pi}{L}Lp},\ \ p\in 0,1,2,...,L-1;$$

where $x\left(L\right)$ have different values and considered the special case in which $U[L]=x'\left(L\right)\ \ \&\ U[L]\in[0,1]$.

In the similar way, we also consider the special case of Inverse Discrete Fourier Cosine Transformation (IDFCT) as below:

$$x\left(p\right)=\frac{1}{L}\sum_{p=0}^{L-1}x'(L)\cos\left(\frac{2\pi}{L}Lp\right),\ \ p\in 0,1,2,...,L-1.$$

Definition 13. The DFCT for $x'\left(L\right):1\leq L\leq L$ is given by matrix in the product form:

$$\begin{bmatrix} x'\left(0\right)\\ x'\left(1\right)\\ x'\left(2\right)\\ .\\ .\\ .\\ x'\left(L-1\right)\end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 & ... & 1\\ 1 & \cos\left(\frac{-2\pi}{L}\right) & \cos\left(\frac{-4\pi}{L}\right) & ... & \cos\left(\frac{-2\pi(L-1)}{L}\right)\\ 1 & \cos\left(\frac{-4\pi}{L}\right) & \cos\left(\frac{-8\pi}{L}\right) & ... & \cos\left(\frac{-4\pi(L-1)}{L}\right)\\ . & . & . & ... & .\\ . & . & . & ... & .\\ . & . & . & ... & .\\ 1 & \cos\left(\frac{-2\pi(L-1)}{L}\right) & \cos\left(\frac{-4\pi(L-1)}{L}\right) & ... & \cos\left(\frac{-2\pi(L-1)^2}{L}\right)\end{bmatrix}\begin{bmatrix} x\left(0\right)\\ x\left(1\right)\\ x\left(2\right)\\ .\\ .\\ .\\ x\left(L-1\right)\end{bmatrix}.$$

But the IDFCT is given by,

$$\begin{bmatrix} x\left(0\right)\\ x\left(1\right)\\ x\left(2\right)\\ .\\ .\\ .\\ x\left(L-1\right)\end{bmatrix}=\frac{1}{L}\begin{bmatrix} 1 & 1 & 1 & ... & 1\\ 1 & \cos\left(\frac{2\pi}{L}\right) & \cos\left(\frac{4\pi}{L}\right) & ... & \cos\left(\frac{2\pi(L-1)}{L}\right)\\ 1 & \cos\left(\frac{4\pi}{L}\right) & \cos\left(\frac{8\pi}{L}\right) & ... & \cos\left(\frac{4\pi(L-1)}{L}\right)\\ . & . & . & ... & .\\ . & . & . & ... & .\\ . & . & . & ... & .\\ 1 & \cos\left(\frac{2\pi(L-1)}{L}\right) & \cos\left(\frac{4\pi(L-1)}{L}\right) & ... & \cos\left(\frac{2\pi(L-1)^2}{L}\right)\end{bmatrix}\begin{bmatrix} x'\left(0\right)\\ x'\left(1\right)\\ x'\left(2\right)\\ .\\ .\\ .\\ x'\left(L-1\right)\end{bmatrix}.$$

Next, with the help of the above definitions, we propose a new methodology to detect a particular signals among the various signals received by the receiver.
Suppose $l(u_1(L),\ u_2(L)\ u_3(L), ..., u_l(L))$ be the number of electromagnetic signals received by the receiver and each of these signals is noted L times. Suppose $x_l(L)$ be the l^{th} $(1 \leq l \leq L)$ signal, then the Discrete Fourier cosine transformation is given by:

$$u_l(L) = \frac{1}{L}\sum_{p=0}^{L-1} U[L]\cos\left(\frac{2\pi}{L}Lp\right);\ L, p = 0, 1, 2, ..., L-1, \tag{2.2.7}$$

where $U[L] \in [0, 1]$.

Here, $U[L] = \theta'_S(p)$ and $\frac{2\pi}{L}Lp = w_S(p)$ are called the amplitude and phase term respectively. Thus, equation (2.2.7) denotes the model for signal representation.

Now, we construct a particular kind of matrix to detect the particular signal among the different signals received by the receiver. For this, we consider a reference signal R which has also been noted L times and its DFCT is given as below:

$$R(L) = \frac{1}{L}\sum_{p=0}^{L-1} \theta'(p)\cos\left(\frac{2\pi}{L}Lp\right);\ L, p = 0, 1, 2, ..., L-1, \tag{2.2.8}$$

where $\theta'(p) \in [0, 1]$.
The procedural steps of the proposed methodology in order to compare the similarity between the two signals have been listed as follows:
Step 1:
Expanding $u_l(L) = \frac{1}{L}\sum_{p=0}^{L-1} U[L]\cos\left(\frac{2\pi}{L}Lp\right)$; for $p = 0, 1, 2, ..., L-1$ leads to,

$$u_l(L) = \frac{1}{L}\left[U[0]\cos\left(\frac{2\pi}{L}L(0)\right) + U[1]\cos\left(\frac{2\pi}{L}L(1)\right) + U[2]\cos\left(\frac{2\pi}{L}L(2)\right) + ... + U[L-1]\cos\left(\frac{2\pi}{L}L(L-1)\right)\right]. \tag{2.2.9}$$

Now, we put the values of $L = 0, 1, 2, ..., L-1$ in equation (2.2.9), through which we obtain the L^{th} sample of the signal which are being explained by taking individual discrete cases:
For $L = 0$ case:

$$u_l(0) = \frac{1}{L}\left[U[0]\cos\left(\frac{2\pi}{L}(0)(0)\right) + U[1]\cos\left(\frac{2\pi}{L}(0)(1)\right) + U[2]\cos\left(\frac{2\pi}{L}(0)(2)\right) + ... + U[L-1]\cos\left(\frac{2\pi}{L}(0)(L-1)\right)\right].$$

$$u_l(0) = \frac{1}{L}\left[U[0].1 + U[1]1 + U[2].1 + ... + U[L-1].1\right]. \tag{2.2.10}$$

For $L = 1$ case:

$$u_l(1) = \frac{1}{L}\left[U[1].1 + U[1]\cos\left(\frac{2\pi}{L}(1)(1)\right) + U[2]\cos\left(\frac{2\pi}{L}(1)(2)\right) + ... + U[L-1]\sin\left(\frac{2\pi}{L}(1)(L-1)\right)\right]. \tag{2.2.11}$$

For $L=2$ case:

$$u_l\left(2\right)=\frac{1}{L}]\left[U[1].1+U[1]\cos\left(\frac{2\pi}{L}(2)(1)\right)+U[2]\cos\left(\frac{2\pi}{L}(2)(2)\right)+...+U[L-1]\cos\left(\frac{2\pi}{L}(2)(L-1)\right)\right]. \tag{2.2.12}$$

Similarly, **for $L=L-1$ case:**

$$u_l\left(L-1\right)=\frac{1}{L}\left[U[1].1+U[1]\cos\left(\frac{2\pi}{L}(L-1)(1)\right)+U[2]\cos\left(\frac{2\pi}{L}(L-1)(2)\right)+...+U[L-1]\cos\left(\frac{2\pi}{L}(L-1)^2\right)\right]. \tag{2.2.13}$$

In similar manner, we obtain the values for L samples of the reference signal.

Step 2:

Now, we construct the matrix for L-samples of signal $u_l\left(L\right)$ & the reference signal as follows:

$$\begin{bmatrix} u_l\left(0\right) \\ u_l\left(1\right) \\ u_l\left(2\right) \\ . \\ . \\ . \\ u_l\left(L-1\right) \end{bmatrix} = \frac{1}{L} \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & \cos\left(\frac{2\pi}{L}\right) & \cos\left(\frac{4\pi}{L}\right) & . & . & . & \cos\left(\frac{2\pi(L-1)}{L}\right) \\ 1 & \cos\left(\frac{4\pi}{L}\right) & \cos\left(\frac{8\pi}{L}\right) & . & . & . & \cos\left(\frac{4\pi(L-1)}{L}\right) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & \cos\left(\frac{2\pi(L-1)}{L}\right) & \cos\left(\frac{4\pi(L-1)}{L}\right) & . & . & . & \cos\left(\frac{2\pi(L-1)^2}{L}\right) \end{bmatrix} \begin{bmatrix} U\left(0\right) \\ U\left(1\right) \\ U\left(2\right) \\ . \\ . \\ . \\ U\left(L-1\right) \end{bmatrix}$$

&

$$\begin{bmatrix} \theta\left(0\right) \\ \theta\left(1\right) \\ \theta\left(2\right) \\ . \\ . \\ . \\ \theta\left(L-1\right) \end{bmatrix} = \frac{1}{L} \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & \cos\left(\frac{2\pi}{L}\right) & \cos\left(\frac{4\pi}{L}\right) & . & . & . & \cos\left(\frac{2\pi(L-1)}{L}\right) \\ 1 & \cos\left(\frac{4\pi}{L}\right) & \cos\left(\frac{8\pi}{L}\right) & . & . & . & \cos\left(\frac{4\pi(L-1)}{L}\right) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & \cos\left(\frac{2\pi(L-1)}{L}\right) & \cos\left(\frac{4\pi(L-1)}{L}\right) & . & . & . & \cos\left(\frac{2\pi(L-1)^2}{L}\right) \end{bmatrix} \begin{bmatrix} \theta'\left(0\right) \\ \theta'\left(1\right) \\ \theta'\left(2\right) \\ . \\ . \\ . \\ \theta'\left(L-1\right) \end{bmatrix}.$$

It may be noted that the first matrix equation given above represents that transmitted signal has been obtained by multiplying the phase term matrix and amplitude matrix. Similarly, the second matrix equation given above represents the components of reference signal.

Step 3:

In view of the above two matrix equations and for the desired analysis, we take the absolute values of all the obtained values to bring them in the range of the disk of radius one in complex plane. These absolute values are given as below:

$$\begin{bmatrix} |\theta(0)| \\ |\theta(1)| \\ |\theta(2)| \\ . \\ . \\ . \\ |\theta(L-1)| \end{bmatrix} \& \begin{bmatrix} |u_l(0)| \\ |u_l(1)| \\ |u_l(2)| \\ . \\ . \\ . \\ |u_l(L-1)| \end{bmatrix}.$$

Step 4:

Finally, we select the maximum absolute cosine value among all the cases of $u_l(l)$ & reference signal. Then, the most similar values will be considered to be reference signal.

Example 1.

Suppose that there are four different electromagnetic waves $(u_1(L), u_2(L), u_3(L) \,\&\, u_4(L))$ which have been detected by the receiver. Then, the sample of each signal is to be taken four times. Assume that $\theta(L)$ is the reference signal. Then, the Discrete Fourier Cosine Transformation (DFCT) of these signals $u_l(L)$ and the reference signal $\theta(L)$ is given by:

$$u_l(L) = \frac{1}{4}\sum_{p=0}^{3} U_l[L]\cos\left(\frac{2\pi}{4}Lp\right);\ L, p = 0, 1, 2, 3; \tag{2.2.14}$$

and

$$\theta(L) = \frac{1}{4}\sum_{p=0}^{3} \theta'[L]\cos\left(\frac{2\pi}{4}Lp\right);\ L, p = 0, 1, 2, 3; \tag{2.2.15}$$

where $U_l(L), \theta'(L) \in [0, 1]$. Further, the equation (2.2.14) gives

$$u_l(L) = \frac{1}{4}[U[0]\cos\left(\frac{2\pi}{4}L(0)\right) + U[1]\cos\left(\frac{2\pi}{4}L(1)\right) + U[2]\cos\left(\frac{2\pi}{4}L(2)\right) + U[2]\cos\left(\frac{2\pi}{4}L(3)\right). \tag{2.2.16}$$

Now, we take the values of $L = 0, 1, 2, 3$ and subsequently obtain the foloowing equations1:

$$u_l(0) = \frac{1}{4}[U[0]\cos\left(\frac{2\pi}{4}(0)(0)\right) + U[1]\cos\left(\frac{2\pi}{4}(0)(1)\right) + U[2]\cos\left(\frac{2\pi}{4}(0)(2)\right) + U[2]\cos\left(\frac{2\pi}{4}(0)(3)\right).$$

$$u_l(0) = \frac{1}{4}[U[0].1 + U[1].1 + U[2].0 + U[2].1. \tag{2.2.17}$$

$$u_l\left(1\right)=\frac{1}{4}[U[0]\cos\left(\frac{2\pi}{4}(1)(0)\right)+U[1]\cos\left(\frac{2\pi}{4}(1)(1)\right)+U[2]\cos\left(\frac{2\pi}{4}(1)(2)\right)+U[2]\cos\left(\frac{2\pi}{4}(1)(3)\right). \tag{2.2.18}$$

$$u_l\left(2\right)=\frac{1}{4}[U[0]\cos\left(\frac{2\pi}{4}(2)(0)\right)+U[1]\cos\left(\frac{2\pi}{4}(2)(1)\right)+U[2]\cos\left(\frac{2\pi}{4}(2)(2)\right)+U[2]\cos\left(\frac{2\pi}{4}(2)(3)\right). \tag{2.2.19}$$

$$u_l\left(3\right)=\frac{1}{4}[U[0]\cos\left(\frac{2\pi}{4}(3)(0)\right)+U[1]\cos\left(\frac{2\pi}{4}(3)(1)\right)+U[2]\cos\left(\frac{2\pi}{4}(3)(2)\right)+U[2]\cos\left(\frac{2\pi}{4}(3)(3)\right). \tag{2.2.20}$$

Next, from all the above equations (2.2.17)-(2.2.20), we obtain:

$$\begin{bmatrix} u_l\left(0\right) \\ u_l\left(1\right) \\ u_l\left(2\right) \\ u_l\left(3\right) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \cos\left(\frac{2\pi}{4}\right) & \cos\left(\frac{4\pi}{4}\right) & \cos\left(\frac{6\pi}{4}\right) \\ 1 & \cos\left(\frac{4\pi}{4}\right) & \cos\left(\frac{8\pi}{4}\right) & \cos\left(\frac{12\pi}{4}\right) \\ 1 & \cos\left(\frac{6\pi}{4}\right) & \cos\left(\frac{12\pi}{4}\right) & \cos\left(\frac{18\pi}{4}\right) \end{bmatrix}\begin{bmatrix} U_1\left(0\right) \\ U_1\left(1\right) \\ U_1\left(2\right) \\ U_1\left(3\right) \end{bmatrix}.$$

In similar manner, for the case of the reference signal, the matrix equation obtained as follows:

$$\begin{bmatrix} \theta\left(0\right) \\ \theta\left(1\right) \\ \theta\left(2\right) \\ \theta\left(3\right) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \cos\left(\frac{2\pi}{4}\right) & \cos\left(\frac{4\pi}{4}\right) & \cos\left(\frac{6\pi}{4}\right) \\ 1 & \cos\left(\frac{4\pi}{4}\right) & \cos\left(\frac{8\pi}{4}\right) & \cos\left(\frac{12\pi}{4}\right) \\ 1 & \cos\left(\frac{6\pi}{4}\right) & \cos\left(\frac{12\pi}{4}\right) & \cos\left(\frac{18\pi}{4}\right) \end{bmatrix}\begin{bmatrix} \theta'\left(0\right) \\ \theta'\left(1\right) \\ \theta'\left(2\right) \\ \theta'\left(3\right) \end{bmatrix}.$$

Suppose that the provided values for the reference signal are as below:

$$\theta'[p]=\begin{cases} 0; & p=0 \\ 0; & p=1 \\ 0.2; & p=2 \\ 1; & p=3 \end{cases} \tag{2.2.21}$$

Then, putting equation(2.2.21) in the above references matrix equation. We get,

$$\begin{bmatrix} \theta\left(0\right) \\ \theta\left(1\right) \\ \theta\left(2\right) \\ \theta\left(3\right) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 1 \end{bmatrix}.$$

Now, the absolute value matrix of reference signal is given as:

$$\begin{bmatrix} |\theta(0)| \\ |\theta(1)| \\ |\theta(2)| \\ |\theta(3)| \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}.$$

The maximum value in the above matrix is 0.3.

Now, for the signal $u_1(L); L = 0, 1, 2, 3$

$$U_1[p] = \begin{cases} 0.5; & p = 0 \\ 0.7; & p = 1 \\ 0.8; & p = 2 \\ 1; & p = 3 \end{cases} \tag{2.2.22}$$

$$\begin{bmatrix} u_1(0) \\ u_1(1) \\ u_1(2) \\ u_1(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.7 \\ 0.8 \\ 1 \end{bmatrix}.$$

Now, the absolute value matrix of reference signal $u_1(L)$ is,

$$\begin{bmatrix} |u_1(0)| \\ |u_1(1)| \\ |u_1(2)| \\ |u_1(3)| \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

The maximum value in the above matrix is 0.8.

Now, for the signal $u_2(L); L = 0, 1, 2, 3$

$$U_2[p] = \begin{cases} 0.4; & p = 0 \\ 0.6; & p = 1 \\ 0.8; & p = 2 \\ 1; & p = 3 \end{cases} \tag{2.2.23}$$

$$\begin{bmatrix} u_2(0) \\ u_2(1) \\ u_2(2) \\ u_2(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \\ 0.8 \\ 1 \end{bmatrix}.$$

Now, the absolute value matrix of reference signal $u_2(L)$ is,

$$\begin{bmatrix} |u_2(0)| \\ |u_2(1)| \\ |u_2(2)| \\ |u_2(3)| \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

The maximum value in the above matrix is 0.7.

Now, for the signal $u_3(L); L = 0, 1, 2, 3$

$$U_3[p] = \begin{cases} 0.6; & p = 0 \\ 1; & p = 1 \\ 0.9; & p = 2 \\ 0.8; & p = 3 \end{cases} \tag{2.2.24}$$

$$\begin{bmatrix} u_3(0) \\ u_3(1) \\ u_3(2) \\ u_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.6 \\ 1 \\ 0.9 \\ 0.8 \end{bmatrix}.$$

Now, the absolute value matrix of reference signal $u_3(L)$ is,

$$\begin{bmatrix} |u_3(0)| \\ |u_3(1)| \\ |u_3(2)| \\ |u_3(3)| \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

The maximum value in the above matrix is 0.8.

Now, for the signal $u_4(L); L = 0, 1, 2, 3,$

$$U_4[p] = \begin{cases} 0.8; & p = 0 \\ 0.5; & p = 1 \\ 0; & p = 2 \\ 0; & p = 3 \end{cases} \tag{2.2.25}$$

$$\begin{bmatrix} u_4(0) \\ u_4(1) \\ u_4(2) \\ u_4(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}.$$

Now, the absolute value matrix of reference signal $u_4(L)$ is,

$$\begin{bmatrix} |u_4(0)| \\ |u_4(1)| \\ |u_4(2)| \\ |u_4(3)| \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.3 \\ 0.2 \end{bmatrix}.$$

The maximum value in the above matrix is 0.3.

Now, listing all the maximum values and tabulating with the reference value, we get

$$\begin{bmatrix} \theta(L) \\ u_1(L) \\ u_2(L) \\ u_3(L) \\ u_4(L) \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.8 \\ 0.7 \\ 0.8 \\ 0.3 \end{bmatrix}.$$

Based on the above, we determine that the signal $u_4(L)$ is the reference signal.

2.3 Cohesive Fuzzy Sets in Solar Activities/Cycles

The planning a space mission requires a good prediction of favourable situations for which a large amount of data related to the solar cycles are required. With the help of estimation based on this data, the best time interval for the space mission may accordingly be predicted. In other words, the time interval and the favourable situations both conditions play a vital role in the success of a particular mission. The most important real-life example is the satellite on Mars (Mangalyaan) which was launched in the year 2013 and got planted in orbit of Mars in the year 2014. In that case, the scientist considered all the possible situations and the particular time was selected according to the data collected regarding the solar cycles. Thus, we can say that the planning under the given set of uncertainties which may include the fuzziness is an essential component behind the success of any mission.

In this section, we will consider the situations which can affect the solar activity/cycles and propose a brief outline of the methodology which could effectively help in the planning of such missions related to the solar activities in the complex plane. In reference to the above discussions, Yazdanbaksh et.al [104] proposed the concept of Adaptive Neuro Complex Fuzzy Inferential System (ANCFIS) which played a significant role in the area of solar energy and also compared the results with the help of two techniques viz. the Adaptive Neuro-Fuzzy Inference System (ANFIS) and Radial Basis Function Networks (RBFNs). In this way the proposed method and the obtained results were validated.

The idea behind implementing the cohesive fuzzy sets in the planning of the solar activities is being given by Figure 2.3 for the better understanding of the concept.

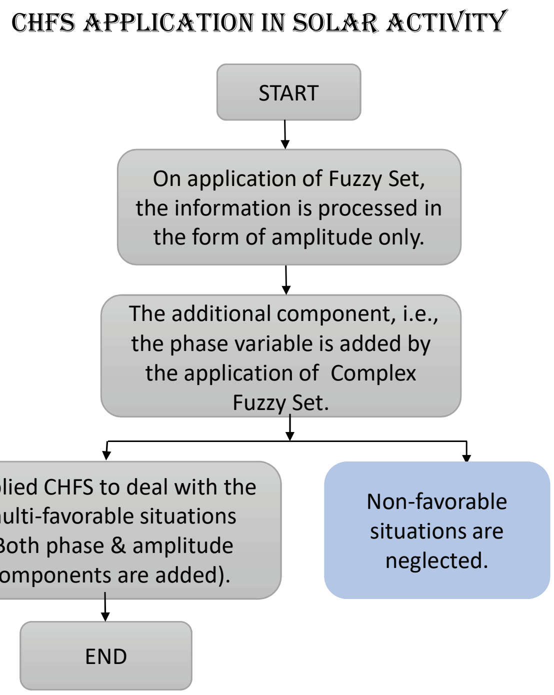

Figure 2.3: Methodology for Solar Activities

Further, the important role of CHFS in the case of solar activity is explained with the help of the following example.

Example 2. In every eleven years, the sun undergoes a period of activity called the "*solar maximum*" followed by a period known as "*solar minimum*". During the solar maximum, large number of sunspots, solar flares and coronal mass ejections are noticed, which can affect communications and weather on earth. During the solar minimum, the less number of sunspots are observed. This implies that one way of tracking the solar activity is by observing the amplitude of sunspots. In this the dark blemishes observed on the face of sun signifies the sunspots and the sites where solar flares are observed to occur. As per the data available with Solar Science resource (NASA) [128], the data collected shows the monthly average of the number of sunspots observed since year 1749 as shown in Figure 2.4.

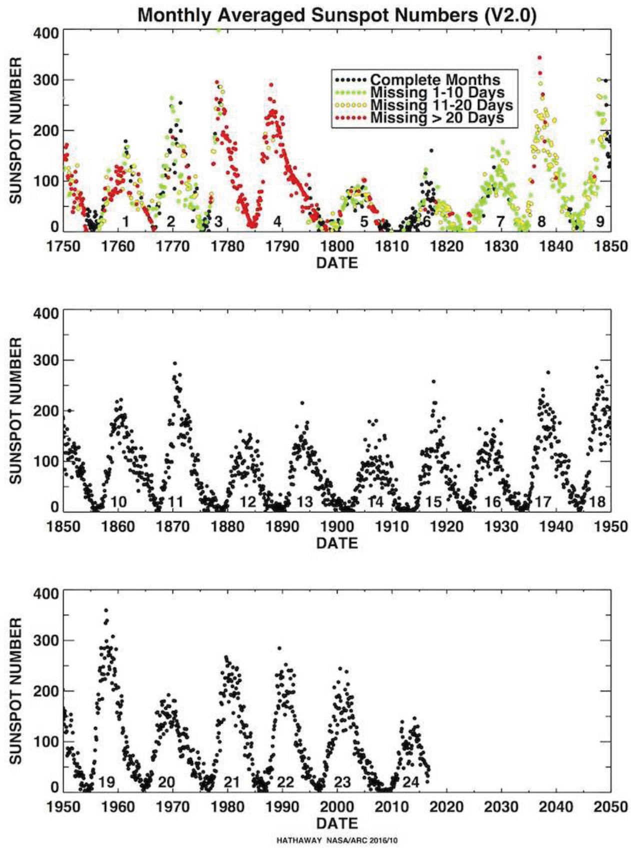

Figure 2.4: Monthly Average of Sunspot Observed [128]

In the case of solar activity, the simple fuzzy set denoted by T is efficient in collecting the data regarding the amplitude of sunspot whereas in case of complex fuzzy set (CFS) one additional information regarding the phase of the sunspot is obtained. This additional information helps to track the solar cycle with amplitude.

This helps us to understand that the notion of complex fuzzy set gives an added advantage over fuzzy set. In the present work, the proposed notion of Cohesive fuzzy set (CHFS) would certainly have another extra advantage over the complex fuzzy set. It may be noted that when CHFS is used in place of CFS then in case of solar activity it encounters the information regarding the interval in which the maximum number of sunspots are obtained. Since the implementation of CHFS will be able to deal with the favorable set of situations in the unit circle on complex plane, therefore, this will not only neglect the useless data, but also every element in the favorable set will be considered.

Now this is being explained in detail with the help of empirical values. Consider an ordinary fuzzy set T with high solar activity, which implies that the set T consists of large number of sunspots. However, the average number of sunspots observed during the month is used to derive the grade of membership in a particular month. Clearly, the grade of membership is totally dependent on average number of sunspots, i.e., if the number of sunspots is 200, it signifies the large grade of membership whereas 2 (number of sunspots) is associated with the small grade of membership. If grade of membership is 0.25 in set T, then it signifies the average number of sunspots in that particular month, say, 50, which can vary considerably if the solar cycle is considered. Therefore, the grade of membership to be 0.25 may be treated as inefficient. For example, it is been noted that the maximum number of sunspots in the months of year 1805 and 1956 was 50 and barely quarter of the way up respectively in solar cycle. Thus, to plan a space mission in these kind of years was not supposed to be possible. This signifies that it requires long term planning to execute a mission related to the space.

Ramot et al. [28] introduced the notion of complex fuzzy set which was able to deal with the phase variable with the explanation of the use of phase in tracking the cycle of solar activity. Further, they explained that the degree of membership can accordingly be increased by using the phase element. Now, the degree of membership depends on both the amplitude and phase variables. The limitation of using CFS is that it only deals with the maximum value of membership of sunspots whereas the nearby values are sometimes neglected which can also play an important role in the tracking of the solar activity.

Therefore, in order to overcome such limitations, it would always be better and advantageous to apply the proposed notion of Cohesive fuzzy set (CHFS) which deals with the set of favorable values which not only counter the limitation of ordinary fuzzy set but also provide an added feature over CFS. Hence, we can assert that CHFS plays a very important role in the planning of any solar activity. It is important to consider following three conditions for achieving a favorable sets in planning a solar activity:

- In first condition, the amplitude will be in the range of [0.5, 1] and no restriction will be applied on the phase element.

- Secondly, the phase element will be in range of $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and no restriction is applied on the amplitude.

- Thirdly, the amplitude and phase element will always lie in the range [0.5, 1] and $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ respectively.

It may be noted that all the above conditions can always be better dealt with the help of cohesive fuzzy sets. Now, the first condition is only applicable for the year in which the average number of sunspots is in between 100 to 200, which will lead to the grade of membership between [0.5, 1]. This will automatically increase the grade of membership irrespective of the membership of phase likely in year 1990-1994 (according to the data given in Figure 2.4). Such condition will automatically neglect the unfavorable data for any solar activity.

In second case, we will be restricting the phase parameter and will select the years in which the average number of sunspots is very less (like in year 1995 - according to Figure 2.4). In those years, due to the decrease in the average number of sunspots, the degree of membership will also decrease. Therefore, in order to increase the membership degree, it is advisable to increase the phase element. In this way, in those years where there are less amplitude of sunspot, a space mission can also be planned.

In third case, this condition relates to the set of most favorable situations in which we will be restricting both amplitude and phase terms in the intervals in which both are increasing. Hence, the degree of membership in this set will be maximum for almost all the data. Thus, to plan a space mission in this interval of the years will increase the chances of success. In this manner, all the nearby situations cannot be neglected and each of the elements of the sets in all the above cases can be dealt with the help of CHFS. The cases explained above can accordingly be worked out depending on the place of experiment. Therefore, the researchers must collect the data related to the solar cycles according to the places and then plan any solar activity.

2.4 Advantages & Limitations of Proposed Methodology

The advantages of CHFS in contrast with the utilization of fuzzy sets and complex fuzzy sets have been explained with the help of table given below:

Table 2.1: Comparative Advantages of Cohesive Fuzzy Sets with the Existing Extensions

Degree of membership	**FS**	**CFS**	**HFS**	**CHFS**
Amplitude	$\surd$	$\surd$	$\surd$	$\surd$
Phase		$\surd$		$\surd$
Advantages	Degree of membership in case of amplitude is obtained	Degree of membership in case of amplitude and phase is obtained.	Degree of membership in case of favorable situation is obtained	Degree of membership in case of amplitude and phase is obtained.
Advantages over other	It is not able to track the solar cycle	It contains both the useful data as well as the non useful data which consumes time. Secondly, it also misses some of the useful data as only the max value is considered.	It contain the favourable data but in range [0,1]	It only contains the favorable values in the set and also all the values are considered.

- The advantage of CHFS is that it contains the properties of both Complex Fuzzy Set (CFS) and Hesitant Fuzzy Set (HFS) which enhances the efficiency of the proposed set in solving the problems more efficiently.
- The proposed notion of CHFS deals with the favorable set, i.e., in case of signals a favorable set of Cosine Transformation is considered, but the Sine Transformation is rejected due to the limitation of the problem under consideration.

This limitation of the proposed methodology may be resolved in future by introducing some new concepts with some other examples.

- Similarly, in case of solar cycles, the different particular favorable cases have been selected on the basis of the structure of the problem.

2.5 Conclusions

A new extension set coined as Cohesive fuzzy set (CHFS) has been successfully proposed which has the dual benefits of complex fuzzy set with coverage of hesitant fuzzy set. We have studied the various operations, several useful identities on the CHFS and duly explained the process of selection of the best among the available multiple favorable situations with the possibility of its range in the extended unit circle of the complex plane. We have successfully established the relationship between the cohesive fuzzy set and complex intuitionistic fuzzy set and also validated the obtained results. The identification process of the reference signal among various transmitted electromagnetic signals has been successfully accomplished by utilizing the feature of cohesive fuzzy set and Fourier cosine transform/inverse Fourier cosine transform. Also, the process of identifying maximum number of sunspots in a particular interval under a solar activity has been discussed and explained with suitable reference. The advantageous features of the proposed methodology have been tabulated for a better readability and a quick glance.

The proposed notion of Cohesive fuzzy sets appears to be a promising one which has the capability to address certain real life situations which can not be dealt with complex fuzzy sets and other extensions of fuzzy sets. Various other properties of CHFS need to be explored to fully comprehend its potential. The concepts of aggregation operators and complex hesitant fuzzy relations for CHFS can further be worked out for solving various types of decision-making problems.

Chapter 3

Complex Neutrosophic Matrix, Operations & Properties

In this chapter, the novel notion of the complex neutrosophic matrix has been proposed and studied in detail. Different algebraic operations and properties related to the proposed matrix along with norm convergence have also been studied. In addition to this, two novel neutrosophic similarity matrices have been successfully introduced and validated. Various properties and results related to the positive definiteness of the proposed matrices have been described. The systematic procedure and outline of the proposed methodology utilizing the similarity measures matrices have been detailed. A numerical example of medical diagnosis consisting of empirical data available in literature has been illustrated for showing the implementation of the proposed methodology. Further, the description of the complex fuzzy matrix and its theoretic algebraic operations have been given. The new similarity measure for the complex fuzzy matrices has been additionally proposed along with a numerical example. Further, the problem of identification of the reference signal has been considered in order to demonstrate the implementation of the proposed methodology.

3.1 Notion of Complex Neutrosophic Matrix

We propose the formal definition of a new kind of a complex neutrosophic matrix along with illustrative example, its complement, union and intersection for a better understanding of the concept.

Definition 24 *A CNM, $S_{m\times n}$ defined on a universe of discourse U, which can be characterized by a truth membership function $\Gamma_S(y_{ij})$, an indeterminacy membership function $I_S(y_{ij})$ and a falsity membership function $\Pi_S(y_{ij})$ that assign complex value functions of the form,*

$$\Gamma_S(y_{ij}) = P_S(y_{ij})\, e^{i\alpha_S(y_{ij})},$$

$$I_S(y_{ij}) = Q_S(y_{ij})\, e^{i\beta_S(y_{ij})},$$

and

$$\Pi_S(y_{ij}) = R_S(y_{ij})\, e^{i\gamma_S(y_{ij})}.$$

in $S_{m\times n}$ for any $y_{ij}\epsilon U$, where $P_S, Q_S, R_S \in [0,1]$ s.t. $0^- \leq P_S + Q_S + R_S \leq 3^+$. The values and the sum of Γ_S, I_S and π_S may always lie within the unit circle in the complex plane. Then, the complex neutrosophic fuzzy matrix $S_{m\times n}$ is represented as

$$S_{m\times n} = [\Gamma_S(y_{ij}), I_S(y_{ij}), \pi_S(y_{ij})]_{m\times n} | y_{ij} \in U.$$

where $|\Gamma_S| \leq 1,\ |I_S| \leq 1, |\pi_S| \leq 1\, \&\, |\Gamma_S + I_S + \pi_S| \leq 3.$

Example: The matrix representaion of complex neutrosophic set of order 3×1 is given below:

$$S_{3\times 1} = \begin{bmatrix} \left(\frac{3}{5}e^{i0.8}, \frac{2}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right) \\ \left(\frac{3}{10}e^{i0.1}, \frac{2}{5}e^{i\frac{3\pi}{4}}, \frac{1}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{10}e^{i\frac{5\pi}{4}}, \frac{2}{5}e^{i\frac{\pi}{4}}\right) \end{bmatrix}$$

Definition 25 *(Complement of the CNM) The complement of the complex neutrosophic matrix can be written in the form of*

$$(S_{m\times n})^c = [\Gamma_S(y_{ij}), I_S(y_{ij}), \pi_S(y_{ij})]^c_{m\times n} = [\Gamma^c_S(y_{ij}), I^c_S(y_{ij}), \pi^c_S(y_ij)]_{m\times n}$$

$$= [\left(P_S(y_{ij})\, e^{i\alpha_S(y_{ij})}\right)^c, \left(Q_S(y_{ij})\, e^{i\beta_S(y_{ij})}\ \right)^c, \left(R_S(y_{ij})\, e^{i\gamma_S(y_{ij})}\right)^c]_{m\times n}$$

where $(P_S(y_{ij}))^c = R_S(y_{ij})$ *and* $\left(e^{i\alpha_S(y_{ij})}\right)^c = e^{i(2\pi-\alpha_S(y_{ij}))}$.
Similarly, $(R_S(y_{ij}))^c = P_S(y_{ij})$ *and* $\left(e^{i\gamma_S(y_{ij})}\right)^c = e^{i(2\pi-\gamma_S(y_{ij}))}$.
Finally, $(Q_S(y_{ij}))^c = 1 - Q_S(y_{ij})$ *and* $\left(e^{i\beta_S(y_{ij})}\right)^c = e^{i(2\pi-\beta_S(y_{ij}))}$.

Example: Suppose $S_{3\times 1}$ be a CNM. Then, $(S_{3\times 1})^c$ will be given by

$$S_{3\times 1} = \begin{bmatrix} \left(\frac{3}{5}e^{i0.8}, \frac{2}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right) \\ \left(\frac{3}{10}e^{i0.1}, \frac{2}{5}e^{i\frac{3\pi}{4}}, \frac{1}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{10}e^{i\frac{5\pi}{4}}, \frac{2}{5}e^{i\frac{\pi}{4}}\right) \end{bmatrix}, (S_{3\times 1})^c = \begin{bmatrix} \left(\frac{2}{5}e^{i\left(2\pi-\frac{\pi}{225}\right)}, \frac{3}{5}e^{i\frac{7\pi}{4}}, \frac{1}{2}e^{i\frac{5\pi}{4}}\right) \\ \left(\frac{7}{10}e^{i\left(2\pi-\frac{\pi}{1800}\right)}, \frac{3}{5}e^{i\frac{5\pi}{4}}, \frac{9}{10}e^{i\frac{7\pi}{4}}\right) \\ \left(\frac{4}{5}e^{i\left(2\pi-\frac{7\pi}{1800}\right)}, \frac{9}{10}e^{i\frac{3\pi}{4}}, \frac{3}{5}e^{i\frac{7\pi}{4}}\right) \end{bmatrix}$$

Definition 26 *(Union of the complex neutrosophic matrix) Consider two complex neutrosophic matrices* $S^1_{m\times n} = [\Gamma^1_S(y_{ij}), I^1_S(y_{ij}), \pi^1_S(y_{ij})]_{m\times n}$ *and* $S^2_{m\times n} = [\Gamma^2_S(y_{ij}), I^2_S(y_{ij}), \pi^2_S(y_{ij})]_{m\times n}$ *respectively. Then, the union of these two matrices will be given by*

$$S^1_{m\times n} \cup S^2_{m\times n} = [\max\{\Gamma^1_S(y_{ij}), \Gamma^2_S(y_{ij})\}, \min\{I^1_S(y_{ij}), I^2_S(y_{ij})\}, \min\{\pi^1_S(y_{ij}), \pi^2_S(y_{ij})\}]_{m\times n}$$

where

$$\max\left\{\Gamma^1_S(y_{ij}), \Gamma^2_S(y_{ij})\right\} = \max\left\{P^1_S(y_{ij}), P^2_S(y_{ij})\right\} e^{i\max\left\{\alpha^1_S(y_{ij}), \alpha^2_S(y_{ij})\right\}},$$

$$\min\{I^1_S(y_{ij}), I^2_S(y_{ij})\} = \min\{Q^1_S(y_{ij}), Q^2_S(y_{ij})\} e^{i\min\{\beta^1_S(y_{ij}), \beta^2_S(y_{ij})\}}$$

and

$$\min\left\{\Pi^1_S(y_{ij}), \Pi^2_S(y_{ij})\right\} = \min\left\{R^1_S(y_{ij}), R^2_S(y_{ij})\right\} e^{i\min\left\{\gamma^1_S(y_{ij}), \gamma^2_S(y_{ij})\right\}}$$

Example: Consider two complex neutrosophic matrices

$$S^1_{3\times 1} = \begin{bmatrix} \left(\frac{3}{5}e^{i0.8}, \frac{2}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right) \\ \left(\frac{1}{10}e^{i0.7}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{9}{10}e^{i\frac{5\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{10}e^{i\frac{5\pi}{4}}, \frac{2}{5}e^{i\frac{\pi}{4}}\right) \end{bmatrix}, \quad S^2_{3\times 1} = \begin{bmatrix} \left(\frac{1}{10}e^{i0.2}, \frac{3}{10}e^{i\frac{3\pi}{4}}, \frac{7}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.5}, \frac{1}{2}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{3}{5}e^{i0.7}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right) \end{bmatrix}$$

$$S^1_{3\times 1} \cup S^2_{3\times 1} = \begin{bmatrix} \left(\frac{3}{5}e^{i0.8}, \frac{2}{5}e^{i\frac{3\pi}{4}}, \frac{7}{10}e^{i\frac{3\pi}{4}}\right) \\ \left(\frac{1}{10}e^{i0.5}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{10}e^{i\frac{\pi}{4}}, \frac{2}{5}e^{i\frac{\pi}{4}}\right) \end{bmatrix}$$

Definition 27 *(Intersection of the complex neutrosophic matrix) Consider two complex neutrosophic matrices* $S^1_{m\times n} = [\Gamma^1_S(y_{ij}), I^1_S(y_{ij}), \pi^1_S(y_{ij})]_{m\times n}$ *and* $S^2_{m\times n} = [\Gamma^2_S(y_{ij}), I^2_S(y_{ij}), \pi^2_S(y_{ij})]_{m\times n}$ *respectively. Then, the intersection of these two matrices will be given by*

$$S^1_{m\times n}\cap S^2_{m\times n} = [\min\{\Gamma^1_S(y_{ij}), \Gamma^2_S(y_{ij})\}, \max\{I^1_S(y_{ij}), I^2_S(y_{ij})\}, \max\{\pi^1_S(y_{ij}), \pi^2_S(y_{ij})\}]_{m\times n}$$

where,

$$\min\left\{\Gamma^1_S(y_{ij}), \Gamma^2_S(y_{ij})\right\} = \min\left\{P^1_S(y_{ij}), P^2_S(y_{ij})\right\}e^{i\min\left\{\alpha^1_S(y_{ij}),\alpha^2_S(y_{ij})\right\}},$$

$$\max\{I^1_S(y_{ij}), I^2_S(y_{ij})\} = \max\{Q^1_S(y_{ij}), Q^2_S(y_{ij})\}e^{i\max\{\beta^1_S(y_{ij}),\beta^2_S(y_{ij})\}}$$

and

$$\max\left\{\Pi^1_S(y_{ij}), \Pi^2_S(y_{ij})\right\} = \max\left\{R^1_S(y_{ij}), R^2_S(y_{ij})\right\}e^{i\max\left\{\gamma^1_S(y_{ij}),\gamma^2_S(y_{ij})\right\}}$$

Example: Consider two complex neutrosophic matrices

$$S^1_{3\times 1} = \begin{bmatrix} \left(\frac{3}{5}e^{i0.8}, \frac{2}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right) \\ \left(\frac{1}{10}e^{i0.7}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{9}{10}e^{i\frac{5\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{10}e^{i\frac{5\pi}{4}}, \frac{2}{5}e^{i\frac{\pi}{4}}\right) \end{bmatrix}, \quad S^2_{3\times 1} = \begin{bmatrix} \left(\frac{1}{10}e^{i0.2}, \frac{3}{10}e^{i\frac{3\pi}{4}}, \frac{7}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.5}, \frac{1}{2}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{3}{5}e^{i0.7}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right) \end{bmatrix}$$

$$S^1_{3\times 1}\cap S^2_{3\times 1} = \begin{bmatrix} \left(\frac{1}{10}e^{i0.2}, \frac{3}{10}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{2}e^{i\frac{\pi}{4}}, \frac{9}{10}e^{i\frac{5\pi}{4}}\right) \\ \left(\frac{3}{5}e^{i0.7}, \frac{1}{5}e^{i\frac{5\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right) \end{bmatrix}$$

3.2 Algebraic Operations on Complex Neutrosophic Matrix

In this section, we have discussed the theoretical operations of the complex neutrosophic set. This section begins with the basic definition related to the concept and followed by the theorem, multiplication and additive identity.

Definition 28 *A be a* $m\times n$ *neutrosophic matrix. If all of its entries are* $< 0, 0, 1e^{i0} >$, *then A is called zero complex neutrosophic matrices and denoted by* 0. *If all of its entries are* $< 1e^{i0}, 1e^{i0}, 0 >$, *then A is called universal complex neutrosophic matrix and denoted by J.*

Theorem 3 *The matrix $S_{m\times n}$ are a complex neutrosophic fuzzy algebra under the component-wise addition and multiplication operations $(+,\odot)$ represented as:*
For $S_1 = [\Gamma_S^1(y_{ij}), I_S^1(y_{ij}), \pi_S^1(y_{ij})]_{m\times n}$ *and* $S_2 = [\Gamma_S^2(y_{ij}), I_S^2(y_{ij}), \pi_S^2(y_{ij})]_{m\times n}$ *in* $S_{m\times n}$,

$S_1+S_2 = (\sup\{S_1,S_2\}) = \left(\sup\{\Gamma_{S_1}(y_{ij}), \Gamma_{S_2}(y_{ij})\}, \sup\{I_S^1(y_{ij}), I_S^2(y_{ij})\}, \inf\{\pi_S^1(y_{ij}), \pi_S^2(y_{ij})\}\right)$

$S_1 \odot S_2 = (\inf\{S_1,S_2\}) = (\inf\{\Gamma_{S_1}(y_{ij}), \Gamma_{S_2}(y_{ij})\}, \inf\{I_S^1(y_{ij}), I_S^2(y_{ij})\}, \sup\{\pi_S^1(y_{ij}), \pi_S^2(y_{ij})\})$

where

$$S_1 = [\Gamma_S^1(y_{ij}), I_S^1(y_{ij}), \pi_S^1(y_{ij})]_{m\times n}$$

or

$$S_1 = \left[P_{S_1}(y_{ij})\, e^{i\alpha_{S_1}(y_{ij})}, Q_{S_1}(y_{ij})\, e^{i\beta_{S_1}(y_{ij})}, R_{S_1}(y_{ij})\, e^{i\gamma_{S_1}(y_{ij})}\right]_{m\times n}$$

and

$$S_2 = \left[P_{S_2}(y_{ij})\, e^{i\alpha_{S_2}(y_{ij})}, Q_{S_2}(y_{ij})\, e^{i\beta_{S_2}(y_{ij})}, R_{S_2}(y_{ij})\, e^{i\gamma_{S_2}(y_{ij})}\right]_{m\times n}$$

Proof: Every matrix in complex neutrosophic algebra should also satisfy the properties of fuzzy algebra. Therefore, $S_1 + O = S_1 \text{and} S_1 \odot J = S_1 \forall\ S_1 \epsilon S_{m\times n}$, hence the zero matrix O is the additive identity and the universal matrix J is the multiplicative identity. Thus, the identity element relative to the operations $(+,\odot)$ exist. Further, $S_1 + J = J$ and $S_1 \odot O = O$. This proves that universal bound holds. Similarly, we can prove for Idempotence, Commutativity, Associative and Absorption properties. Now, in the case of Distributivity property, we have to prove

$$S_1 \odot (S_2 + S_3) = (S_1 \odot S_2) + (S_1 \odot S_3)$$

where

$$S_1 = \left[P_{S_1}(y_{ij})\, e^{i\alpha_{S_1}(y_{ij})}, Q_{S_1}(y_{ij})\, e^{i\beta_{S_1}(y_{ij})}, R_{S_1}(y_{ij})\, e^{i\gamma_{S_1}(y_{ij})}\right]_{m\times n},$$

$$S_2 = \left[P_{S_2}(y_{ij})\, e^{i\alpha_{S_2}(y_{ij})}, Q_{S_2}(y_{ij})\, e^{i\beta_{S_2}(y_{ij})}, R_{S_2}(y_{ij})\, e^{i\gamma_{S_2}(y_{ij})}\right]_{m\times n}$$

and

$$S_3 = \left[P_{S_3}(y_{ij})\, e^{i\alpha_{S_3}(y_{ij})}, Q_{S_3}(y_{ij})\, e^{i\beta_{S_3}(y_{ij})}, R_{S_3}(y_{ij})\, e^{i\gamma_{S_3}(y_{ij})}\right]_{m\times n}.$$

Next, if $S_1 \le S_2 (or) S_3 i.e. \Gamma_{S_1}(y_{ij}) \le \Gamma_{S_2}(y_{ij})\ or\ \Gamma_{S_3}(y_{ij})$, $I_S^1(y_{ij}) \le I_S^2(y_{ij})$ or $I_S^3(y_{ij}), \pi_S^1(y_{ij}) \ge \pi_S^2(y_{ij})$ or $\pi_S^3(y_{ij})$ then in both cases

$$\inf\{S_1, \sup\{S_2,\ S_3\}\} = S_1 \text{and} \sup\{\inf\{S_1,\ S_2\}, \inf\{S_1, S_3\}\} = S_1.$$

In a similar manner,

$$S_1 + (S_2 \odot S_3) = (S_1 + S_2) \odot (S_1 + S_3)$$

Next, if $S_1 \geq S_2(or)S_3$ then in both cases

$$\sup\{\{S_1, \inf\{S_2,\ S_3\}\}\} = S_1 \text{and} \inf\{\sup\{S_1,\ S_2\}, \sup\{S_1, S_3\}\} = S_1.$$

Hence, all the properties are proved.

Definition 29 *Multiplication of two complex neutrosophic matrices. Consider two complex neutrosophic matrices given by $S^1_{3\times 2}$ and $S^2_{2\times 1}$ on the unit circle in complex plane i.e.*

$$S^1_{3\times 2} = \begin{bmatrix} \left(\breve{a}_1 e^{i\theta_1},\ \breve{a}_2 e^{i\theta_2}, \breve{a}_3 e^{i\theta_3}\right) & \left(\breve{a}_4 e^{i\theta_4}, \breve{a}_5 e^{i\theta_5}, \breve{a}_6 e^{i\theta_6}\right) \\ \left(\breve{b}_1 e^{i\sigma_1}, \breve{b}_2 e^{i\sigma_2}, \breve{b}_3 e^{i\sigma_3}\right) & \left(\breve{b}_4 e^{i\sigma_4}, \breve{b}_5 e^{i\sigma_5}, \breve{b}_6\ e^{i\sigma_6}\right) \\ \left(\breve{c}_1 e^{i\rho_1}, \breve{c}_2 e^{i\rho_2}, \breve{c}_3 e^{i\rho_3}\right) & \left(\breve{c}_4 e^{i\rho_4}, \breve{c}_5 e^{i\rho_5}, \breve{c}_6 e^{i\rho_6}\right) \end{bmatrix},$$

$$S^2_{2\times 1} = \begin{bmatrix} \left(\breve{p}_1 e^{i\alpha_1},\ \breve{p}_2 e^{i\alpha_2}, \breve{p}_3 e^{i\alpha_3}\right) \\ \left(\breve{q}_1 e^{i\beta_1}, \breve{q}_2 e^{i\beta_2}, \breve{q}_3 e^{i\beta_3}\right) \\ \\ \end{bmatrix}$$

Now the product of two matrices is given by

$$S^1_{3\times 2}.\ S^2_{2\times 1} = \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \end{bmatrix}$$

where,

$$d_{11} = \{\sup\{\inf\left(\breve{a}_1 e^{i\theta_1}, \breve{p}_1 e^{i\alpha_1}\right), \inf\left(\breve{a}_4 e^{i\theta_4}, \breve{q}_1 e^{i\beta_1}\right)\}, \sup\{\inf\left(\breve{a}_2 e^{i\theta_2}, \breve{p}_2 e^{i\alpha_2}\right), \inf\left(\breve{a}_5 e^{i\theta_5}, \breve{q}_2 e^{i\beta_2}\right)\},$$
$$\inf\{\sup\left(\breve{a}_3 e^{i\theta_3}, \breve{p}_3 e^{i\alpha_3}\right), \sup(\breve{a}_6 e^{i\theta_6}, \breve{q}_3 e^{i\beta_3})\}\}$$

Similarly, for d_{21} & d_{31}.

Example: Let us consider two matrices given below

$$S^1_{3\times 2} = \begin{bmatrix} \left(\frac{3}{5}e^{i0.8}, \frac{2}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right) & \left(\frac{1}{2}e^{i0.4}, \frac{1}{5}e^{i\frac{3\pi}{4}}, \frac{1}{10}e^{i\frac{5\pi}{4}}\right) \\ \left(\frac{1}{10}e^{i0.7}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{9}{10}e^{i\frac{5\pi}{4}}\right) & \left(\frac{7}{10}e^{i0.3}, \frac{1}{10}e^{i\frac{3\pi}{4}}, \frac{1}{2}\ e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.7}, \frac{1}{10}e^{i\frac{5\pi}{4}}, \frac{2}{5}e^{i\frac{\pi}{4}}\right) & \left(\frac{7}{10}e^{i0.1}, \frac{9}{10}e^{i\frac{\pi}{4}}, \frac{1}{5}e^{i\frac{3\pi}{4}}\right) \end{bmatrix},$$

$$S^2_{2\times 1} = \begin{bmatrix} \left(\frac{1}{10}e^{i0.2}, \frac{3}{10}e^{i\frac{3\pi}{4}}, \frac{7}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.5}, \frac{1}{2}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{\pi}{4}}\right) \\ \end{bmatrix}$$

$$S^1_{3\times 2}.\ S^2_{2\times 1} = \begin{bmatrix} \left(\sup\left\{\frac{1}{10}e^{i0.2}, \frac{1}{5}e^{i0.4}\right\}, \sup\left\{\frac{3}{10}e^{i\frac{\pi}{4}}, \frac{1}{5}e^{i\frac{\pi}{4}}\right\}, \inf\left\{\frac{7}{10}e^{i\frac{3\pi}{4}}, \frac{3}{10}e^{i\frac{5\pi}{4}}\right\}\right) \\ \left(\sup\left\{\frac{1}{10}e^{i0.2}, \frac{1}{5}e^{i0.3}\right\}, \sup\left\{\frac{1}{5}e^{i\frac{\pi}{4}}, \frac{1}{10}e^{i\frac{\pi}{4}}\right\}, \inf\left\{\frac{9}{10}e^{i\frac{5\pi}{4}}, \frac{1}{2}\ e^{i\frac{\pi}{4}}\right\}\right) \\ \left(\sup\left\{\frac{1}{10}e^{i0.2}, \frac{1}{5}e^{i0.1}\right\}, \sup\left\{\frac{1}{10}e^{i\frac{3\pi}{4}}, \frac{1}{2}e^{i\frac{\pi}{4}}\right\}, \inf\left\{\frac{7}{10}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{3\pi}{4}}\right\}\right) \end{bmatrix}$$

$$S^1_{3\times 2}.\ S^2_{2\times 1} = \begin{bmatrix} \left(\frac{1}{5}e^{i0.4}, \frac{3}{10}e^{i\frac{\pi}{4}}, \frac{3}{10}e^{i\frac{3\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.3}, \frac{1}{5}e^{i\frac{\pi}{4}}, \frac{1}{2}\ e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.2}, \frac{1}{2}e^{i\frac{3\pi}{4}}, \frac{3}{10}e^{i\frac{\pi}{4}}\right) \end{bmatrix}$$

Definition 30 *The identity element for addition.*

Consider two neutrosophic matrices $S_{2\times 2}$ and $I_{2\times 2}$ respectively, where $I_{2\times 2}$ is an identity matrix. Then,

$$S_{2\times 2} = \begin{bmatrix} \left(\frac{1}{10}e^{i0.3}, \frac{7}{10}e^{i\frac{\pi}{4}}, \frac{1}{5}e^{i\frac{5\pi}{4}}\right) & \left(\frac{7}{10}e^{i0.4}, \frac{3}{5}e^{i\frac{5\pi}{4}}, \frac{1}{10}e^{i\frac{\pi}{4}}\right) \\ \left(\frac{1}{5}e^{i0.2}, \frac{4}{5}e^{i\frac{5\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right) & \left(\frac{3}{5}e^{i0.7}, \frac{1}{2}e^{i\frac{\pi}{4}}, \frac{2}{5}e^{i\frac{3\pi}{4}}\right) \end{bmatrix}$$

$$I_{2\times 2} = \begin{bmatrix} (0, 0, 1e^{i0}) & (0, 0, 1e^{i0}) \\ (0, 0, 1e^{i0}) & (0, 0, 1e^{i0}) \end{bmatrix}$$

$$S_{2\times 2} + I_{2\times 2} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = S_{2\times 2}$$

where,

$$d_{11} = \left(\sup\left(\frac{1}{10}e^{i0.3}, 0\right), \sup\left(\frac{7}{10}e^{i\frac{\pi}{4}}, 0\right), \inf\left(\frac{1}{5}e^{i\frac{5\pi}{4}}, 1\right)\right) = \left(\frac{1}{10}e^{i0.3}, \frac{7}{10}e^{i\frac{\pi}{4}}, \frac{1}{5}e^{i\frac{5\pi}{4}}\right)$$

$$d_{12} = \left(\sup\left(\frac{7}{10}e^{i0.4}, 0\right), \sup\left(\frac{3}{5}e^{i\frac{5\pi}{4}}, 0\right), \inf\left(\frac{1}{10}e^{i\frac{\pi}{4}}, 1\right)\right) = \left(\frac{7}{10}e^{i0.4}, \frac{3}{5}e^{i\frac{5\pi}{4}}, \frac{1}{10}e^{i\frac{\pi}{4}}\right)$$

$$d_{21} = \left(\sup\left(\frac{1}{5}e^{i0.2}, 0\right), \sup\left(\frac{4}{5}e^{i\frac{5\pi}{4}}, 0\right), \inf\left(\frac{1}{2}e^{i\frac{3\pi}{4}}, 1\right)\right) = \left(\frac{1}{5}e^{i0.2}, \frac{4}{5}e^{i\frac{5\pi}{4}}, \frac{1}{2}e^{i\frac{3\pi}{4}}\right)$$

$$d_{22} = \left(\sup\left(\frac{3}{5}e^{i0.7}, 0\right), \sup\left(\frac{1}{2}e^{i\frac{\pi}{4}}, 0\right), \inf\left(\frac{2}{5}e^{i\frac{3\pi}{4}}, 1\right)\right) = \left(\frac{3}{5}e^{i0.7}, \frac{1}{2}e^{i\frac{\pi}{4}}, \frac{2}{5}e^{i\frac{3\pi}{4}}\right)$$

3.3 Matrix Norm Convergence for Complex Neutrosophic Matrix

This section includes the norm convergence of the complex neutrosophic matrix, followed by some basic properties, definitions and theorem.

Definition 31 *[70] "Suppose $\check{F} = R\, or\, C$, V in linear space over F. If the real vector function $||*||$ on V verify the properties given below:*

- *For arbitrary $u \in V, ||u|| \geq 0$, and $||u|| = 0 \implies u = 0$.*
- *For arbitrary $a \in \check{F}, u \in V$ get $||au|| = |a|.||u||$,*
- *For arbitrary $u,\ v \in V$, get $||u + v|| \leq ||u|| + ||v||$,*

Then, $||u||$ is called the vector norm of X in V."

Definition 32 *Consider $||*||$ is a non-negative real function on $\check{F}^{n\times n}$, if*

- $$||C_1C_2R\Gamma_S(y_{ij})|| \leq ||C_1R\Gamma_S(y_{ij})||.||C_2R\Gamma_S(y_{ij})||$$
- $$||C_1C_2\tau\Gamma_S(y_{ij})|| \leq ||C_1\tau\Gamma_S(y_{ij})||.||C_2\tau\Gamma_S(y_{ij})||$$

where $R(\Gamma_S(y_{ij}))$ & $\tau(\Gamma_S(y_{ij}))$ is the real and imaginary part of the CNM (Complex neutrosophic matrix).
Similarly, for $I_S(y_{ij})\&\pi_S(y_{ij})$. Then, this known as $||*||$ is CNFM(n, m).

Definition 33 *Consider $||*||$ is a non-negative real function on $F^{n\times n}$, if*

$$||S_1S_2R(\Gamma(x_{ij}))|| \leq ||S_1R(\Gamma(x_{ij}))||.||S_2R(\Gamma(x_{ij}))||;$$

$$||S_1S_2\tau(\Gamma(x_{ij}))|| \leq ||S_1\tau\Gamma(x_{ij})||.||S_2\tau\Gamma(x_{ij})||;$$

where $R(\Gamma(x_{ij}))$ & $\tau(\Gamma(x_{ij}))$ is the real and imaginary part of the complex neutrosophic matrix. Similarly, we can obtain for $I(x_{ij})\&\pi(x_{ij})$ functions.
Then, called $||*||$ is $CNFM(n, m)$.

Definition 34 *Suppose $(V, ||*||)$ is a n-dimensional normed linear space, $p_1, p_2, \ldots,$ $p_k, \ldots$ is a vector sequence and δ is a fixed vector of V, if*

$$\lim_{k\to\infty} ||p_k - \delta|| = 0.$$

Then, we can say that vector sequence convergence in the norm, δ is the limit of a sequence, given as:

$$\lim_{k\to\infty} p_k = \delta \ or \ p_k \to \delta.$$

Definition 35 *Suppose $(V, ||*||)$ is a n-dimensional normed linear space, $p_1, p_2, \ldots,$ $p_k, \ldots$ $\left(where \ \ p_u = \left(\Gamma_p^u(x_{ij}), I_p^u(x_{ij}), \pi_p^u(x_{ij})\right), u = 1, 2, 3,\right)$ is a complex neutroso -phic matrix sequence of V, $p(k)$ $\{p(k)$ is of form $\left(\Gamma_p^k(x_{ij}), I_p^k(x_{ij}), \pi_p^k(x_{ij})\right)\}$ constitutes a fixed complex neutrosophic function $\delta = (\Gamma_\delta(x_{ij}), I_\delta(x_{ij}), \pi_\delta(x_{ij}))$ of V, if*

$$\lim_{k\to\infty} ||pR\left(\Gamma_p^k(x_{ij})\right) - \delta R(\Gamma_\delta(x_{ij}))|| = 0,$$
$$\lim_{k\to\infty} ||p\tau\left(\Gamma_p^k(x_{ij})\right) - \delta\tau(\Gamma_\delta(x_{ij}))|| = 0;$$

where $R\left(\Gamma_p^k(x_{ij})\right), R(\Gamma_\delta(x_{ij}))$ denotes the real & $\tau\left(\Gamma_p^k(x_{ij})\right), \tau(\Gamma_\delta(x_{ij}))$ denotes the imaginary parts of the complex neutrosophic matrix respectively.
Similarly, for the case of indeterminacy and falsity components of the matrix can be obtained.

3.3.1 Convergence of Power of Complex Neutrosophic Matrix

Definition 36 *Consider $M(\Gamma_M(y_{ij}), I_M(y_{ij}), \pi_M(y_{ij})) \in CNFM(n, n)$ power K of M is defined as M^k, among them $M^1 = M, M^k = M^{k-1}M$.*

Theorem 4 *Consider $M(\Gamma_M(y_{ij}), I_M(y_{ij}), \pi_M(y_{ij})) \in CNFM(n, n)$, there exists a positive integer a and K, such that $\forall k \geq K$ has $M^{k+a} = M^k$.*

Proof:. Suppose $\forall k \geq 1$,

$$M(\Gamma_M(y_{ij}), I_M(y_{ij}), \pi_M(y_{ij}))$$

$$= \left[R\left(\Gamma_M\left(y_{ij}\right), I_M\left(y_{ij}\right), \pi_M\left(y_{ij}\right)\right) + i\left(\tau\left(\Gamma_M\left(y_{ij}\right), I_M\left(y_{ij}\right), \pi_M\left(y_{ij}\right)\right)\right)\right]_{n\times n}$$

$$M^k = \left[R\left(\Gamma_M\left(y_{ij}\right), I_M\left(y_{ij}\right), \pi_M\left(y_{ij}\right)\right) + i\left(\tau\left(\Gamma_M\left(y_{ij}\right), I_M\left(y_{ij}\right), \pi_M\left(y_{ij}\right)\right)\right)\right]_{n\times n}^k$$

$$R\left(\Gamma_M\left(y_{ij}\right), I_M\left(y_{ij}\right), \pi_M\left(y_{ij}\right)\right) =$$

$$\vee_{1\le p_1,,p_{k-1}\le n}\{(R\left(\Gamma_M\left(y_{ip_1}\right), I_M\left(y_{ip_1}\right), \pi_M\left(y_{ip_1}\right)\right) \wedge \wedge \left(R\left(\Gamma_M\left(y_i, p_{k-1}\right), I_M\left(y_i, p_{k-1}\right), \pi_M\left(y_i, p_{k-1}\right)\right)\right)))\}$$

It is known that $\vee \& \wedge$ are closed, therefore, the number of the elements of $\{M^k, k \ge 1\}$ will not be greater than $\left(n^{4n}\right)^n$. Then, there exists a positive integer a and K, s.t. $M^{k+a} = M^k$, thus $k \ge K$ has

$$M^{k+a} = M^{(k-k)+k+a} = M^{k-k}M^{k+a} = M^{k-k}M^k = M^k$$

3.4 Some New Similarity Measure Matrices

In this section of the current manuscript, the two similarity measure matrices are proposed and the properties are satisfied with the help of theorems.

Definition 37 *Suppose two complex neutrosophic matrices represented as* $M_{m\times n} = \left[T_M\left(x_{ij}\right), I_M\left(x_{ij}\right), F_M\left(x_{ij}\right)\right]_{m\times n}$ *and* $N_{m\times n} = \left[T_N\left(x_{ij}\right), I_N\left(x_{ij}\right), F_N\left(x_{ij}\right)\right]_{m\times n}$. *Then, the proposed similarity measure matrix is given by,*

$$S_1\left(M,\ N\ \right) = \left[1 - \frac{1}{mn}\sum_{i=1}^{m}\sum_{i=1}^{n}\left[\left|T_M T_N^2 - T_N T_N^2\right| + \left|I_M I_N^2 - I_N I_M^2\right| + \left|F_M F_N^2 - F_N F_M^2\right|\right]\right]_{m\times n}.$$

Theorem 5 *Consider* $S\left(M\right)$ *and* $S\left(N\ \right)$ *be two complex neutrosophic matrices. Then, these matrices must satisfy the following conditions:*

(i) $0 \le S\left(M, N\ \right) \le 1$.

(ii) $S\left(M, N\ \right) = 1 \Longleftrightarrow M = N$.

(iii) $S\left(M, N\ \right) = S\left(N,\ M\ \right)$.

(iv) $If\ \ M \subseteq N\ \subseteq o, thenS\left(M, N\ \right) \ge S\left(M,\ o\right)$ *and* $S\left(N,\ o\right) \ge S\left(M,\ o\right)$.

Proof:

(i) It is already known that T_M and T_N are two truth membership functions of complex neutrosophic matrices.
Then, its obvious that

$$|T_M| \leq 1, \quad |T_N| \leq 1 \ \& \ |T_M - T_N| \leq 1.$$

This implies,

$$|T_M^2| \leq 1, \ |T_N^2| \leq 1.$$

This means,

$$|T_M T_N^2| \leq 1 \ \& \ |T_N T_M^2| \leq 1.$$

Similarly, we can obtain it for the neutrality and falsity functions.

$$\Rightarrow \left|T_M T_N^2 - T_N T_M^2\right| \leq 1, \left|I_M I_N^2 - I_N I_M^2\right| \leq 1 and \left|F_M F_N^2 - F_N F_M^2\right| \leq 1.$$

This proves that,

$$0 \leq 1 - \frac{1}{mn} \sum_{i=1}^{m} \sum_{i=1}^{n} \left[\left|T_M T_N^2 - T_N T_M^2\right| + \left|I_M I_N^2 - I_N I_M^2\right| + \left|F_M F_N^2 - F_N F_M^2\right|\right] \leq 1.$$

Hence, $0 \leq S\left(M, N\ \right) \leq 1$.

(ii) If $M = N$, then

$$\left|T_M T_N^2 - T_N T_M^2\right| = \left|I_M I_N^2 - I_N I_M^2\right| = \left|F_M F_N^2 - F_N F_M^2\right| = 0.$$

Thus, $S(M, N) = 1$.

(iii) Replacing M by N then, also we obtain the same relation it simply proves that it satisfies the condition $S(M, N) = S(N, \quad M)$.

(iv) When $M \subseteq N \ \subseteq o$ then,

$$T_M \leq T_N \leq \ T_o, I_M \geq I_N \geq \ I_o \,\text{and}\, F_M \geq F_N \geq \ F_o.$$

$$\Rightarrow |T_M - T_N| \leq |T_M - T_o|\,.$$

$$\Rightarrow \left|T_M T_N^2 - T_N T_M^2\right| \leq \left|T_M T_o^2 - T_o T_M^2\right|.$$

Similarly,

$$\left|I_M I_N^2 - I_N I_M^2\right| \geq \left|I_M I_o^2 - I_o I_M^2\right| \, and \left|F_M F_N^2 - F_N F_M^2\right| \geq \left|F_M F_o^2 - F_o F_M^2\right|.$$

Thus, $S(M, N) \geq S(M, o)$.

Similarly, we can obtain the second condition that is $S(N, o) \geq S(M, o)$.

Hence, all the properties are satisfied.

Definition 38 *Suppose two complex neutrosophic matrices represented as* $M_{m\times n} = [T_M(x_{ij}), I_M(x_{ij}), F_M(x_{ij})]_{m\times n}$ *and* $N_{m\times n} = [T_N(x_{ij}), I_N(x_{ij}), F_N(x_{ij})]_{m\times n}$. *Then, the proposed logarithmic similarity matrix is given by:*

$S_2(M, N) =$

$$\left[\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[log_2 2 - log_2\left(\frac{1}{3}\left(\left|T_M T_N^2 - T_N T_M^2\right| + \left|I_M I_N^2 - I_N I_M^2\right| + \left|F_M F_N^2 - F_N F_M^2\right|\right)\right)\right]\right]_{m\times n}$$

Theorem 6 *Consider* $S(M)$ *and*$S(N)$ *be two complex neutrosophic matrices. Then, these must also satisfy the conditions given in Theorem 5.*

(i) It is already known that T_M and T_N are two truth membership functions of complex neutrosophic matrices. Then, its obvious that

$$|T_M| \leq 1, |T_N| \leq 1 \;\&\; |T_M - T_N| \leq 1.$$

This implies,

$$|T_M^2| \leq 1, \; T_N^2 \leq 1.$$

This means,

$$|T_M T_N^2| \leq 1 \;\&\; |T_N T_M^2| \leq 1.$$

Similarly, it can be proved for the case of neutrality and falsity,

$$\Rightarrow \left|T_M T_N^2 - T_N T_M^2\right| \leq 1, \left|I_M I_N^2 - I_N I_M^2\right| \leq 1 and \left|F_M F_N^2 - F_N F_M^2\right| \leq 1.$$

$$\Rightarrow log[\left|T_M T_N^2 - T_N T_M^2\right| + \left|I_M I_N^2 - I_N I_M^2\right| + \left|F_M F_N^2 - F_N F_M^2\right|] \leq 1.$$

This proves that,

$$0 \leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[log_2 2 - log_2\left(\frac{1}{3}\left(\left|T_M T_N^2 - T_N T_M^2\right| + \left|I_M I_N^2 - I_N I_M^2\right| + \left|F_M F_N^2 - F_N F_M^2\right|\right)\right)\right] \leq 1.$$

Hence, $0 \leq S(M, N) \leq 1$.

(ii) If $M = N$, then

$$\left|T_M T_N^2 - T_N T_M^2\right| = \left|I_M I_N^2 - I_N I_M^2\right| = \left|F_M F_N^2 - F_N F_M^2\right| = 0.$$

Thus, $S\left(M, N\ \right) = 1$.

(iii) Replacing M by N. Then, also the same function relations are obtained which simply proves that it satisfies the condition $S\left(M, N\ \right) = S\left(N,\ M\ \right)$.

(iv) When $M \subseteq N\ \ \subseteq o$. Then, $T_M \leq T_N \leq\ T_o, I_M \geq I_N \geq\ I_o$ and $F_M \geq F_N \geq\ F_o$.

$$\Rightarrow \left|T_M - T_N\right| \leq \left|T_M - T_\wr\right|.$$

$$\Rightarrow \left|T_M T_N^2 - T_N T_M^2\right| \leq \left|T_M T_\wr^2 - T_\wr T_M^2\right|.$$

Similarly,

$$\left|I_M I_N^2 - I_N I_M^2\right| \geq \left|I_M I_o^2 - I_o I_M^2\right| \text{ and } \left|F_M F_N^2 - F_N F_M^2\right| \geq \left|F_M F_o^2 - F_o F_M^2\right|.$$

Next,

$$log\ \left(\left|T_M T_N^2 - T_N T_M^2\right| +\ \left|I_M I_N^2 - I_N I_M^2\right| +\ \left|F_M F_N^2 - F_N F_M^2\right|\right) \geq$$

$$log\left(\left|T_M T_o^2 - T_o T_M^2\right| + \left|I_M I_o^2 - I_o I_M^2\right| + \left|F_M F_o^2 - F_o F_M^2\right|\right).$$

Thus, $S\left(M, N\ \right) \geq S\left(M,\ o\right)$.
Similarly, we can obtain the second condition, that is, $S\left(N,\ o\right) \geq S\left(M,\ o\right)$.
Hence, all the properties are satisfied.

3.5 Positive Definiteness of Similarity Measure Matrices of CNMs

In this section, the positive definiteness of the similarity measure matrix has been proved in detail with the help of theorems.

Let us recall some basic definitions and theorems related to the positive definiteness of complex matrices and the proof of the following definitions are already present in the literature. Therefore, detailed proof of the theorems will not be presented in the given section.

Definition 39 *[58]"Suppose N be a complex hermitian matrix with n dimension. Then, N is known as a positive semidefinite matrix if it satisfies the condition $x{*}Nx \geq 0$, where $x*$ denotes the conjugate of complex matrix x.*
Secondly, if $x * Nx = 0 \Longrightarrow x = 0$.Then, the matrix N will be known as strictly positive definite."

Theorem 7 *[58] The eigenvalues obtained for the case of hermitian matrix N is always real and positive.*

Theorem 8 *[58] Consider two hermitian positive semidefinite matrices N and M. Then, the sum of these two matrices $N + M$ is also positive semidefinite.*

Definition 40 *Suppose $N_1, N_2,\ N_3,\ ..., N_n$ denotes the Complex neutrosophic matrices in the universe of $U = u_1, u_2,\ u_3,\ ..., u_n$ and all the CNSs are hermitian matrices. Then, the similarity measure is described by S and is represented as*

$$S = \begin{bmatrix} S_{11}(N_1, N_1) & S_{12}(N_1, N_2) & \dots & S_{1n}(N_1, N_n) \\ S_{21}(N_2, N_1) & S_{22}(N_2, N_2) & \dots & S_{2n}(N_2, N_n) \\ \cdot & \cdot & \dots & \\ S_{n1}(N_n, N_1) & S_{2n}(N_n, N_2) & \dots & S_{nn}(N_n, N_n) \end{bmatrix}$$

This is already known that $S_{ij}(N_i, N_j) = S_{ji}(N_j,\ N_i)$ as all the matrices took are hermitian matrices and $S_{ii}(\ N_i,\ N_i) = 1$.

Theorem 9 *The hermitian matrix S between CNMs is a non-singular matrix.*

Proof Suppose S be the singular matrix. Then, the two columns must be linearly independent. Then, let N_i and N_j are linearly dependent. Therefore,

$$N_j = p.\ N_k$$

Thus,

$$N_{jq} = p.\ N_{kq} \text{ for all } q = 1,\ 2,\ ...,\ n.$$

If $q = j$ or $q = k$, then

$$N_{jj} = p.\ N_{kj} = 1$$

$$\Longrightarrow p = 1/N_{kj} > 1 \text{ or } \; p = N_{kj} < 1$$

Similarly, for $q = k$. This results in the contradiction of the assumption. Hence, S is a non-singular matrix.

Theorem 10 *The similarity matrix S is a positive definite matrix.*

Proof: Consider the similarity matrix proposed in definition 37. for square matrices with dimensions $n \times n$. Then, the similarity matrix S is defined as

$$S \; = \begin{bmatrix} 1-\alpha_{11} & 1-\alpha_{12} & ... & 1-\alpha_{12} \\ 1-\alpha_{21} & 1-\alpha_{22} & ... & 1-\alpha_{2n} \\ . & . & ... & . \\ 1-\alpha_{n1} & 1-\alpha_{2n} & ... & 1-\alpha_{nn} \end{bmatrix};$$

where

$$\alpha_{ij} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{i=1}^{n}\{\left|T_M(x_{ij})\,T_N^2(x_{ij}) - T_N(x_{ij})\,T_M^2(x_{ij})\right| + \left|I_M(x_{ij})\,I_N^2(x_{ij}) - I_N(x_{ij})\,I_M^2(x_{ij})\right|$$
$$+ \left|F_M(x_{ij})\,F_N^2(x_{ij}) - F_N(x_{ij})\,F_M^2(x_{ij})\right|\}.$$

When $M = N$, then $S(M,N) = 1$ and also $S(M,N) = S(N,\ M)$.
Therefore, $S = S^*$.

$$S \; = \begin{bmatrix} 1 & 1-\alpha_{12} & ... & 1-\alpha_{12} \\ 1-\alpha_{21} & 1 & ... & 1-\alpha_{2n} \\ . & . & ... & . \\ 1-\alpha_{n1} & 1-\alpha_{2n} & ... & 1 \end{bmatrix}.$$

According to theorem 7. given above, all the hermitian matrices have real and positive roots, thus the similarity matrix S will also follow the same condition and have real and positive roots, denoted by $\theta_1,\ \theta_2,\ \theta_3,\ ...,\ \theta_n$. Let us assume that θ be an arbitrary eigenvalue of similarity matrix S. According to Gerschgorin Theorem, we obtain

$$|\theta - 1| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} (S)\ .$$

Then,

$$|\theta - 1| = (n-1) - \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_{ij}) \ .$$

Now, let us consider

$$\sum_{j=1}^{n} (\theta_k) \ = \ \sum_{\substack{j=1 \\ j \neq i}}^{n} (s) \ = n \ \& \ n.\theta_{max} \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} (\theta_k) \ \geq n.\theta_{min},$$

where the min and max of the eigenvalues are denoted by

θ_{min} & θ_{max} $(\theta_{min} \leq 1, \ \theta_{max} \geq 1)$ respectively. Therefore,

$$|\theta_{min} - 1| = 1 - \theta_{min};$$

$$\leq (n-1) - \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_{ij}) \ ;$$

$$\theta_{min} \geq 2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_{ij}) \ - n.$$

No loss of generality is done by varying the values of i & j. Then, consider $n = 2$ we get,

$$\theta_{min} \geq 2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_{ij}) \ - 2 = \theta_{min} \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_{ij}) \ > 0.$$

Similarly, we obtain the values of θ_{min} by varying the values of n $(n = 3, \ 4, \ ...)$. Thus, it is concluded that $\theta_{min} \geq 0$. Finally, we also conclude that all the eigenvalues of S are non-negative. Hence, the similarity matrix is positive semidefinite.
From theorem 9, we have already proved that it is a nonsingular matrix. Therefore, the determinant of the similarity matrix S will not be equal to 0. Thus, all eigenvalues are strictly positive.
Hence, we can say that similarity matrix S is positive definite.
Similarly, we can prove it for the logarithmic similarity matrix given in definition 38.

3.6 Application of the Similarity Measure Matrices in Medical Diagnosis

In this section of the manuscript, we have explained the methodology of the application using proposed measures.

Suppose we consider a set of n patients denoted as $\{P_1,\ P_2,\ P_3,\ \ldots, P_n\ \}$, set of symptoms is denoted by $\{s_1,\ s_2,\ \ldots,\ s_k\}$ and the number of diseases is denoted by $\{d_1,\ d_2,\ d_3,\ \ldots, d_m\ \}$. The procedure for the proposed methodology is explained below.

Methodology

Step 1: We construct a decision matrix between the patients, symptoms and diseases. The matrices should be of the form

$$\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \end{array} \begin{bmatrix} \varkappa_{11} & \varkappa_{12} & \varkappa_{13} & \varkappa_{14} & \ldots & \varkappa_{1k} \\ \varkappa_{21} & \varkappa_{22} & \varkappa_{23} & \varkappa_{24} & \ldots & \varkappa_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \varkappa_{n1} & \varkappa_{n2} & \varkappa_{n3} & \varkappa_{n4} & \ldots & \varkappa_{nk} \end{bmatrix}.$$

$$S_1 \quad S_2 \quad S_3 \quad S_4 \quad \ldots \quad S_k$$

Step 2: Secondly, we construct a decision-making matrix based on symptoms and diseases related to them. The matrix should be of the form

$$\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_k \end{array} \begin{bmatrix} \varkappa'_{11} & \varkappa'_{12} & \varkappa'_{13} & \ldots & \varkappa'_{1m} \\ \varkappa'_{21} & \varkappa'_{22} & \varkappa'_{23} & \ldots & \varkappa'_{2m} \\ \varkappa'_{31} & \varkappa'_{32} & \varkappa'_{33} & \ldots & \varkappa'_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ \varkappa'_{51} & \varkappa'_{52} & \varkappa'_{53} & \ldots & \varkappa'_{km} \end{bmatrix}.$$

$$d_1 \quad d_2 \quad d_3 \quad \ldots \quad d_m$$

Step 3: Then, the similarity matrix S_d is applied to these two matrices.

Step 4: Then, the highest similarity matrix element is noted. Accordingly, the disease is told to the patient.

Step 5: End of the methodology.

3.7 Application of the Proposed Methodology

In this section, a case is considered and the methodology explained in the above section is used to obtain the result. This example not only proves the validity of the methodology but also increases the understanding of the concept.

Suppose three patients denoted by $\{P_1,\ P_2,\ P_3\ \}$ shows the symptoms {Temperature, Headache, Stomach Pain, Cough, Chest Pain} and the set of diseases related to the given set is {Viral, Malaria, Stomach Problem, Chest Problem}. Then, the application using the methodology to detect the type of disease is given below.

Step 1: The following matrix is the relationship between patients and symptoms. In this matrix, the numbers of patients are denoted by the number of rows and the number of columns denotes the number of symptoms that is Temperature, Headache, Stomach pain, Cough, Chest Pain respectively.

$$D_1$$

$$= \begin{bmatrix} \begin{pmatrix} 0.6e^{1.0i}, 0.4e^{1.2i} \\ 0.2e^{0.8i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.2i}, 0.4e^{1.1i} \\ 0.3e^{0.7i} \end{pmatrix} & \begin{pmatrix} 0.3e^{1.0i}, 0.4e^{1.2i}, \\ 0.4e^{0.6i} \end{pmatrix} & \begin{pmatrix} 0.6e^{1.0i}, 0.5e^{1.2i} \\ 0.3e^{0.8i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.0i}, 0.3e^{1.0i} \\ , 0.2e^{0.5i} \end{pmatrix} \\ \begin{pmatrix} 0.7e^{1.3i}, 0.4e^{1.2i}, \\ 0.5e^{0.9i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.5i}, 0.6e^{1.5i} \\ 0.3e^{0.5i} \end{pmatrix} & \begin{pmatrix} 0.5e^{1.4i}, 0.4e^{1.2i}, \\ 0.4e^{1.0i} \end{pmatrix} & \begin{pmatrix} 0.6e^{1.0i}, 0.4e^{1.0i} \\ 0.4e^{0.6i} \end{pmatrix} & \begin{pmatrix} 0.3e^{1.5i}, 0.4e^{1.0i} \\ , 0.5e^{1.0i} \end{pmatrix} \\ \begin{pmatrix} 0.5e^{0.6i}, 0.5e^{1.2i}, \\ 0.5e^{0.9i} \end{pmatrix} & \begin{pmatrix} 0.5e^{1.3i}, 0.4e^{1.2i} \\ 0.4e^{0.4i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.0i}, 0.4e^{1.0i}, \\ 0.2e^{0.6i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.0i}, 0.5e^{1.1i}, \\ 0.2e^{1.2i} \end{pmatrix} & \begin{pmatrix} 0.5e^{1.2i}, 0.2e^{1.2i} \\ , 0.2e^{1.4i} \end{pmatrix} \end{bmatrix}$$

Step 2: Next, the following matrix is the relation between the disease and symptoms. In this case, the number of columns represents the diseases like Viral, Malaria, Stomach Problem, Chest Problem whereas the number of rows represents the symptoms - Temperature, Headache, Stomach Pain, Cough, Chest Pain respectively.

$$D_2$$

$$= \begin{bmatrix} \begin{pmatrix} 0.4e^{1.2i}, 0.4e^{1.4i} \\ 0.3e^{0.6i} \end{pmatrix} & \begin{pmatrix} 0.6e^{1.3i}, 0.4e^{1.4i} \\ 0.2e^{1.5i} \end{pmatrix} & \begin{pmatrix} 0.5e^{1.4i}, 0.5e^{1.5i} \\ 0.2e^{0.6i} \end{pmatrix} & \begin{pmatrix} 0.6e^{1.5i}, 0.4e^{0.6i} \\ 0.5e^{0.7i} \end{pmatrix} \\ \begin{pmatrix} 0.5e^{0.6i}, 0.4e^{0.7i} \\ 0.2e^{0.8i} \end{pmatrix} & \begin{pmatrix} 0.4e^{0.7i}, 0.4e^{0.8i} \\ 0.3e^{0.9i} \end{pmatrix} & \begin{pmatrix} 0.5e^{0.8i}, 0.4e^{0.9i} \\ 0.2e^{1.0i} \end{pmatrix} & \begin{pmatrix} 0.5e^{0.9i}, 0.4e^{1.0i} \\ 0.5e^{0.8i} \end{pmatrix} \\ \begin{pmatrix} 0.4e^{1.0i}, 0.4e^{1.1i} \\ 0.4e^{1.2i} \end{pmatrix} & \begin{pmatrix} 0.5e^{1.1i}, 0.2e^{1.2i} \\ 0.2e^{1.3i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.2i}, 0.4e^{1.3i} \\ 0.5e^{1.4i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.3i}, 0.4e^{1.4i} \\ 0.3e^{1.5i} \end{pmatrix} \\ \begin{pmatrix} 0.3e^{1.4i}, 0.4e^{1.5i} \\ 0.5e^{0.6i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.5i}, 0.5e^{0.6i} \\ 0.3e^{0.7i} \end{pmatrix} & \begin{pmatrix} 0.5e^{0.6i}, 0.4e^{0.7i} \\ 0.3e^{0.8i} \end{pmatrix} & \begin{pmatrix} 0.3e^{0.7i}, 0.4e^{0.8i} \\ 0.4e^{0.9i} \end{pmatrix} \\ \begin{pmatrix} 0.4e^{0.8i}, 0.4e^{0.9i} \\ 0.5e^{1.0i} \end{pmatrix} & \begin{pmatrix} 0.6e^{1.0i}, 0.4e^{1.2i} \\ 0.3e^{1.4i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.2i}, 0.4e^{1.4i} \\ 0.5e^{0.6i} \end{pmatrix} & \begin{pmatrix} 0.4e^{1.4i}, 0.3e^{0.6i} \\ 0.2e^{0.8i} \end{pmatrix} \end{bmatrix}$$

Step 3: Applying similarity measure matrix given in definition 37 among D_1 and D_2 matrices, we get

$$S_d[D_1, D_2] = \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} \begin{bmatrix} \mathbf{0.9217} & 0.9125 & 0.9048 & 0.9046 \\ 0.8930 & 0.8878 & 0.8863 & \mathbf{0.8978} \\ \mathbf{0.9016} & 0.8919 & 0.8965 & 0.9010 \end{bmatrix}$$

$$V \quad M \quad SP \quad C$$

where $P_1,\ P_2$ & P_3 represents the number of patients and $V,\ M,\ S\ P$ & C denotes Viral fever, Malaria, Stomach problem and chest pain respectively.

Step 4: Applying similarity measure matrix given in definition 38 among D_1 and D_2 matrices, we get

$$S_d[D_1, D_2] = \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} \begin{bmatrix} \mathbf{0.7876} & 0.7557 & 0.7424 & 0.7307 \\ 0.6975 & 0.6839 & 0.6801 & \mathbf{0.7107} \\ \mathbf{0.7216} & 0.6946 & 0.7070 & 0.7200 \end{bmatrix}$$

$$V \quad M \quad SP \quad C$$

Step 5: Now results obtained using both the proposed similarities state that

Patients	Disease
P_1	Viral Fever
P_2	Chest Pain
P_3	Viral Fever

The concluded results are also written in bold in the above two similarity matrices to show the difference between the other values and results.

Step 6: Finally, this ends the methodology.

3.8 Comparative Analysis

In literature, many researchers have worked on similarity measures and found solutions to various practical life problems related to the fields like medicine and decision-making problems. In the present manuscript, the concept of similarity measure is studied in detail and two new similarity measure matrices are proposed. This similarity measure is studied in detail with the help of a similarity matrix i.e., the similarity measures are basically in matrix form and applied to the two matrices. The positive definiteness of the proposed similarity matrices is also defined and proved with the help of theorem. This positive definiteness of matrix plays a significant role in its application.

Later, in the work, the application of the proposed measure is described in the field of medical science where it plays a vital role in the identification of diseases with similar symptoms and later, the results are verified with the results already obtained in the literature by Mondal et al. [63]

The comparison values have been tabulated below which contains the results of our proposed similarity measures and other existing measures. On the basis of these values, we observe that the proposed similarity measure matrices are effective in resolving the difficulties related to medical diagnosis.

Used method	CNCSM[30]	CNDSM[30]	CNJSM[30]	Proposed Method 1	Proposed Method 2
Patient 1	Viral Fever (0.9303)	Viral Fever (0.8623)	Viral Fever (0.8595)	Viral Fever (0.9217)	Viral Fever (0.7876)
Patient 2	Chest Prob. (0.8651)	Chest Prob. (0.8307)	Chest Prob. (0.8502)	Chest Prob. (0.8978)	Chest Prob. (0.7107)
Patient 3	Viral Fever (0.9267)	Viral Fever (0.9005)	Viral Fever (0.8708)	Viral Fever (0.9016)	Viral Fever (0.7216)

where
CNCSM - The complex neutrosophic cosine similarity measure,
CNDSM - The complex neutrosophic dice similarity measure,
CNJSM - Complex neutrosophic Jaccard similarity measure,
PM_1 - Proposed Measure 1,
PM_2 - Proposed Measure 2.

3.9 Complex Fuzzy Matrix with Algebraic Operations

In this section, we have extended the concept of the complex fuzzy matrix with its examples. In addition to this, various set-theoretic operations viz. addition, multiplication, union and intersection on the CFM have described to increase the understanding of the basics.

Definition 41 *A complex fuzzy matrix $C_{m\times n}$, defined on a universe of discourse U, is characterized by a membership function $\mu_C(x_{ij})$ that assigns any element $x_{ij} \in U$. All the values of function $\mu_C(x_{ij})$ will lie in the unit disk of complex plane and will be of the form $r_C(x_{ij})\, e^{j\omega_C(x_{ij})}$, where $j = \sqrt{-1}$, $r_C(x_{ij})$ & $\omega_C(x_{ij})$ are both real-valued functions subject to $r_C(x_{ij}) \in [0,1]$.*

Then, CFM ($C_{m\times n}$) can be represented as

$$C_{m\times n} = \left\{ (x_{ij},\ \mu_C(x_{ij}))_{m\times n} \,|x_{ij} \in U \right\}.$$

Example: Suppose that we have an example of a medical situation in which there is a set of three patients, say, $B = (b_1,\ b_2,\ b_3)$, who are suffering from diseases having similar symptoms. Then, the possibility of a patient suffering from the set of particular disease $D = (d_1,\ d_2,\ d_3)$, can be represented through the following matrix, i.e.,

$$\begin{array}{cc} & \begin{array}{ccc} d_1 & d_2 & d_3 \end{array} \\ \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} & \begin{bmatrix} f_{11}e^{ig_{11}} & f_{12}e^{ig_{12}} & f_{13}e^{ig_{13}} \\ f_{21}e^{ig_{21}} & f_{22}e^{ig_{22}} & f_{23}e^{ig_{23}} \\ f_{31}e^{ig_{31}} & f_{32}e^{ig_{32}} & f_{33}e^{ig_{33}} \end{bmatrix} \end{array}$$

where $\left(f_{11}e^{ig_{11}},\ f_{12}e^{ig_{12}},\ \ldots,\ f_{33}e^{ig_{33}}\right)$ represents the degree of membership function for the patients in cases of a particular disease.

Theoretic Algebraic Operations on Complex Fuzzy Matrices

Let us consider two complex fuzzy matrices whose entries are of the form $r_C(x_{ij})\, e^{j\omega_C(x_{ij})}$ and given by

$$C^1_{2\times 2} = \begin{bmatrix} h_{11}e^{i\theta_1} & h_{12}e^{i\theta_2} \\ h_{21}e^{i\theta_3} & h_{22}e^{i\theta_4} \end{bmatrix} \ \&\ \ C^2_{2\times 2} = \begin{bmatrix} J_{11}e^{i\alpha_1} & J_{12}e^{i\alpha_2} \\ J_{21}e^{i\alpha_3} & J_{22}e^{i\alpha_4} \end{bmatrix} \tag{3.9.1}$$

- **Addition Operation of Two Complex Fuzzy Matrices**

  The sum of $C^1_{2\times 2}$ & $C^2_{2\times 2}$ is defined and represented as follows:

$$C^1_{2\times 2} +\ C^2_{2\times 2} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix};$$

  where

$$f_{11} = \max\left\{h_{11}e^{i\theta_1}, J_{11}e^{i\alpha_1}\right\} = \max\{h_{11},\ J_{11}\}e^{i\ max\{\theta_1,\alpha_1\}};$$

$$f_{12} = \max\left\{h_{12}e^{i\theta_2}, J_{12}e^{i\alpha_2}\right\} = \max\{h_{12},\ J_{12}\}e^{i\ max\{\theta_2,\alpha_2\}};$$

$$f_{21} = \max\left\{h_{21}e^{i\theta_3}, J_{21}e^{i\alpha_3}\right\} = \max\{h_{21},\ J_{21}\}e^{i\ max\{\theta_3,\alpha_3\}};$$

$$f_{22} = \max\left\{h_{22}e^{i\theta_4}, J_{22}e^{i\alpha_4}\right\} = \max\{h_{22},\ J_{22}\}e^{i\ max\{\theta_4,\alpha_4\}}.$$

Example: In view of the particular examples, the sum of given two matrices is illustrated as follows:

$$C^1_{2\times 2} = \begin{bmatrix} 0.6e^{i0.3} & 0.1e^{i0.7} \\ 0.2e^{i0.1} & 0.5e^{i0.4} \end{bmatrix} \;\&\; C^2_{2\times 2} = \begin{bmatrix} 0.5e^{i0.1} & 0.4e^{i0.3} \\ 0.8e^{i0.6} & 0.7e^{i0.2} \end{bmatrix}$$

Then

$$C^1_{2\times 2} + C^2_{2\times 2} = \begin{bmatrix} 0.6e^{i0.3} & 0.4e^{i0.7} \\ 0.8e^{i0.6} & 0.7e^{i0.4} \end{bmatrix}.$$

Commutativity and Associativity of Addition for CFMS:

Theorem 11 *Suppose that there are three CFMSs, say, P, Q and R, then the operation of addition is commutative and associative.*

(i) $P + Q = Q + P$. *(Commutative law)*

(ii) $(P + Q) + R = P + (Q + R)$. *(Associative law)*

Proof: Consider the following three complex fuzzy matrices:

$$P = \begin{bmatrix} a_{11}e^{i\theta_1} & a_{12}e^{i\theta_2} \\ a_{21}e^{i\theta_3} & a_{22}e^{i\theta_4} \end{bmatrix}, Q = \begin{bmatrix} b_{11}e^{i\alpha_1} & b_{12}e^{i\alpha_2} \\ b_{21}e^{i\alpha_3} & b_{22}e^{i\alpha_4} \end{bmatrix} \;\&\; R = \begin{bmatrix} g_{11}e^{i\gamma_1} & g_{12}e^{i\gamma_2} \\ g_{21}e^{i\gamma_3} & g_{22}e^{i\gamma_4} \end{bmatrix}.$$

Suppose that

$$P + Q = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = Y;$$

where

$$x_{11} = \max\left\{a_{11}e^{i\theta_1}, b_{11}e^{i\alpha_1}\right\} = \max\{a_{11},\ b_{11}\}e^{i\ max\{\theta_1,\alpha_1\}};$$

$$x_{12} = \max\left\{a_{12}e^{i\theta_2}, b_{12}e^{i\alpha_2}\right\} = \max\{a_{12},\ b_{12}\}e^{i\ max\{\theta_2,\alpha_2\}};$$

$$x_{21} = \max\left\{a_{21}e^{i\theta_3}, b_{21}e^{i\alpha_3}\right\} = \max\{a_{21},\ b_{21}\}e^{i\ max\{\theta_3,\alpha_3\}};$$

$$x_{22} = \max\left\{a_{22}e^{i\theta_4}, b_{22}e^{i\alpha_4}\right\} = \max\{a_{22},\ b_{22}\}e^{i\ max\{\theta_4,\alpha_4\}}.$$

Similarly, $Q + P = Y$. Hence, $P + Q = Q + P$.

Also, in case of associativity,

$$(P + Q) + R = Y + R = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} + \begin{bmatrix} g_{11}e^{i\gamma_1} & g_{12}e^{i\gamma_2} \\ g_{21}e^{i\gamma_3} & g_{22}e^{i\gamma_4} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = K;$$

where

$$k_{11} = \max\left\{x_{11}, g_{11}e^{i\gamma_1}\right\} = \max\{a_{11},\ b_{11},\ g_{11}\}e^{i\ max\{\theta_1,\alpha_1,\gamma_1\}};$$

$$k_{12} = \max\left\{x_{12}, g_{12}e^{i\gamma_2}\right\} = \max\{a_{12},\ b_{12}, g_{12}\}e^{i\ max\{\theta_2,\alpha_2,\gamma_2\}};$$

$$k_{21} = \max\left\{x_{21}, g_{21}e^{i\gamma_3}\right\} = \max\{a_{21},\ b_{21}, g_{21}\}e^{i\ max\{\theta_3,\alpha_3,\gamma_3\}};$$

$$k_{22} = \max\left\{x_{21}, g_{22}e^{i\gamma_4}\right\} = \max\{a_{22},\ b_{22}, g_{22}\}e^{i\ max\{\theta_4,\alpha_4,\gamma_4\}}.$$

Next,

$$Q + R = \begin{bmatrix} b_{11}e^{i\alpha_1} & b_{12}e^{i\alpha_2} \\ b_{21}e^{i\alpha_3} & b_{22}e^{i\alpha_4} \end{bmatrix} + \begin{bmatrix} g_{11}e^{i\gamma_1} & g_{12}e^{i\gamma_2} \\ g_{21}e^{i\gamma_3} & g_{22}e^{i\gamma_4} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = Y';$$

where

$$y_{11} = \max\left\{b_{11}e^{i\alpha_1}, g_{11}e^{i\gamma_1}\right\} = \max\{b_{11},\ g_{11}\}e^{i\ max\{\alpha_1,\gamma_1\}};$$

$$y_{12} = \max\left\{b_{12}e^{i\alpha_2}, g_{12}e^{i\gamma_2}\right\} = \max\{b_{12},\ g_{12}\}e^{i\ max\{\alpha_2,\gamma_2\}};$$

$$y_{21} = \max\left\{b_{21}e^{i\alpha_3}, g_{21}e^{i\gamma_3}\right\} = \max\{b_{21},\ g_{21}\}e^{i\ max\{\alpha_3,\gamma_3\}};$$

$$y_{22} = \max\left\{b_{22}e^{i\alpha_4}, g_{22}e^{i\gamma_4}\right\} = \max\{b_{22},\ g_{22}\}e^{i\ max\{\alpha_4,\gamma_4\}}.$$

Further,

$$P + (Q + R) = P + Y' = \begin{bmatrix} a_{11}e^{i\theta_1} & a_{12}e^{i\theta_2} \\ a_{21}e^{i\theta_3} & a_{22}e^{i\theta_4} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} = L;$$

where

$$l_{11} = \max\left\{a_{11}e^{i\theta_1}, y_{11}\right\} = \max\{a_{11},\ b_{11},\ g_{11}\}e^{i\ max\{\theta_1,\alpha_1,\gamma_1\}};$$

$$l_{12} = \max\left\{a_{12}e^{i\theta_2}, y_{12}\right\} = \max\{a_{12},\ b_{12}, g_{12}\}e^{i\ max\{\theta_2,\alpha_2,\gamma_2\}};$$

$$l_{21} = \max\left\{a_{21}e^{i\theta_3}, y_{21}\right\} = \max\{a_{21},\ b_{21}, g_{21}\}e^{i\ max\{\theta_3,\alpha_3,\gamma_3\}};$$

$$l_{22} = \max\left\{a_{22}e^{i\theta_4}, y_{22}\right\} = \max\{a_{22},\ b_{22}, g_{22}\}e^{i\ max\{\theta_4,\alpha_4,\gamma_4\}}.$$

Hence, $(P + Q) + R = P + (Q + R)$.

- **Multiplication Operation of Two Complex Fuzzy Matrices**

  Suppose $C^1_{2\times 2}$ & $C^2_{2\times 2}$ given by equation 3.9.1 are two CFMs, then their product is defined as follows:

$$C^1_{2\times 2}C^2_{2\times 2} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix};$$

where

$$d_{11} = \left\{\max\left\{\min\left\{h_{11}e^{i\theta_1}, J_{11}e^{i\alpha_1}\right\}, \min\left\{h_{12}e^{i\theta_2}, J_{21}e^{i\alpha_3}\right\}\right\}\right\};$$

$$d_{12} = \left\{\max\left\{\min\left\{h_{11}e^{i\theta_1}, J_{12}e^{i\alpha_2}\right\}, \min\left\{h_{12}e^{i\theta_2}, J_{22}e^{i\alpha_4}\right\}\right\}\right\};$$

$$d_{21} = \left\{\max\left\{\min\left\{h_{21}e^{i\theta_3}, J_{11}e^{i\alpha_1}\right\}, \min\left\{h_{22}e^{i\theta_4}, J_{21}e^{i\alpha_3}\right\}\right\}\right\};$$

$$d_{22} = \left\{\max\left\{\min\left\{h_{21}e^{i\theta_3}, J_{12}e^{i\alpha_2}\right\}, \min\left\{h_{22}e^{i\theta_4}, J_{22}e^{i\alpha_4}\right\}\right\}\right\};$$

Example: The product of given two matrices is obtained as follows:

$$C^1_{2\times 2} = \begin{bmatrix} 0.6e^{i0.3} & 0.1e^{i0.7} \\ 0.2e^{i0.1} & 0.5e^{i0.4} \end{bmatrix} \ \& \ C^2_{2\times 2} = \begin{bmatrix} 0.5e^{i0.1} & 0.4e^{i0.3} \\ 0.8e^{i0.6} & 0.7e^{i0.2} \end{bmatrix}$$

Then

$$C^1_{2\times 2}C^2_{2\times 2} = \begin{bmatrix} 0.5e^{i0.6} & 0.4e^{i0.3} \\ 0.5e^{i0.4} & 0.5e^{i0.2} \end{bmatrix}.$$

- **Union of Two Complex Fuzzy Matrices**

Now again, taking the value of $C^1_{2\times 2}$ & $C^2_{2\times 2}$ from equation 3.9.1 and then, the union of these matrices is given as

$$C^1_{2\times 2} \cup C^2_{2\times 2} = \begin{bmatrix} \check{d}_{11} & \check{d}_{12} \\ \check{d}_{21} & \check{d}_{22} \end{bmatrix};$$

where

$$\check{d}_{11} = \left\{\max\{h_{11},\ J_{11}\}e^{i\min\{\theta_1,\alpha_1\}}\right\};$$

$$\check{d}_{12} = \left\{\max\{h_{12},\ J_{12}\}e^{i\min\{\theta_2,\alpha_2\}}\right\};$$

$$\check{d}_{21} = \left\{\max\{h_{21},\ J_{21}\}e^{i\min\{\theta_3,\alpha_3\}}\right\};$$

$$\check{d}_{22} = \left\{\max\{h_{11},\ J_{11}\}e^{i\min\{\theta_1,\alpha_1\}}\right\}.$$

Example: The union of given two matrices is obtained as follows:

$$C^1_{2\times 2} = \begin{bmatrix} 0.6e^{i0.3} & 0.1e^{i0.7} \\ 0.2e^{i0.1} & 0.5e^{i0.4} \end{bmatrix} \ \& \ C^2_{2\times 2} = \begin{bmatrix} 0.5e^{i0.1} & 0.4e^{i0.3} \\ 0.8e^{i0.6} & 0.7e^{i0.2} \end{bmatrix}.$$

Then

$$C^1_{2\times 2} \cup C^2_{2\times 2} = \begin{bmatrix} 0.6e^{i0.1} & 0.4e^{i0.3} \\ 0.8e^{i0.1} & 0.7e^{i0.2} \end{bmatrix}$$

- **Intersection of Two Complex Fuzzy Matrices**

  Similarly, suppose that $C^1_{2\times 2}$ & $C^2_{2\times 2}$ are two CFMs, then the intersection of these matrices is defined as follows:

$$C^1_{2\times 2} \cap C^2_{2\times 2} = \begin{bmatrix} \check{\check{d}}_{11} & \check{\check{d}}_{12} \\ \check{\check{d}}_{21} & \check{\check{d}}_{22} \end{bmatrix};$$

where

$$\check{\check{d}}_{11} = \left\{ \min\{h_{11},\ J_{11}\} e^{i\max\{\theta_1,\alpha_1\}} \right\};$$

$$\check{\check{d}}_{12} = \left\{ \min\{h_{12},\ J_{12}\} e^{i\max\{\theta_2,\alpha_2\}} \right\};$$

$$\check{\check{d}}_{21} = \left\{ \min\{h_{21},\ J_{21}\} e^{i\max\{\theta_3,\alpha_3\}} \right\};$$

$$\check{\check{d}}_{22} = \left\{ \min\{h_{11},\ J_{11}\} e^{i\max\{\theta_1,\alpha_1\}} \right\}.$$

Example: The intersection of given two matrices is obtained as follows:

$$C^1_{2\times 2} = \begin{bmatrix} 0.6e^{i0.3} & 0.1e^{i0.7} \\ 0.2e^{i0.1} & 0.5e^{i0.4} \end{bmatrix} \ \& \ C^2_{2\times 2} = \begin{bmatrix} 0.5e^{i0.1} & 0.4e^{i0.3} \\ 0.8e^{i0.6} & 0.7e^{i0.2} \end{bmatrix}$$

Then

$$C^1_{2\times 2} \cap C^2_{2\times 2} = \begin{bmatrix} 0.5e^{i0.3} & 0.1e^{i0.7} \\ 0.2e^{i0.6} & 0.5e^{i0.4} \end{bmatrix}.$$

- **Commutativity, Associativity and Distributivity Properties of CFMs:**

Theorem 12 *Suppose that there are three CFMSs, say, P, Q and R, then the union operation of CFMs is commutative and associative and distributive over intersection.*

(i) $P \cup Q = Q \cup P$ *(Commutative law)*

(ii) $P \cup (Q \cup R) = (P \cup Q) \cup R$ *(Associative law)*

(iii) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$ *(Distributive law)*

Proof: Consider the following three complex fuzzy matrices:

$$P = \begin{bmatrix} a_{11}e^{i\theta_1} & a_{12}e^{i\theta_2} \\ a_{21}e^{i\theta_3} & a_{22}e^{i\theta_4} \end{bmatrix}, Q = \begin{bmatrix} b_{11}e^{i\alpha_1} & b_{12}e^{i\alpha_2} \\ b_{21}e^{i\alpha_3} & b_{22}e^{i\alpha_4} \end{bmatrix} \ \& \ R = \begin{bmatrix} g_{11}e^{i\gamma_1} & g_{12}e^{i\gamma_2} \\ g_{21}e^{i\gamma_3} & g_{22}e^{i\gamma_4} \end{bmatrix}$$

Commutative Property:

$$P \cup Q = \begin{bmatrix} \max(a_{11}, b_{11})e^{i\min(\theta_1,\alpha_1)} & \max(a_{12}, b_{12})e^{i\min(\theta_2,\alpha_2)} \\ \max(a_{21}, b_{21})e^{i\min(\theta_3,\alpha_3)} & \max(a_{22}, b_{22})e^{i\min(\theta_4,\alpha_4)} \end{bmatrix} = Q \cup P.$$

Associative Property:

$$P \cup (Q \cup R) = \begin{bmatrix} \check{e}_{11} & \check{e}_{12} \\ \check{e}_{21} & \check{e}_{22} \end{bmatrix} = (P \cup Q) \cup R;$$

where

$$\check{e}_{11} = \max(a_{11},\ b_{11},\ g_{11})e^{i\min(\theta_1,\alpha_1,\ \gamma_1)};$$

$$\check{e}_{12} = \max(a_{12},\ b_{12},\ g_{12})e^{i\min(\theta_2,\alpha_2,\ \gamma_2)};$$

$$\check{e}_{21} = \max(a_{21},\ b_{21},\ g_{21})e^{i\min(\theta_3,\alpha_3,\ \gamma_3)};$$

$$\check{e}_{22} = \max(a_{22},\ b_{22},\ g_{22})e^{i\min(\theta_4,\alpha_4,\ \gamma_4)}.$$

Distributive Property:

$$P \cup (Q \cap R) = P \cup \check{Q};$$

where

$$\check{Q} = \begin{bmatrix} \check{\check{e}}_{11} & \check{\check{e}}_{12} \\ \check{\check{e}}_{21} & \check{\check{e}}_{22} \end{bmatrix};$$

and

$$\check{\check{e}}_{11} = \min(\ b_{11},\ g_{11})e^{i\max(\alpha_1,\ \gamma_1)};$$

$$\check{\check{e}}_{12} = \min(b_{12},\ g_{12})e^{i\max(\alpha_2,\ \gamma_2)};$$

$$\check{\check{e}}_{21} = \min(\ b_{21},\ g_{21})e^{i\max(\alpha_3,\ \gamma_3)};$$

$$\check{\check{e}}_{22} = \min(b_{22},\ g_{22})e^{i\max(\alpha_4,\ \gamma_4)}.$$

Next, remaining part of the equation is calculated that is

$$P \cup \check{Q} = \begin{bmatrix} \check{\check{p}}_{11} & \check{\check{p}}_{12} \\ \check{\check{p}}_{21} & \check{\check{p}}_{22} \end{bmatrix};$$

where

$$\check{\check{p}}_{11} = a_{11}e^{i\theta_1} \cup \check{\check{e}}_{11};$$

$$\check{\check{p}}_{12} = a_{12}e^{i\theta_2} \cup \check{\check{e}}_{12};$$

$$\check{\check{p}}_{21} = a_{21}e^{i\theta_3} \cup \check{\check{e}}_{21};$$

$$\check{\check{p}}_{22} = a_{22}e^{i\theta_4} \cup \check{\check{e}}_{22}.$$

Similarly, we will obtain the right-hand side of the identity and after calculation, it is observed that the desired values are obtained. In this manner, the distributive property is satisfied.

Now, the above three identities are being illustrated with the help of numerical examples for better understanding of the concept.

Suppose the matrices are of following form:

$$P = \begin{bmatrix} 0.5e^{i0.7} & 0.3e^{i0.1} \\ 06e^{i0.3} & 02e^{i0.5} \end{bmatrix}, \; Q = \begin{bmatrix} 0.7e^{i0.1} & 0.5e^{i0.3} \\ 02e^{i0.4} & 09e^{i0.2} \end{bmatrix} \,\&\, R = \begin{bmatrix} 0.4e^{i0.3} & 0.3e^{i0.5} \\ 06e^{i0.1} & 07e^{i0.2} \end{bmatrix}.$$

Commutative law:

$$P \cup Q = \begin{bmatrix} 0.7e^{i0.1} & 0.5e^{i0.1} \\ 0.6e^{i0.3} & 0.9e^{i0.2} \end{bmatrix} = Q \cup P.$$

Associative law:

$$(P \cup Q) \cup R = \begin{bmatrix} 0.7e^{i0.1} & 0.5e^{i0.1} \\ 0.6e^{i0.3} & 0.9e^{i0.2} \end{bmatrix} \cup \begin{bmatrix} 0.4e^{i0.3} & 0.3e^{i0.5} \\ 0.6e^{i0.1} & 0.7e^{i0.2} \end{bmatrix} = \begin{bmatrix} 0.7e^{i0.1} & 0.5e^{i0.1} \\ 0.6e^{i0.1} & 0.9e^{i0.2} \end{bmatrix}.$$

$$(P \cup Q) \cup R = \begin{bmatrix} 0.7e^{i0.1} & 0.5e^{i0.1} \\ 0.6e^{i0.3} & 0.9e^{i0.2} \end{bmatrix} \cup \begin{bmatrix} 0.4e^{i0.3} & 0.3e^{i0.5} \\ 0.6e^{i0.1} & 0.7e^{i0.2} \end{bmatrix} = \begin{bmatrix} 0.7e^{i0.1} & 0.5e^{i0.1} \\ 0.6e^{i0.1} & 0.9e^{i0.2} \end{bmatrix}.$$

$$P \cup (Q \cup R) = (P \cup Q) \cup R.$$

Distributive law:

$$P \cup (Q \cap R) = \begin{bmatrix} 0.5\ e^{i0.7} & 0.3e^{i0.1} \\ 0.6e^{i0.3} & 0.2e^{i0.5} \end{bmatrix} \cup \begin{bmatrix} 0.4e^{i0.3} & 0.3e^{i0.5} \\ 0.2e^{i0.4} & 0.7e^{i0.2} \end{bmatrix} = \begin{bmatrix} 0.5e^{i0.3} & 0.3e^{i0.1} \\ 0.6e^{i0.3} & 0.7e^{i0.2} \end{bmatrix}.$$

$$(P \cup Q) \cap (P \cup R) = \begin{bmatrix} 0.7e^{i0.1} & 0.5e^{i0.1} \\ 0.6e^{i0.3} & 0.9e^{i0.2} \end{bmatrix} \cap \begin{bmatrix} 0.5e^{i0.3} & 0.3e^{i0.1} \\ 0.6e^{i0.1} & 0.7e^{i0.2} \end{bmatrix} = \begin{bmatrix} 0.5e^{i0.3} & 0.3e^{i0.1} \\ 0.6e^{i0.3} & 0.7e^{i0.2} \end{bmatrix}.$$

3.10 Similarity Measure for Complex Fuzzy Matrix

In this section, we have proposed a new similarity measure for the complex fuzzy matrix and studied its computational feature with the help of a suitable numerical example. In literature, it may be noted that the following necessary conditions for the proposed similarity measure must be satisfied:

Definition 42 *A real valued mapping:* $\hat{S} : P \times Q \to [0,1]$ *is known as a similarity measure between two complex fuzzy matrices* $P = \mu_P(x_{ij}) = r_P(x_{ij})\, e^{i\omega_P(x_{ij})}$ *and* $Q = \mu_Q(x_{ij}) = r_Q(x_{ij})\, e^{i\omega_Q(x_{ij})}$*, if* $\hat{S}$ *satisfies the following axioms:*

(i) $\hat{S}(P,Q) = \hat{S}(Q,P)$;

(ii) $\hat{S}(P,Q) = 1 \Longleftrightarrow (P,Q) = (Q,P)$;

(iii) $\hat{S}(P,Q) = 0 \Longleftrightarrow x_{ij} \in U$,
where $r_P(x_{ij}) = 1, \quad r_Q(x_{ij}) = 0$ *or* $r_P(x_{ij}) = 0, \quad r_Q(x_{ij}) = 1$ *and*
$\omega_P(x_{ij}) = 2\pi, \quad \omega_Q(x_{ij}) = 0$ *or* $\omega_P(x_{ij}) = 0, \quad \omega_Q(x_{ij}) = 2\pi$

(iv) For three modified complex fuzzy matrices P*,* Q *and* R *subject to* $P \subseteq Q \subseteq R$*,*

then, $\hat{S}(P,Q) \leq \hat{S}(P,R)$ *or* $\hat{S}(P,Q) \leq \hat{S}(R,Q)$.

Further, we propose a new similarity measure for the complex fuzzy matrices which is supposed to be very helpful in obtaining the solutions to various decision making problems as follows:

Definition 43 *Suppose there are two complex fuzzy matrices* P *and* Q *on the universe of discourse* U*. The complex form of* P *and* Q *can be written as follows:* $P = \mu_P(x_{ij}) = r_P(x_{ij})\, e^{i\omega_P(x_{ij})}, Q = \mu_Q(x_{ij}) = r_Q(x_{ij})\, e^{i\omega_Q(x_{ij})}$*. The similarity measure of two CFMs* P *and* Q*, denoted by* $\hat{S}(P,Q)$*, is defined as follows:*

$$\hat{S}(P,Q) = \frac{1}{2mn} \sum_{j=1}^{n} \sum_{i=1}^{m} \left(\frac{|P \cap Q|}{|P \cup Q|} \left[\hat{S}^r(P,Q) + \frac{\hat{S}^\omega(P,Q)}{2\pi} \right] \right); \tag{3.10.1}$$

where

$$\hat{S}^r(P,Q) = 1 - \sum_{j=1}^{n} \sum_{i=1}^{m} \max\left(|r_P(x_{ij}) - r_Q(x_{ij})| \right);$$

$$\hat{S}^\omega(P,Q) = 2\pi - \sum_{j=1}^{n} \sum_{i=1}^{m} \max\left(|\omega_P(x_{ij}) - \omega_Q(x_{ij})| \right).$$

Theorem 13 *The proposed similarity measure* $\hat{S}(P,Q)$ *given by equation (3.10.1) is a valid similarity measure.*

Proof: In view of the axioms listed in Definition 4.1 and also to validate the proposed similarity measure, we prove the axioms one by one below:

(i) $|P \cap Q| = |Q \cap P|$ & $|P \cup Q| = |Q \cup P|$.

$$\hat{S}^r(P,Q) = 1 - \sum_{j=1}^{n}\sum_{i=1}^{m} \max\left(\left|r_P\left(x_{ij}\right) - r_Q\left(x_{ij}\right)\right|\right).$$

$$= 1 - \sum_{j=1}^{n}\sum_{i=1}^{m} \max\left(\left|r_Q\left(x_{ij}\right) - r_P\left(x_{ij}\right)\right|\right) = \hat{S}^r(Q,P).$$

Similarly, $\hat{S}^\omega(P,Q) = \hat{S}^\omega(Q,P)$.

$\implies \hat{S}\left(P,Q\right) = \hat{S}\left(Q,P\right).$

(ii) Let $P = Q$. Then,

$$\frac{|P \cap Q|}{|P \cup Q|} = 1.$$

$$\hat{S}^r(P,Q) = 1 - \sum_{j=1}^{n}\sum_{i=1}^{m} \max\left(\left|r_P\left(x_{ij}\right) - r_P\left(x_{ij}\right)\right|\right) = 1.$$

$$\hat{S}^\omega(P,Q) = 1 - \sum_{j=1}^{n}\sum_{i=1}^{m} \max\left(\left|\omega_P\left(x_{ij}\right) - \omega_P\left(x_{ij}\right)\right|\right) = 2\pi.$$

Substituting all the values in the proposed similarity measure. Then,

$\hat{S}\left(P,Q\right) = 1.$

(iii) Substitute $r_P\left(x_{ij}\right) = 1, \quad r_Q\left(x_{ij}\right) = 0$ and $\omega_P\left(x_{ij}\right) = 2\pi, \quad \omega_Q\left(x_{ij}\right) = 0$. Then,

$$\hat{S}^r(P,Q) = \hat{S}^\omega(P,Q) = 0.$$

$$\implies \hat{S}\left(P,Q\right) = 0.$$

(iv) When $P \subseteq Q \subseteq R$.

Then, $r_P\left(x_{ij}\right) \le r_Q\left(x_{ij}\right) \le r_R\left(x_{ij}\right)$ and $\omega_P\left(x_{ij}\right) \le \omega_Q\left(x_{ij}\right) \le \omega_R\left(x_{ij}\right).$

$$\implies \max\left(\left|r_P\left(x_{ij}\right) - r_R\left(x_{ij}\right)\right|\right) \le \max\left(\left|r_P\left(x_{ij}\right) - r_Q\left(x_{ij}\right)\right|\right).$$

$$\implies \max\left(\left|\omega_P\left(x_{ij}\right) - \omega_R\left(x_{ij}\right)\right|\right) \le \max\left(\left|\omega_P\left(x_{ij}\right) - \omega_Q\left(x_{ij}\right)\right|\right).$$

$$\implies \hat{S}\left(P,Q\right) \le \hat{S}\left(P,R\right).$$

Hence, all the axioms for similarity measure is satisfied.

Example: Suppose P and Q be two complex fuzzy matrices defined as :

$$P_{2\times 2} = \begin{bmatrix} 0.6e^{i0.3\pi} & 0.1e^{i0.7\pi} \\ 0.2e^{i0.1\pi} & 0.5e^{i0.4\pi} \end{bmatrix} \ \& \ Q_{2\times 2} = \begin{bmatrix} 0.5e^{i0.1\pi} & 0.4e^{i0.3\pi} \\ 0.8e^{i0.6\pi} & 0.7e^{i0.2\pi} \end{bmatrix}$$

Then

$$\hat{S}\left(P,Q\right) = \frac{1}{2\times 2\times 2}\sum_{j=1}^{2}\sum_{i=1}^{2}\left(\frac{|P\cap Q|}{|P\cup Q|}\left[\hat{S}^{r}(P,Q)+\frac{\hat{S}^{\omega}(P,Q)}{2\pi}\right]\right)$$

$$= \frac{1}{8}\sum_{j=1}^{2}\sum_{i=1}^{2}\left(\frac{|r_P(x_{ij})\exp^{i\omega_P(x_{ij})}\cap r_Q(x_{ij})\exp^{i\omega_Q(x_{ij})}|}{|r_P(x_{ij})\exp^{i\omega_P(x_{ij})}\cup r_Q(x_{ij})\exp^{i\omega_Q(x_{ij})}|}\left(\hat{S}^{r}(P(x_{ij}),Q(x_{ij}))+\hat{S}^{\omega}(P(x_{ij}),Q(x_{ij})\right)\right)$$

$$= \frac{1}{8}\{\frac{|r_P(x_{11})\exp^{i\omega_P(x_{11})}\cap r_Q(x_{11})\exp^{i\omega_Q(x_{11})}|}{|r_P(x_{11})\exp^{i\omega_P(x_{11})}\cup r_Q(x_{11})\exp^{i\omega_Q(x_{11})}|}\left(\hat{S}^{r}(P(x_{11}),Q(x_{11}))+\hat{S}^{\omega}(P(x_{11}),Q(x_{11})\right)+$$

$$\frac{|r_P(x_{12})\exp^{i\omega_P(x_{12})}\cap r_Q(x_{12})\exp^{i\omega_Q(x_{12})}|}{|r_P(x_{12})\exp^{i\omega_P(x_{12})}\cup r_Q(x_{12})\exp^{i\omega_Q(x_{12})}|}\left(\hat{S}^{r}(P(x_{12}),Q(x_{12}))+\hat{S}^{\omega}(P(x_{12}),Q(x_{12})\right)+$$

$$\frac{|r_P(x_{21})\exp^{i\omega_P(x_{21})}\cap r_Q(x_{21})\exp^{i\omega_Q(x_{21})}|}{|r_P(x_{21})\exp^{i\omega_P(x_{21})}\cup r_Q(x_{21})\exp^{i\omega_Q(x_{21})}|}\left(\hat{S}^{r}(P(x_{21}),Q(x_{ij}))+\hat{S}^{\omega}(P(x_{21}),Q(x_{21})\right)+$$

$$\frac{|r_P(x_{22})\exp^{i\omega_P(x_{22})}\cap r_Q(x_{22})\exp^{i\omega_Q(x_{22})}|}{|r_P(x_{22})\exp^{i\omega_P(x_{22})}\cup r_Q(x_{22})\exp^{i\omega_Q(x_{22})}|}\left(\hat{S}^{r}(P(x_{22}),Q(x_{22}))+\hat{S}^{\omega}(P(x_{22}),Q(x_{22})\right)\}$$

Thus, the value of the proposed similarity measure has been computed to be

$$\hat{S}\left(P,Q\right) = 0.3647$$

.

3.11 Application of Complex Fuzzy Matrix in the Identification of the Signal

In this segment, we have used the concept of the complex fuzzy matrix in detecting the appropriate signal among the various signals transmitted by the transmitter. The methodology used to detect the reference signal is explained below and the applicability of the following methodology is described with the help of an example.

Methodology

Step1. Suppose p $(S_1(x),\ S_2(x),\ S_3(x), \ldots,\ S_p(x))$ number of signals are sent by the transmitter, then each of the p signals are sampled Q times by the receiver. Then, the appropriate signal $S_l(x)$ $(l\ varies\ from\ 1\ to\ p)$ is selected with the help of reference signal R, whose value is already known. Let both the signals $S_l(x)$ and R are considered Q times. The absolute value of each j-th signal i.e., $S_j(u)$ $(1 \leq u \leq p)$ in terms of discrete complex fuzzy transform is given by $x_{j,S} = \varepsilon_{j,S} e^{i\delta_{j,S}}$, $(\varepsilon_{j,S}, \delta_{j,S} \in \mathbb{R}\ \&\ \varepsilon_{j,S} \geq 1 \forall\ S\ (1 \leq S \leq p))$, where $x_{j,S}$ is the complex Fourier coefficients of signals $S_j(u)$.

Step 2. Now the signals are expressed in form of matrix and is given by $E_{m\times n} = [|S_j(u)|]_{Q\times p}$, where signals are denoted by the column of the matrix and Q samples of each signal is considered.

$$E = \begin{bmatrix} |S_1(1)| & |S_2(Q)| & \ldots & |S_p(1)| \\ |S_1(2)| & |S_2(Q)| & \ldots & |S_p(2)| \\ . & . & \ldots & . \\ |S_1(Q)| & |S_2(Q)| & \ldots & |S_p(Q)| \end{bmatrix}$$

Step 3. Similarly, the second matrix is given by $F_{m\times n} = \left[\left|S'_j(u)\right|\right]_{Q\times p}$

$$F = \begin{bmatrix} |S'_1(1)| & |S'_2(Q)| & \ldots & \left|S'_p(1)\right| \\ |S'_1(2)| & |S'_2(Q)| & \ldots & \left|S'_p(2)\right| \\ . & . & \ldots & . \\ |S'_1(Q)| & |S'_2(Q)| & \ldots & \left|S'_p(Q)\right| \end{bmatrix}$$

Step 4. Next, the product of the above two matrices $(E\ \&\ F)$ is obtained.

Step 5. Now, the complex fuzzy max-min decision matrix is obtained.

Step 6. Finally, the optimal fuzzy set is obtained.

In order to have a summarized overall view of the proposed methodology, we present the procedural steps in the form of the following Figure 3.1 given below:

We utilize the notion of complex fuzzy matrix and the proposed methodology given above in finding the reference signals among the five signals obtained by the receiver. Assume that there is a set of five signals $S = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\}$ and every signal is sampled

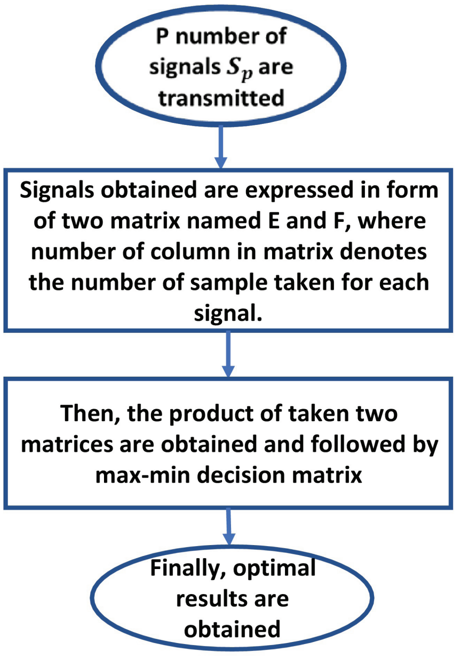

Figure 3.1: Procedural Steps of Proposed Methodology

five times each. Let R denotes the reference signal from which each of the signals is accordingly compared to obtain the high degree of resemblance between the signal and the reference signal.

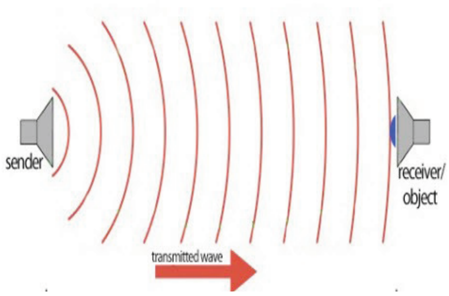

Figure 3.2: Signal Transfer from Transmitter to Receiver

According to the step 2, the matrix E and second matrix F are obtained. Both the obtained matrices are given below.

$$E = \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.1 & 0.2 \\ 0.5 & 0.4 & 0.4 & 0.1 & 0.2 \\ 0.4 & 0.2 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.3 & 0.2 & 0.4 \\ 0.1 & 0.2 & 0.1 & 0.1 & 0.3 \end{pmatrix} \quad \& \quad F = \begin{pmatrix} 0.2 & 0.1 & 0.3 & 0.1 & 0.1 \\ 0.4 & 0.2 & 0.3 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.5 & 0.3 & 0.5 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.4 & 0.3 \end{pmatrix}$$

Next, according to step 4 the following matrix is obtained:

$$E \times F = \begin{pmatrix} 0.35 & 0.22 & 0.26 & 0.28 & 0.27 \\ 0.47 & 0.28 & 0.42 & 0.34 & 0.35 \\ 0.32 & 0.25 & 0.34 & 0.34 & 0.31 \\ 0.38 & 0.28 & 0.35 & 0.4 & 0.37 \\ 0.21 & 0.15 & 0.2 & 0.24 & 0.22 \end{pmatrix}$$

Now, as per given in step 5, the max-min of matrices is calculated as follows:

$$Mm\,[E \times F] = [D_{i1}] = (d_{i1})\,\forall i \in \{1, 2, 3, 4, 5\}$$

subject to

$$d_{11} = \min\{u_{11}, u_{21},\ u_{31},\ u_{41},\ u_{51}\};$$

where

$$u_{11} = 0.35, u_{21} = 0.47,\ u_{31} = 0.32,\ u_{41} = 0.38,\ u_{51} = 0.21.$$

Then,

$$d_{11} = \min\{0.35, 0.47,\ 0.32,\ 0.38,\ 0.21\} = 0.21.$$

Similarly,

$$d_{21} = \min\{0.22, 0.28,\ 0.25,\ 0.28,\ 0.15\} = 0.15.$$

$$d_{31} = \min\{0.26, 0.42,\ 0.34,\ 0.35,\ 0.2\} = 0.2.$$

$$d_{41} = \min\{0.28, 0.34,\ 0.34,\ 0.4,\ 0.24\} = 0.24.$$

$$d_{51} = \min\{0.27, 0.35,\ 0.31,\ 0.37,\ 0.22\} = 0.22.$$

Finally, we obtain the min-max decision matrix which is given below:

$$mM\,(E \times F) = \begin{pmatrix} d_{11} = 0.21 \\ d_{21} = 0.15 \\ d_{31} = 0.2 \\ d_{41} = 0.24 \\ d_{51} = 0.22 \end{pmatrix}$$

In the end, the optimum fuzzy set is obtained $[S]$

$$optMm\left(E \times F\right)\left(\alpha\right) = \left\{\frac{0.21}{\alpha_1}, \frac{0.15}{\alpha_2}, \frac{0.2}{\alpha_3}, \frac{0.24}{\alpha_4}, \frac{0.22}{\alpha_5}\right\}.$$

Hence, α_4 is the signal.

In this manner, the reference signal is identified among the various number of signals obtained by the receiver. This is also validates the proposed methodology.

3.12 Conclusions

In the current study, a novel concept of complex neutrosophic matrices is presented and explained with the help of a few algebraic operations and properties, which will be of great help for the researchers to understand the basics of the concept. The norm and power convergence of the complex neutrosophic matrix have been discussed thoroughly. Further, new similarity measures have been proposed and the property of positive definiteness for the proposed measures has been studied. Later, applicability of the proposed theory has been presented in case of medical diagnosis for better clarity. Various set-theoretic properties of the fundamental operations related to commutativity, associativity and distributivity in case of complex fuzzy matrix have been established. Some suitable numerical examples to illustrate these computations have also been included. A new similarity measure for the complex fuzzy matrix has been proposed with the proof of its validity. The proposed methodology has been duly implemented in the process of identification of reference signal from a set of signals transmitted from the transmitter.

Chapter 4

Information Measures of Neutrosophic Sets

In this chapter, we have proposed some new exponential similarity measures with proof of their validity and also presented several counter-intuitive cases to show the efficacy of the exponential measures. In order to show the applicability of the exponential similarity measures, we have presented two illustrative examples - one related to the classification problem (pattern recognition) and other related to the evaluation problem of decision-making . In add ition to this, some important comparative remarks have been enumerated. We have also proposed entropy for the single valued neutrosophic information measure. The 'useful' divergence of neutrosophic information measure is described. The concept of hybrid ambiguity 'useful' measure is also defined. The 'useful' neutrosophic information improvement measure of neutrosophic measure is explained.

4.1 Similarity Measure of Neutrosophic Sets

Now, the new similarity measures under neutrosophic environment have been presented below:
Let U be the universe of discourse.

Definition 44 *Consider* $P = \{(T_P(u_i),\ I_P(u_i),\ F_P(u_i))|u_i \epsilon U\}$ *and* $Q = \{(T_Q(u_i),\ I_Q(u_i),\ F_Q(u_i))|u_i \epsilon U,\ i = 1, 2, \ldots, n\}$ *be two valued neutrosophic sets, then the* *similarity measure $SM_1(P,\ Q)$ between P and Q is defined as:*

$$SM_1\left(P,\ Q\right)=\frac{1}{n}\sum_{i=1}^{n}\left(SM_i^T\left(u_i\right)\times SM_i^I\left(u_i\right)\times SM_i^F\left(u_i\right)\right);$$

$$SM_1^w\left(P,\ Q\right)=\sum_{i=1}^{n}w_i\times\left(SM_i^T\left(u_i\right)\times SM_i^I\left(u_i\right)\times SM_i^F\left(u_i\right)\right);$$

$$SM_2\left(P,\ Q\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{SM_i^T\left(u_i\right)+SM_i^I\left(u_i\right)+SM_i^F\left(u_i\right)}{3}\right);$$

$$SM_2^w\left(P,\ Q\right)=\sum_{i=1}^{n}w_i\times\left(\frac{SM_i^T\left(u_i\right)+SM_i^I\left(u_i\right)+SM_i^F\left(u_i\right)}{3}\right);$$

where

$$SM_i^T\left(u_i\right)=e^{-\left|T_P(u_i)-T_Q(u_i)\right|};$$

$$SM_i^I\left(u_i\right)=e^{-\left|I_P(u_i)-I_Q(u_i)\right|}$$

&

$$SM_i^F\left(u_i\right)=e^{-\left|F_P(u_i)-F_Q(u_i)\right|}.$$

Theorem 14 *The measure proposed in Definition 44 is a valid similarity measure.*

Proof: For this, we need to show that the similarity measure $SM_1(P,Q)$ between two neutrosophic sets P and Q holds the conditions as defined in Definition 44.

- We know that $T_P\left(u_i\right), T_Q\left(u_i\right)\leq 1$, which implies $\left|T_P\left(u_i\right)-T_Q\left(u_i\right)\right|\leq 1$. This can also be written as $-1\leq\left|T_P\left(u_i\right)-T_Q\left(u_i\right)\right|\leq 0$.
  Hence,
  $0\leq e^{-\left|T_P(u_i)-T_Q(u_i)\right|}\leq 1\Rightarrow 0\leq SM_i^T\left(u_i\right)\leq 1$. Also $0\leq SM_i^I\left(u_i\right), SM_i^F\left(u_i\right)\leq 1$.
  Therefore, from equation given in 44 we conclude that $0\leq SM_1\left(P,Q\right)\leq 1$.

- We know that $SM_i^T\left(u_i\right)=1,\ \ SM_i^I\left(u_i\right)=1$ and $SM_i^F\left(u_i\right)=1$ if and only if $P=Q$, so we have $SM_1\left(P,Q\right)=1\Longleftrightarrow P=Q$.

- As $SM_i^T\left(u_i\right),\ SM_i^I\left(u_i\right),\ SM_i^F\left(u_i\right)$ are symmetric for neutrosophic sets. Hence, we observe that $SM_1\left(P,Q\right)=SM_1\left(Q,P\right)$.

- If $P \subseteq Q \subseteq O$, then for $u_i \in U$ we have,

$$0 \leq T_P(u_i) \leq T_Q(u_i) \leq T_O(u_i) \leq 1;$$

$$0 \geq I_P(u_i) \geq I_Q(u_i) \geq I_O(u_i) \geq 1;$$

and

$$0 \leq F_P(u_i) \leq F_Q(u_i) \leq F_O(u_i) \leq 1.$$

It means that

$$-|T_P(u_i) - T_Q(u_i)| \leq \min\{|T_P(u_i) - T_Q(u_i)|, |T_Q(u_i) - T_O(u_i)|\};$$

$$-|I_P(u_i) - I_Q(u_i)| \leq \min\{|I_P(u_i) - I_Q(u_i)|, |I_Q(u_i) - I_O(u_i)|\};$$

and

$$-|F_P(u_i) - F_Q(u_i)| \leq \min\{|F_P(u_i) - F_Q(u_i)|, |F_Q(u_i) - F_O(u_i)|\};$$

This implies that

$$SM_i^T(P,Q) \leq \min\left\{SM_i^T(P,Q), SM_i^T(Q,O)\right\};$$

$$SM_i^I(P,Q) \leq \min\left\{SM_i^I(P,Q), SM_i^I(Q,O)\right\};$$

and

$$SM_i^F(P,Q) \leq \min\left\{SM_i^F(P,Q), SM_i^F(Q,O)\right\}.$$

Thus, based on this, equation in definition 44 becomes $SM_1(P,Q) \leq SM_1(P,Q)$ and $SM_1(P,Q) \leq SM_1(Q,O)$.
Hence, the proposed measure in the Definition 44 is the valid similarity measure over two neutrosophic sets.

Theorem 15 *The measure proposed in the Definition 44 is a valid similarity measure.*

Proof: For this, we need to show the similarity measure $SM_1(P,Q)$ between two neutrosophic sets P and Q holds the conditions defined in Definition 44.
We know that $T_P(u_i)$, $T_Q(u_i) \leq 1$, which implies $|T_P(u_i) - T_Q(u_i)| \leq 1$. This can also be written as

$$-1 \leq |T_P(u_i) - T_Q(u_i)| \leq 0.$$

Hence, $0 \leq e^{-\left|T_P(u_i)-T_Q(u_i)\right|} \leq 1 \Rightarrow 0 \leq SM_i^T(u_i) \leq 1$. Also, $0 \leq SM_i^I(u_i), SM_i^F(u_i) \leq 1$. Therefore, from equation in defintion44 we conclude that

$$0 \leq SM_1^w(P,Q) \leq \sum_{i=1}^{n} w_i = 1.$$

We know that $SM_i^T(u_i) = 1$, $SM_i^I(u_i) = 1$ and $SM_i^F(u_i) = 1$ if only if $P = Q$ because, $\sum_{i=1}^{n} w_i = 1$, so we have , $SM_1^w(P,\ Q) = 1 \Longleftrightarrow P = Q$.
As $SM_i^T(u_i)$, $SM_i^I(u_i)$, $SM_i^F(u_i)$ are symmetric for neutrosophic sets. Hence, we observe that

$$SM_1^w(P,\ Q) = SM_1(Q,\ P).$$

For $P \subseteq Q \subseteq O$ and $u_i \in U$, we have

$$SM_i^T(P,\ Q) \leq \min\left\{SM_i^T(P,\ Q), SM_i^T(Q,\ O)\right\};$$

$$SM_i^I(P,\ Q) \leq \min\left\{SM_i^I(P,\ Q), SM_i^I(Q,\ O)\right\};$$

and

$$SM_i^F(P,\ Q) \leq \min\left\{SM_i^F(P,\ Q), SM_i^F(Q,\ O)\right\}.$$

Thus, based on this, equation in definition 44 becomes $SM_1^w(P,\ Q) \leq SM_1^w(P,\ Q)$ and $SM_1^w(P,\ Q) \leq SM_1^w(Q,O)$.
Hence, the proposed measure in the Definition 44 is the valid similarity measure over two neutrosophic sets.

Comparison with Existing Similarity Measures

In order to show the effectiveness, performance and advantages of the proposed similarity measures, we present the following comparative analysis with existing measures presented by equations given in definition 44.
Thus, to carry out the comparison of the proposed similarity measures with the existing ones in the literature, we consider five different cases consisting of two neutrosophic sets as follows:

Case 1: $A = \{0.2, 0.3, 0.4\}$ & $B = \{0.2,\ 0.3,\ 0.4\}$
Case 2: $A = \{0.3, 0.2, 0.4\}$ & $B = \{0.4,\ 0.2,\ 0.3\}$
Case 3: $A = \{1, 0.0, 0.0\}$ & $B = \{0.0,\ 1,\ 1\}$
Case 4: $A = \{1, 0.0, 0.0\}$ & $B = \{0.0,\ 0.0,\ 0.0\}$
Case 5: $A = \{0.4, 0.2, 0.6\}$ & $B = \{0.2,\ 0.1,\ 0.3\}$

Based on the computational analysis, the values obtained by the proposed similarity measures and existing similarity measures for each case have been tabulated in the Table 4.1.

Table 4.1: Comparison of Proposed Similarity Measure with Existing Ones

	Case 1	Case 2	Case 3	Case 4	Case 5
SM_1	1	0.8187	0.0497	0.3678	0.5488
SM_2	1	0.978	0.3678	0.7892	0.8214
$S_i[28]$	1	0.93	0.0	0.0	0.666
$S_D[28]$	1	0.965	0.0	0.0	0.8
$S_C[28]$	1	0.965	0.0	Null	1
$S_T[28]$	1	-2.10	0.954	0.984	0.259

In view of the computed values obtained by the different measures, we can conclude that the proposed similarity measures are quite effective and give distinguished result whereas the existing ones are not able to perform good in some cases (indicated by the bold values).
Remark : Null represents the case when the degree of similarity can not be computed due to the problem division by zero.

4.2 Applications of Neutrosophic Similarity Measures

4.2.1 Classification Problem

Consider a standard classification problem where we have m different classes (say) $C_1,\ C_2,\ C_3,\ldots,C_m$ of known patterns over the universe of discourse $U=\{u_1,u_2,u_3,\ldots,u_n\}$. Suppose we choose one sample (say) $P_1,\ P_2,\ P_3,\ldots,P_m$ from each class and have an unknown sample Q where the information in each known and unknown pattern is featured under the neutrosophic environment. Thus, our main objective is to classify the unknown sample into one of the known classes. In order to solve this classification problem, we calculate the similarity measure of unknown sample Q with each known pattern $P_i(i=1,\ 2,\ 3,\ldots,m)$ and then allocate the unknown sample to one of the classes which has highest similarity index.
Example: Let us consider three existing patterns $P_1,\ P_2$ and P_3 being described by the neutrosophic sets in $U=\{u_1,\ \ u_2,\ u_3\}$ as following:

$$P_1=\{(u_1,\ 0.5,\ 0.4,0.2),\ (u_2,\ 0.4,\ 0.3,\ 0.4),\ (u_3,\ 0.4,\ 0.5,\ 0.1)\};$$

$$P_2 = \{(u_1,\ 0.6,\ 0.5, 0.1),\ (u_2,\ 0.5,\ 0.1,\ 0.3),\ (u_3,\ 0.5,\ 0.5,\ 0.1)\};$$

$$P_3 = \{(u_1,\ 0.4,\ 0.4, 0.2),\ (u_2,\ 0.4,\ 0.5,\ 0.2),\ (u_3,\ 0.3,\ 0.3,\ 0.4)\};$$

Let us take an unknown pattern Q given by

$$Q = \{(u_1,\ 0.4,\ 0.4, 0.2),\ (u_2,\ 0.5,\ 0.6,\ 0.1),\ (u_3,\ 0.3,\ 0.4,\ 0.4)\}.$$

Now, the main objective of the problem is to find the class to which Q belongs.
We present the computational procedure of solving the classification problem under consideration with the help of following Figure 4.1.

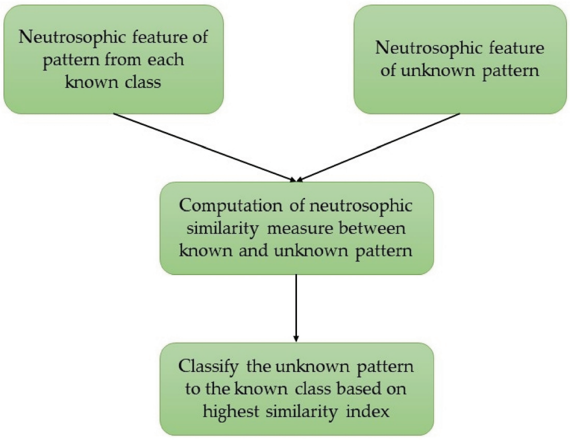

Figure 4.1: Computational Procedure for Classification Problem

With the help of proposed similarity measures given by equations in definition 44, and choosing the arbitrary weight vector **w=(0.3,0.4,0.3)** (may be selected on the decision makers choice) of the elements of U, we compute the desired values and tabulate them in Table 4.2.

Table 4.2: Computed Values of Similarity Measures

	$(\boldsymbol{P_1},\mathbf{Q})$	$(\boldsymbol{P_2},\mathbf{Q})$	$(\boldsymbol{P_3},\mathbf{Q})$
$\boldsymbol{SM_1}$	0.6725	0.5611	0.5322
$\boldsymbol{SM_1^w}$	0.6659	0.5656	0.5530
$\boldsymbol{SM_2}$	0.880	0.8226	0.804
$\boldsymbol{SM_2^w}$	0.876	0.824	0.814

Based on the obtained values in Table 4.2, we conclude that the unknown pattern Q belongs to the class P_1. The results obtained by utilizing the proposed similarity measures are certain

-ly found to be consistent with the results obtained in [101]. The values obtained are also more prominent and decisive in nature.

4.2.2 Evaluation Process in Decision Making

In view of the general format of a decision-making problem, we consider a set of available alternatives (say) $Z_1, Z_2, \ldots, Z_m$ and the set of criteria (say) $O_1,\ O_2, \ldots,\ O_n$. The main goal of the problem is to select the optimal and the best alternatives out of the m available alternatives with respect to n criteria. The procedure for ranking of the alternatives is based on transforming the neutrosophic decision matrix and computing the similarity index between the alternatives and the ideal solution which has been clearly represented with the he -lp of the following block diagram given in Figure 4.2:

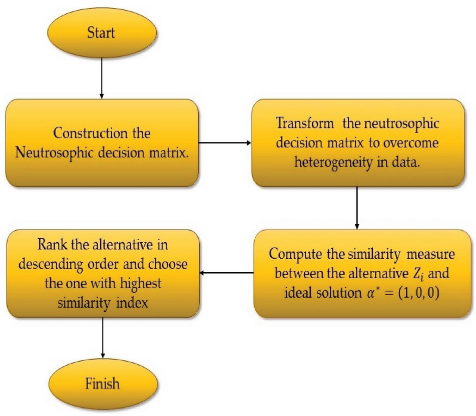

Figure 4.2: Ranking Procedure for Decision Making with Similarity Measures

Example: Consider there is a financial private limited firm whose objective is to invest a significant amount of money in the best possible sector. Suppose there are four possible investment sectors selected on the basis of an initial survey, say,
Z_1 : Automobile Sector,
Z_2 : Food & Beverages Service Sector,
Z_3 : Information Technology Sector,
Z_4: Ammunition Production Sector.

The investment company must take a decision according to the following three important criteria:

O_1 : Risk Factor,
O_2 : Growth Prospects,
O_3 : Ecological Impact.
Suppose that the management and the decision-makers assign suitable weights to each criteria based on their experience and risk bearing capability given by $w = (0.35, 0.25, 0.4)$. The necessary information has been taken from the experts/decision makers for the sake of evaluation of the alternatives $Z_i{\prime}s$ with respect to each criterion $O_j{\prime}s$.

The opinion values of each alternative with respect to each criteria have been expressed as a neutrosophic information, and the following neutrosophic decision matrix has been provided :

$$R = \begin{array}{c} \\ \mathcal{Z}_1 \\ \mathcal{Z}_2 \\ \mathcal{Z}_3 \\ \mathcal{Z}_4 \end{array} \begin{array}{c} \begin{array}{cc} \quad\mathcal{O}_1 & \qquad\qquad \mathcal{O}_2 \end{array} \\ \left(\begin{array}{cc} (0.4, 0.2, 0.3) & (0.4, 0.2, \\ (0.6, 0.1, 0.2) & (0.6, 0.1, \\ (0.3, 0.2, 0.3) & (0.5, 0.2, \\ (0.7, 0.0, 0.1) & (0.6, 0.1, \end{array} \right. \end{array}$$

The ideal solution in such decision-making problems can be as $\alpha^* = (1,\ 0,\ 0)$. However, it may be noted that the ideal solution generally does not exist in practice but a closer value is accepted. Our decision can be obtained by calculating the values proposed similarity measures between each alternative $Z_i(i = 1, 2, 3, 4)$ and the ideal solution α^*. In view of the procedure presented in Figure 4.2, these values have been computed and tabulated in the Table 4.3.

Table 4.3: Obtained Results Using the Proposed Similarity Measures

	SM_1	SM_1^w	SM_2	SM_2^w
(Z_1, α^*)	0.2962	0.2889	0.6768	0.6716
(Z_2, α^*)	0.4665	0.4605	0.7813	0.7779
(Z_3, α^*)	0.3456	0.3445	0.7098	0.7092
(Z_4, α^*)	0.6703	0.4919	0.7942	0.7892

On the basis of the computed values, the ranking order of the four alternatives in the above problem is

$$Z_4 > Z_2 > Z_3 > Z_1$$

Thus, we have that the alternative Z_4 is the best choice among all the alternatives. The results obtained by utilizing the proposed similarity measures are consistent with the results obtained by Ye [59] and Wang et al. [153].

4.3 Entropy of Single Valued Neutrosophic Information Measure

In this section of the study, the entropy of the single valued neutrosophic information measure is explained in detail with the help of theorem. The properties of single valued neutrosophic entropy are explained below.

Definition 45 *[33]"Consider M' to be a set of all SVNSs and $A \in M'$. Then, the entropy of A, denoted by $E_{M'}(A)$, satisfies*

1. $E_{M'}(A) = 0$ *iff* $\Gamma_A(y) = \Lambda_A(y) = \theta_A(y) = 0 \, or \, 1.$

2. $E_{M'}(A) = 1$, *when* $\Gamma_A(y) = \Lambda_A(y) = \theta_A(y) = 0.5.$

3. $E_{M'}(A_1) \geq E_{M'}(A_2)$ *if* $A_1 \subset A_2$ *i.e.,*
   $\Gamma_{A_1}(y_i) \leq \Gamma_{A_2}(y_i)$, $\Lambda_{A_1}(y_i) \geq \Lambda_{A_2}$ & $\theta_{A_1}(y_i) \leq \theta_{A_2}(y_i)$.

4. $E_{M'}(A) = E_{M'}(\tilde{A})$ *where $\tilde{A}$ is complement of A."*

Next, the entropy measure given by Luca et al.[1] is extended and modified to find the entropy for the single value neutrosophic information measure. This measure takes the following form,

$$\begin{aligned}
E_{M'_\Gamma}(A) &= \frac{1}{2n\log(0.5)}\sum_{i=1}^{n}[\Gamma_A(y_i)\log(\Gamma_A(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_A(y_i)))];\\
E_{M'_\Lambda}(A) &= \frac{1}{2n\log(0.5)}\sum_{i=1}^{n}[\Lambda_A(y_i)\log(\Lambda_A(y_i)) + (1-\Lambda_A(y_i))\log((1-\Lambda_A(y_i)))]; \quad (4.3.1)\\
E_{M'_\theta}(A) &= \frac{1}{2n\log(0.5)}\sum_{i=1}^{n}[\theta_A(y_i)\log(\theta_A(y_i)) + (1-\theta_A(y_i))\log((1-\theta_A(y_i)))].
\end{aligned}$$

Similarly, the'useful' single valued neutrosophic information measure is obtained with some modification in 'useful' fuzzy information measure given by Hooda and Bajaj [31] earlier in literature and this measure takes the following form for truth, falsity and neutral membership functions i.e.,

$$E_{M'_\Gamma}(A;P;U)=\frac{1}{2n\log(0.5)}\frac{\sum_{i=1}^{n}u_ip_i[\Gamma_A(y_i)\log(\Gamma_A(y_i))+(1-\Gamma_A(y_i))\log((1-\Gamma_A(y_i)))]}{\sum_{i=1}^{n}u_ip_i};$$
$$E_{M'_\Lambda}(A;P;U)=\frac{1}{2n\log(0.5)}\frac{\sum_{i=1}^{n}u_ip_i[\Lambda_A(y_i)\log(\Lambda_A(y_i))+(1-\Lambda_A(y_i))\log((1-\Lambda_A(y_i)))]}{\sum_{i=1}^{n}u_ip_i};$$
$$E_{M'_\theta}(A;P;U)=\frac{1}{2n\log(0.5)}\frac{\sum_{i=1}^{n}u_ip_i[\theta_A(y_i)\log(\theta_A(y_i))+(1-\theta_A(y_i))\log((1-\theta_A(y_i)))]}{\sum_{i=1}^{n}u_ip_i};\tag{4.3.2}$$

where $u_i>0$.

Theorem 16 *The measure 4.3.2 must satisfy the neutrosophic entropy measure properties given by 45.*

Proof: **<u>Axiom 1</u>**: $E_{M'_\Gamma}(A;P;U)=0$.
Then,

$$\frac{1}{2n\log(0.5)}\frac{\sum_{i=1}^{n}u_ip_i[\Gamma_A(y_i)\log(\Gamma_A(y_i))+(1-\Gamma_A(y_i))\log((1-\Gamma_A(y_i)))]}{\sum_{i=1}^{n}u_ip_i}=0.$$

$$\Gamma_A(y_i)\log(\Gamma_A(y_i))+(1-\Gamma_A(y_i))\log((1-\Gamma_A(y_i)))]=0.$$

Next, either $\Gamma_A(y_i)=0$ or $1\,\forall\, i=1,2,...,n$.
This proves that it satisfies the crisp set property.

<u>Axiom 2</u>: $\Gamma_A(y)=\Lambda_A(y)=\theta_A(y)=0.5$.
Putting this in equation 4.3.2, we get

$$\frac{1}{2n\log(0.5)}\frac{\sum_{i=1}^{n}u_ip_i[0.5\log(0.5)+(1-0.5)\log((1-0.5)]}{\sum_{i=1}^{n}u_ip_i}=1.$$

Hence, $E_{M'}(A)=1$, when $\Gamma_A(y)=\Lambda_A(y)=\theta_A(y)=0.5$.

<u>Axiom 3</u>: If $A_1\subset A_2$, then, $\Gamma_{A_1}\le\Gamma_{A_2}$.
Also,

$$\Gamma_{A_1}(y_i)\log(\Gamma_{A_1}(y_i))+(1-\Gamma_{A_1}(y_i))\log((1-\Gamma_{A_1}(y_i)))\le\Gamma_{A_2}(y_i)\log(\Gamma_{A_2}(y_i))$$
$$+(1-\Gamma_{A_2}(y_i))\log((1-\Gamma_{A_2}(y_i))).$$

This implies

$$M'_\Gamma(A_1;P;U)\le M'_\Gamma(A_2;P;U).$$

Similarly, we can prove that

$$M'_\Lambda(A_1;P;U) \geq M'_\Lambda(A_2;P;U) \,\&\, M'_\theta(A_1;P;U) \leq M'_\theta(A_2;P;U).$$

Hence, this proves $E_{M'}(A_1) \geq E_{M'}(A_2)$ if $A_1 \subset A_2$.

Axiom 4: For the complement,
$M'_\Gamma(\tilde{A};P;U)$

$$= \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [\Gamma_A^c(y_i)\log(\Gamma_A^c(y_i)) + (1-\Gamma_A(y_i))^c \log(1-\Gamma_A(y_i))^c]}{\sum_{i=1}^{n} u_i p_i}.$$

$$\implies M'_\Gamma(\tilde{A};P;U)$$

$$= \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [(1-\Gamma_A(y_i))\log(1-\Gamma_A(y_i)) + (\Gamma_A(y_i))\log((\Gamma_A(y_i)))]}{\sum_{i=1}^{n} u_i p_i}.$$

$$\implies M'_\Gamma(\tilde{A};P;U) = M'_\Gamma(A;P;U).$$

In similar manner, the condition can be obtained for the neutral and falsity membership functions.
Thus, we observe that all the axioms have been satisfied and this is a valid entropy measure. It may be noted that these properties can also be satisfied with the help of some numerical examples. We consider Table 4.4 which shows the behavior of the proposed measure in case of the crisp set. In this case, the values of degree of truth membership function is 1 when membership is maximum whereas the values of indeterminacy and falsity are zero.

Table 4.4: Behavior of proposed measure on crisp set in case of maximum membership

u_i	p_i	$(\Gamma_A(y_i), \Lambda_A(y_i), \theta_A(y_i))$	$(M'(\Gamma_A(y_i)), M'(\Lambda_A(y_i)), M'(\theta_A(y_i)))$
u_4	0.4	(1,0,0)	0
u_3	0.3	(1,0,0)	0
u_2	0.2	(1,0,0)	0
u_1	0.1	(1,0,0)	0

Secondly, in Table 4.5 when degree of membership is minimum then falsity component is 1 and the values of rest of the components are equal to zero i.e., $A = (1,0,0)$ or $(0,0,1)$. Consider the universe $U = (1,2,3,4)$ with utilities $u_i = (u_1,u_2,u_3,u_4)$ and probabilities $p(A) = (0.1,0.2,0.3,0.4)$. Note: In similar manner, all the above properties of single valued neutrosophic information measure can be satisfied using the numerical example.

Table 4.5: Behavior of proposed measure on crisp set in case of minimum membership

u_i	p_i	$(\Gamma_A(y_i), \Lambda_A(y_i), \theta_A(y_i))$	$(M'(\Gamma_A(y_i)), M'(\Lambda_A(y_i)), M'(\theta_A(y_i)))$
u_4	0.4	(0,0,1)	0
u_3	0.3	(0,0,1)	0
u_2	0.2	(0,0,1)	0
u_1	0.1	(0,0,1)	0

4.4 'Useful' Divergence Measure of Single Valued Neutrosophic Information Measure

In this segment of the study, we have described the properties of divergence and the divergence between the proposed neutrosophic information measures.
The divergence of two single valued neutrosophic sets is defined on the basis of the following parameters:

- Two positive and symmetric single valued neutrosophic sets are compared.
- Divergence is equal to zero when these two sets coincide.
- Divergence is inversely proportion to the similarity between the two sets. As the similarity increases the divergence decreases.

Consider two single valued neutrosophic sets A and B on the same similarity points y_i $(i = 1, 2, ..., n)$ and with neutrosophic vectors
$(\Gamma_A(y_i), \Lambda_A(y_i), \theta_A(y_i))$ and $(\Gamma_B(y_i), \Lambda_B(y_i), \theta_B(y_i))$ where $(i = 1, 2, ..., n)$.

The simplest form of fuzzy divergence was introduced by Bhandari and Pal [25] as

$$I(A,B) = \sum_{i=1}^{n}[\Gamma_A(y_i)\log\frac{\Gamma_A(y_i)}{\Gamma_B(y_i)} + (1-\Gamma_A(y_i))\log\frac{(1-\Gamma_A(y_i))}{(1-\Gamma_B(y_i))}]. \tag{4.4.1}$$

Next, we consider two single valued neutrosophic fuzziness of A from B and the 'useful' measure for truth membership component of these two single valued neutrosophic directed divergence measure of A from B is given by:

$$I(A,B;P;U) = \frac{1}{2n\log(0.5)}\frac{\sum_{i=1}^{n} u_i p_i[\Gamma_A(y_i)\log(\Gamma_B(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_B(y_i)))]}{\sum_{i=1}^{n} u_i p_i}. \tag{4.4.2}$$

Then, the 'useful' neutrosophic symmetric divergence measure is defined as,

$$\mathbf{J(A,B;P;U)=I(A,B;P;U)+I(B,A;P;U)}. \tag{4.4.3}$$

Theorem 17 *The proposed measure $I(A,B,P,U)$,i.e., $I(A,B;P;U) \geq 0$ if $\Gamma_A(y_i) = \Gamma_B(y_i)$ where $i = 1,2,...,n$ is a valid information measure.*

Proof: Suppose

$$\sum_{i=1}^{n} \Gamma_A(y_i) = e, \quad \sum_{i=1}^{n} \Gamma_B(y_i) = f \quad \& \quad \sum_{i=1}^{n} u_i p_i = u.$$

Then,

$$\frac{1}{2n\log(0.5)}[\sum_{i=1}^{n} u_i p_i (\Gamma_A(y_i) \log \frac{\Gamma_A(y_i)}{\Gamma_B(y_i)}] \geq \frac{ue}{2n\log(0.5)} \log \frac{e}{f}. \tag{4.4.4}$$

In similar manner, we can prove that

$$\frac{1}{2n\log(0.5)}[\sum_{i=1}^{n} u_i p_i (1-\Gamma_A(y_i)) \log \frac{(1-\Gamma_A(y_i))}{(1-\Gamma_B(y_i))}] \geq \frac{u(n-e)}{2n\log(0.5)} \log \frac{n-e}{n-f}. \tag{4.4.5}$$

Adding equations 4.4.4 and 4.4.5,

$$\begin{aligned} \frac{1}{2n\log(0.5)}[\sum_{i=1}^{n} u_i p_i (\Gamma_A(y_i) \log \frac{\Gamma_A(y_i)}{\Gamma_B(y_i)} + \sum_{i=1}^{n} u_i p_i (1-\Gamma_A(y_i) \log \frac{(1-\Gamma_A(y_i)}{(1-\Gamma_B(y_i))}] \geq \\ \frac{u}{2n\log(0.5)}[e \log \frac{e}{f} + (n-e) \log \frac{n-e}{n-f}]. \end{aligned} \tag{4.4.6}$$

Suppose

$$\begin{aligned} f(e) &= \frac{1}{2n\log(0.5)}[e\log\frac{e}{f} + (n-e)\frac{n-e}{n-f}], \\ f'(e) &= \frac{1}{2n\log(0.5)}[\log\frac{e}{f} + \frac{n-e}{n-f}], \\ f''(e) &= \frac{1}{2n\log(0.5)}[\frac{1}{e} + \frac{1}{n-e}] > 0. \end{aligned}$$

Thus $f''(e) > 0$, which proves that $f(e)$ is a complex function and have a minimum value when $e = f$. Secondly, $\sum_{i=1}^{n} u_i p_i > 0$.

Hence, $I(A,B;P;U) \geq 0$.

4.5 Notion of Hybrid Ambiguity 'Useful' Measure

In this segment of the current manuscript, hybrid ambiguity of the single valued neutrosophic information measure under utility distribution has been obtained.

Consider two single valued neutrosophic information sets A and B. The two conditions to be satisfied to find the hybrid ambiguity are given below:

- Then, entropy of set A defines ambiguity for the given set.

- Secondly, the difference between A and directed divergence of B is calculated by $I(A, B)$.

Hybrid ambiguity= Entropy of $A + I(A, B)$.

The hybrid ambiguity for the truth membership function is given below:

$$HA_\Gamma = \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [\Gamma_A(y_i)\log(\Gamma_A(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_A(y_i)))]}{\sum_{i=1}^{n} u_i p_i} - \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [\Gamma_A(y_i)\log\frac{\Gamma_A(y_i)}{\Gamma_B(y_i)} + (1-\Gamma_A(y_i))\log\frac{(1-\Gamma_A(y_i))}{1-\Gamma_B(y_i)}]}{\sum_{i=1}^{n} u_i p_i}.$$

$$= \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [\Gamma_A(y_i)\log(\Gamma_B(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_B(y_i)))]}{\sum_{i=1}^{n} u_i p_i}.$$

$$\implies [I(A, B)]_\Gamma = \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [\Gamma_A(y_i)\log(\Gamma_B(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_B(y_i)))]}{\sum_{i=1}^{n} u_i p_i}.$$

Similarly, we can find for other two components for neutrosophic theory.

Remarks: It may be noted that we can establish a relation between entropy and directed divergence of two single valued neutrosophic sets as follows.

If $\Gamma_B = 0.5$, then

$$[I(A, B)]_\Gamma = \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [\Gamma_A(y_i)\log\frac{\Gamma_A(y_i)}{0.5} + (1-\Gamma_A(y_i))\log\frac{(1-\Gamma_A(y_i))}{0.5}]}{\sum_{i=1}^{n} u_i p_i}.$$

$$= \frac{1}{2} - \frac{1}{2n\log(0.5)} \frac{\sum_{i=1}^{n} u_i p_i [\Gamma_A(y_i)\log(\Gamma_A(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_A(y_i)))]}{\sum_{i=1}^{n} u_i p_i}.$$

Then, the relation obtained is given by

$= \frac{1}{2}$-(Entropy of SVNS A).

4.6 'Useful' Single Valued Neutrosophic Information Improvement Measure

The 'useful' information improvement measure for the three single valued neutrosophic sets under consideration can be explained as follows,

Consider sets A and B, where set A is estimated from set B and was revised to set C. Then, the original and final ambiguity is given by $I'(A,B)$ and $I'(A,C)$. Then, the reduced ambiguity for truth membership function is given by

$$I'_{\Gamma}(A,B,C) = I'_{\Gamma}(A,B) - I'_{\Gamma}(A,C).$$

$$= \frac{1}{2n\log(0.5)}[\frac{\sum_{i=1}^{n} u_i p_i[\Gamma_A(y_i)\log(\Gamma_B(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_B(y_i)))]}{\sum_{i=1}^{n} u_i p_i}$$
$$- \frac{\sum_{i=1}^{n} u_i p_i[\Gamma_A(y_i)\log(\Gamma_C(y_i)) + (1-\Gamma_A(y_i))\log((1-\Gamma_C(y_i)))]}{\sum_{i=1}^{n} u_i p_i}].$$
$$= \frac{1}{2n\log(0.5)}[\frac{\sum_{i=1}^{n} u_i p_i[\Gamma_A(y_i)\log\frac{\Gamma_B(y_i)}{\Gamma_C(y_i)} + (1-\Gamma_A(y_i))\log\frac{(1-\Gamma_B(y_i))}{(1-\Gamma_C(y_i))})]}{\sum_{i=1}^{n} u_i p_i}].$$

Similarly, we can obtain for the improved measure of the neutrality and falsity function of neutrosophic information measure.

This is called the 'useful' single valued neutrosophic improved information measure.

4.7 Conclusions

We have successfully introduced somenew measures of similarity for the neutrosophic sets in terms of the exponential functions of the truth membership,vindeterminacy-membership and falsity-membership. The efficiency of the proposed measure has been validated by presenting few counter-intuitive cases which show that the existing measures fail under some certain cases, while the proposed measures classify them more accurately and precisely. Furth -ermore, to illustrate the applicability of the proposed similarity measures, an example of classification problem and an example of decision-making problem under neutrosophic enviro -nment have been successfully solved. Finally, we conclude that the proposed types of exponential similarity measures are better than the existing measures. The proposed measures produce a reasonable and distinguishable results which is the main outcome and advantage in contrast with other existing methods. Also, it may clearly be observed that the proposed measures are very simple and have the minimum computational burden as compared with other existing methods. The proposed exponential similarity measure for the the neutrosophic sets can be extended for single and multi-valued neutrosophic hyper soft set also along with the relevant application which will certainly give an added advantage in the literature. The proposed strategy utilizing the exponential similarity measure can further be applied in various other decision-making problems, e.g., supplier selection, pattern recognition, cluster analysis, medical diagnosis, weaver selection, fault diagnosis, data mining, logistic centre location selection etc. Later, we have successfully established the validity of the proposed measures named as the probabilistic single valued neutrosophic 'useful' information measure, 'useful' divergence measure, hybrid ambiguity and 'useful' information improvement measure of single-valued neutrosophic sets. All these measures have been explained and validated with the help of well established axioms and numerical example.

Chapter 5

Energy of Picture Fuzzy Graph in Site Selection

In this chapter, we have considered the fact that picture fuzzy graph has the sufficient streng -th to formulate the impreciseness, vagueness and incompleteness embedded in the information of an application. Therefore, we have proposed the definition of adjacency matrix of such graph, its spectrum and energy/Laplacian energy with upper/lower bounds in the curre -nt chapter. In reference with picture fuzzy directed graph, similar studies and results have been presented. Further, we have also presented a new algorithm to solve hydropower plant site selection problem by utilizing the notion of energy/Laplacian energy of picture fuzzy graph. Some comparative findings and advantages of the proposed approach have also been provided.

5.1 Notion of Energy of Picture Fuzzy Graph

In this section, we have proposed some novel concepts of adjacency matrix, spectrum , energy and Laplacian energy of picture fuzzy graph as follows:
Let $G = (S, R)$ be a picture fuzzy graph, where S is the picture fuzzy vertex set and R is the picture fuzzy edge set.

Definition 46 *The* ***adjacency matrix*** $A(G)$ *of the graph* G *is a square matrix defined as*

$$A(G) = [a_{ij}],\ \textit{where}\ a_{ij} = (\mu_R(\alpha_i, \alpha_j), \eta_R(\alpha_i, \alpha_j), \nu_R(\alpha_i, \alpha_j))\,.$$

Here, $\mu_R(\alpha_i, \alpha_j)$, $\eta_R(\alpha_i, \alpha_j)$ & $\nu_R(\alpha_i, \alpha_j)$ are the degree of membership, degree of neutral membership (abstain) and degree non-membership respectively.

Definition 47 *The **spectrum** of the adjacency matrix $A(G)$ of the picture fuzzy graph $G = (S, R)$ is given by (Θ, Φ, Ψ), where Θ, Φ and Ψ are the set of the eigenvalues of matrices*

$$A(\mu_R(\alpha_i, \alpha_j)) = [\mu_R(\alpha_i, \alpha_j)]\,;$$

$$A(\eta_R(\alpha_i, \alpha_j)) = [\eta_R(\alpha_i, \alpha_j)]\,;$$

and

$$A(\nu_R(\alpha_i, \alpha_j))) = [\nu_R(\alpha_i, \alpha_j)]\,;$$

respectively.

Definition 48 *The **energy** $E(G)$ of the picture fuzzy graph G is defined as*

$$E(G) = (E(\mu_R(\alpha_i, \alpha_j)), E(\eta_R(\alpha_i, \alpha_j)), E(\nu_R(\alpha_i, \alpha_j))) = \left(\sum_{i=1, \theta_i \in \Theta}^{m} |\theta_i|, \sum_{i=1, \phi_i \in \Phi}^{m} |\phi_i|, \sum_{i=1, \psi_i \in \Psi}^{m} |\psi_i| \right).$$

For illustrating the proposed definitions, we consider the following example of a picture fuzzy graph:

Example: Suppose $G = (S, R)$ be a picture fuzzy graph given in Figure 5.1.

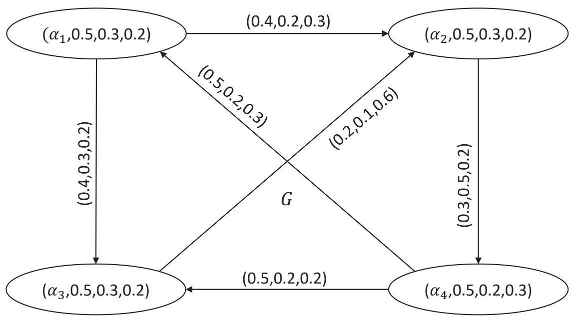

Figure 5.1: Graph $G = (S, R)$ for Energy

In view of the definitions proposed above, the adjacency matrix $A(G)$ is given by

$$A(G) = \begin{pmatrix} (0.0, 0.0, 0.0) & (0.4, 0.3, 0.2) & (0.0, 0.0, 0.0) & (0.4, 0.2, 0.3) \\ (0.4, 0.3, 0.2) & (0.0, 0.0, 0.0) & (0.5, 0.2, 0.2) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.5, 0.2, 0.2) & (0.0, 0.0, 0.0) & (0.3, 0.5, 0.2) \\ (0.4, 0.2, 0.3) & (0.0, 0.0, 0.0) & (0.3, 0.5, 0.2) & (0.0, 0.0, 0.0) \end{pmatrix}.$$

Using the adjacency matrix $A(G)$, the spectrum of the picture fuzzy graph G can be evaluated as

$$Spec(\mu_R(\alpha_i,\alpha_j)) = \{-0.8063, 0.8063, -0.0992, 0.0992\};$$
$$Spec(\eta_R(\alpha_i,\alpha_j)) = \{-0.5114, 0.5114, -0.0917, 0.0917\};$$
$$Spec(\nu_R(\alpha_i,\alpha_j)) = \{-0.4561, 0.4561, -0.0438, 0.0438\}.$$

Hence, the spectrum of G may be presented as

$$\begin{aligned} Spec(G) = \{&(-0.8063, -0.5114, -0.4561), (0.8063, 0.5114, 0.4561), \\ &(-0.0992, -0.0917, -0.0438), (0.0992, 0.0917, 0.0438)\}. \end{aligned}$$

Now, the calculation of energy of the picture fuzzy graph can be done as

$$E(\mu_R(\alpha_i,\alpha_j)) = 1.811;\ E(\eta_R(\alpha_i,\alpha_j)) = 1.2062;\ E(\nu_R(\alpha_i,\alpha_j)) = 0.998.$$

Hence, the energy of G is

$$E(G) = (1.811, 1.2062, 0.998).$$

Next, we present the studies and various important results related to eigenvalues of adjacency matrix, upper bound and lower bound of energy of picture fuzzy graph.

Theorem 18 *Let $G = (S, R)$ be a picture fuzzy graph and $A(G)$ be its adjacency matrix. If $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_m$, $\phi_1 \geq \phi_2 \geq \ldots \geq \phi_m$ and $\psi_1 \geq \psi_2 \geq \ldots \geq \psi_m$ are the absolute eigenvalues of $A(\mu_R(\alpha_i,\alpha_j))$, $A(\eta_R(\alpha_i,\alpha_j))$ and $A(\nu_R(\alpha_i,\alpha_j))$, respectively, then:*

(i) $\sum\limits_{i=1,\theta_i\in\Theta}^{m} \theta_i = 0$, $\sum\limits_{i=1,\phi_i\in\Phi}^{m} \phi_i = 0$ *and* $\sum\limits_{i=1,\psi_i\in\Psi}^{m} \psi_i = 0$;

(ii) $\sum\limits_{i=1,\theta_i\in\Theta}^{m} \theta_i^2 = 2\sum\limits_{1\leq i<j\leq m} (\mu_R(\alpha_i,\alpha_j))^2$;
$\sum\limits_{i=1,\phi_i\in\Phi}^{m} \phi_i^2 = 2\sum\limits_{1\leq i<j\leq m} (\eta_R(\alpha_i,\alpha_j))^2$; *and*
$\sum\limits_{i=1,\psi_i\in\Psi}^{m} \psi_i^2 = 2\sum\limits_{1\leq i<j\leq m} (\nu_R(\alpha_i,\alpha_j))^2$.

Proof :

(i) Since the diagonal entries of adjacency matrix $A(G)$ are zero, therefore the trace of the matrix is zero. As the trace of a matrix is equal to sum of its eigenvalues, the proof is obvious.

(ii) By the trace property of a matrix, we have

$$tr((A(\mu_R(\alpha_i,\alpha_j)))^2) = \sum_{i=1,\theta_i\in\Theta}^{m} \theta_i^2;$$

where

$$\begin{aligned} tr((A(\mu_R(\alpha_i,\alpha_j)))^2) = &\Big(0 + (\mu_R(\alpha_1,\alpha_2))^2 + \dots (\mu_R(\alpha_1,\alpha_m))^2\Big) \\ &+ \Big((\mu_R(\alpha_2,\alpha_1))^2 + 0 + \dots (\mu_R(\alpha_2,\alpha_m))^2\Big) \\ &\vdots \\ &+ \Big((\mu_R(\alpha_n,\alpha_1))^2 + 0 + (\mu_R(\alpha_m,\alpha_m))^2 + \dots + 0\Big) \\ &= 2\sum_{1\le i<j\le n} (\mu_R(\alpha_i,\alpha_j))^2. \end{aligned}$$

Hence,

$$\sum_{i=1,\theta_i\in\Theta}^{m} \theta_i^2 = 2\sum_{1\le i<j\le n} (\mu_R(\alpha_i,\alpha_j))^2.$$

Similarly, we can show that

$$\sum_{i=1,\phi_i\in\Phi}^{m} \phi_i^2 = 2\sum_{1\le i<j\le n} (\eta_R(\alpha_i,\alpha_j))^2 \text{ and } \sum_{i=1,\psi_i\in\Psi}^{m} \psi_i^2 = 2\sum_{1\le i<j\le n} (\nu_R(\alpha_i,\alpha_j))^2.$$

Hence, the results of the theorem are proved.

Further, throughout the manuscript we denote

$$M_\mu =: \sum_{1\le i<j\le n} (\mu_R(\alpha_i,\alpha_j))^2;\ M_\eta =: \sum_{1\le i<j\le n} (\eta_R(\alpha_i,\alpha_j))^2;\ \text{and } M_\nu =: \sum_{1\le i<j\le n} (\nu_R(\alpha_i,\alpha_j))^2.$$

Also, we denote

$$|A_\mu| =: det(A(\mu_R(\alpha_i,\alpha_j)));\ |A_\eta| =: det(A(\eta_R(\alpha_i,\alpha_j)));\ \text{and } |A_\nu| =: det(A(\nu_R(\alpha_i,\alpha_j))).$$

Theorem 19 *Let $G = (S, R)$ be the picture fuzzy graph with m vertices and $A(G)$ be its adjacency matrix. Then,*

(i) $\sqrt{2M_\mu + m(m-1)|A_\mu|^{\frac{2}{m}}} \le E(\mu_R(\alpha_i,\alpha_j)) \le \sqrt{2mM_\mu}$;

(ii) $\sqrt{2M_\eta + m(m-1)|A_\eta|^{\frac{2}{m}}} \le E(\eta_R(\alpha_i,\alpha_j)) \le \sqrt{2mM_\eta}$;

(iii) $\sqrt{2M_\nu + m(m-1)|A_\nu|^{\frac{2}{m}}} \le E(\nu_R(\alpha_i,\alpha_j)) \le \sqrt{2mM_\nu}$.

Proof : Using Cauchy-Schwarz inequality for the vectors $(1, 1, \dots, 1)$ and $(|\theta_1|, |\theta_2|, \dots, |\theta_m|)$ with m entries, we get

$$\sum_{i=1}^{m} |\theta_i| \leq \sqrt{m}\sqrt{\sum_{i=1}^{m} |\theta_i|^2}. \tag{5.1.1}$$

Also,

$$\left(\sum_{i=1}^{m} |\theta_i|\right)^2 = \sum_{i=1}^{m} |\theta_i|^2 + 2\sum_{1\leq i<j\leq m}^{m} |\theta_i\theta_j| = 0. \tag{5.1.2}$$

The characteristic polynomial of $A(G)$ is given by

$$\prod_{i=1}^{m}(\lambda - \theta_i) = |A(G) - \lambda I|.$$

Now, comparing the coefficients of λ^{n-2} in above polynomial, we get

$$\sum_{1\leq i<j\leq m}^{m} |\theta_i\theta_j| = -M_\mu. \tag{5.1.3}$$

Using equation (5.1.3) in equation (5.1.2), we have

$$\sum_{i=1}^{m} |\theta_i|^2 = 2M_\mu. \tag{5.1.4}$$

Substituting equation (5.1.4) in equation (5.1.1), we have

$$\sum_{1\leq i<j\leq m}^{m} |\theta_i| \leq \sqrt{m}\sqrt{2M_\mu} = \sqrt{2mM_\mu}.$$

Therefore ,

$$E(\mu_R(\alpha_i, \alpha_j)) \leq \sqrt{2mM_\mu}. \tag{5.1.5}$$

Next,

$$\begin{aligned}(E(\mu_R(\alpha_i, \alpha_j)))^2 = \left(\sum_{i=1}^{m} |\theta_i|\right)^2 &= \sum_{i=1}^{m} |\theta_i|^2 + 2\sum_{1\leq i<j\leq m}^{m} |\theta_i\theta_j| \\ &= 2M_\mu + \frac{2m(m-1)}{2} AM\{|\theta_i\theta_j|\}.\end{aligned}$$

Since, the arithmetic mean is greater than or equal to the geometric mean, i.e., $AM\{|\theta_i\theta_j|\} \geq GM\{|\theta_i\theta_j|\}$, therefore,

$$E(\mu_R(\alpha_i, \alpha_j)) \geq \sqrt{2M_\mu + m(m-1)GM\{|\theta_i\theta_j|\}}. \tag{5.1.6}$$

Also,

$$GM\{|\theta_i\theta_j|\} = \left(\prod_{1\le i<j\le m} |\theta_i\theta_j|\right)^{\frac{2}{m(m-1)}} = \left(\prod_{1\le i<j\le m} |\theta_i|^{m-1}\right)^{\frac{2}{m(m-1)}}$$

$$= \left(\prod_{i=1}^{m} |\theta_i|\right)^{\frac{2}{m}} = |A_\mu|^{\frac{2}{m}}. \tag{5.1.7}$$

Substituting equation (5.1.7) in equation (5.1.6), we get

$$E(\mu_R(\alpha_i,\alpha_j)) \ge \sqrt{2M_\mu + m(m-1)|A_\mu|^{\frac{2}{m}}}. \tag{5.1.8}$$

Thus, from equations (5.1.5) and (5.1.8), we have

$$\sqrt{2M_\mu + m(m-1)|A_\mu|^{\frac{2}{m}}} \le E(\mu_R(\alpha_i,\alpha_j)) \le \sqrt{2mM_\mu}.$$

On similar lines, we can show that

$$\sqrt{2M_\eta + m(m-1)|A_\eta|^{\frac{2}{m}}} \le E(\eta_R(\alpha_i,\alpha_j)) \le \sqrt{2mM_\eta}$$

and

$$\sqrt{2M_\nu + m(m-1)|A_\nu|^{\frac{2}{m}}} \le E(\nu_R(\alpha_i,\alpha_j)) \le \sqrt{2mM_\nu}.$$

Theorem 20 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices and $A(G)$ be its adjacency matrix. If $m \le 2M_\mu$, $m \le 2M_\eta$ and $m \le 2M_\nu$, then:*

(i) $E(\mu_R(\alpha_i,\alpha_j)) \le \frac{2M_\mu}{m} + \sqrt{(m-1)\left\{2M_\mu - \left(\frac{2M_\mu}{m}\right)^2\right\}}$;

(ii) $E(\eta_R(\alpha_i,\alpha_j)) \le \frac{2M_\eta}{m} + \sqrt{(m-1)\left\{2M_\eta - \left(\frac{2M_\eta}{m}\right)^2\right\}}$;

(iii) $E(\nu_R(\alpha_i,\alpha_j)) \le \frac{2M_\nu}{m} + \sqrt{(m-1)\left\{2M_\nu - \left(\frac{2M_\nu}{m}\right)^2\right\}}$.

Proof : Since $A(G)$ of the picture fuzzy graph G is symmetric with trace zero, therefore,

$$\theta_1 \ge \frac{2\sum\limits_{1\le i<j\le m} \mu_R(\alpha_i,\alpha_j)}{m},$$

where $\theta_1, \theta_2, \ldots, \theta_m$ are the eigenvalues of $A(G)$. In view of the results obtained in Theorem 18, we write

$$\sum_{i=2}^{m} \theta_i^2 = 2M_\mu - \theta_1^2. \tag{5.1.9}$$

Using Cauchy- Schwarz inequality for the vectors $(1, 1, \dots, 1)$ & $(|\theta_1|, |\theta_2|, \dots, |\theta_m|)$ with $(m-1)$ entries, we get

$$E(\mu_R(\alpha_i, \alpha_j)) - \theta_1 = \sum_{i=2}^{m} \theta_i^2 \le \sqrt{(m-1) \sum_{i=2}^{m} \theta_i^2}. \qquad (5.1.10)$$

Substituting equation (5.1.9) in equation (5.1.10) and after rearranging, we have

$$E(\mu_R(\alpha_i, \alpha_j)) \le \theta_1 + \sqrt{(m-1)\left(2M_\mu - \theta_1^2\right)}. \qquad (5.1.11)$$

Since the function $F(\alpha) = \alpha + \sqrt{(m-1)(2M_\mu - \alpha^2)}$ decreases on the interval $(\sqrt{\frac{2M_\mu}{m}}, \sqrt{2M_\mu})$, as $1 \le \frac{2M_\mu}{m}$, therefore,

$$\sqrt{\frac{2M_\mu}{m}} \le \frac{2M_\mu}{m} \le \frac{2(\mu_R(\alpha_i, \alpha_j))}{m} \le \theta_1 \le \sqrt{2M_\mu}.$$

Thus, the equation (5.1.11) implies

$$E(\mu_R(\alpha_i, \alpha_j)) \le \frac{2M_\mu}{m} + \sqrt{(m-1)\left\{2M_\mu - \left(\frac{2M_\mu}{m}\right)^2\right\}}.$$

On similar lines, we can also show that

$$E(\eta_R(\alpha_i, \alpha_j)) \le \frac{2M_\eta}{m} + \sqrt{(m-1)\left\{2M_\eta - \left(\frac{2M_\eta}{m}\right)^2\right\}};$$

and

$$E(\nu_R(\alpha_i, \alpha_j)) \le \frac{2M_\nu}{m} + \sqrt{(m-1)\left\{2M_\nu - \left(\frac{2M_\nu}{m}\right)^2\right\}}.$$

It may be noted that the results obtained in the above theorem provide the upper bound for the energy of the picture fuzzy graph, with the conditions $m \le 2M_\mu$, $m \le 2M_\eta$ and $m \le 2M_\nu$.

Theorem 21 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices. Then$E(G) \le \frac{m}{2}(1 + \sqrt{m})$.*

Proof : If $n \le 2M\mu$, then by calculus it is easy to show that

$$f(M_\mu) = \frac{2M_\mu}{m} + \sqrt{(m-1)\left\{2M_\mu - \left(\frac{2M_\mu}{m}\right)^2\right\}}$$

obtains a maximum value when $M_\mu = \frac{m^2+m\sqrt{m}}{4}$. Substituting this value of M_μ in the above Theorem 20, we get $E(\mu_R(\alpha_i, \alpha_j)) \leq \frac{m(1+\sqrt{m})}{2}$. Similarly, the results for other energy components can be obtained. Hence, the theorem is proved.

Next, we study another important notion of energy of graph, known as **Laplacian Energy of Picture Fuzzy Graph** and discuss its various graph-theoretic aspects.

Definition 49 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices. The **degree matrix** $D(G) = [d_{ij}]$, of G is a $m \times m$ diagonal matrix defined as:*

$$d_{ij} = \begin{cases} d_G(\alpha_i) & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Example: Let $G = (S, R)$ be picture fuzzy matrix 4 vertices. Then, the degree of matrix $D(G)$ is a 4×4 diagonal matrix defined as

$$D(G) = \begin{pmatrix} (0.4, 0.2, 0.1) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.5, 0.7, 0.6) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.2, 0.3, 0.4) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.1, 0.2, 0.5) \end{pmatrix}.$$

Definition 50 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices. The Laplacian matrix of a picture fuzzy graph G is defined as $L(G) = D(G) - A(G)$; where $D(G)$ & $A(G)$ are degree and adjacency matrix of the picture fuzzy graph G, respectively.*

Example: Let $G = (S, R)$ be the picture fuzzy graph on 4 vertices. The laplacian matrix of a picture fuzzy graph G is defined as

$$L(G) = D(G) - A(G) = \begin{pmatrix} (0.4, 0.2, 0.1) & (-0.4, -0.2, -0.3) & (-0.4, -0.3, -0.2) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.5, 0.7, 0.6) & (0.0, 0.0, 0.0) & (-0.3, -0.5, -0.2) \\ (0.0, 0.0, 0.0) & (-0.2, -0.1, -0.6) & (0.2, 0.3, 0.4) & (0.0, 0.0, 0.0) \\ (-0.5, -0.2, -0.3) & (0.0, 0.0, 0.0) & (-0.5, -0.2, -0.2) & (0.1, 0.2, 0.5) \end{pmatrix}.$$

Definition 51 *The spectrum of the Laplacian matrix $L(G)$ of the picture fuzzy graph $G = (S, R)$ is given by $\{(\Delta, \Upsilon, \Omega)\}$, where Δ, Υ and Ω are the set of the eigenvalues of $L(\mu_R(\alpha_i, \alpha_j))$, $L(\eta_R(\alpha_i, \alpha_j))$ and $L(\nu_R(\alpha_i, \alpha_j)))$, respectively.*

Example: Using the example given in the definition 51 we calculate the spectrum of $L(G)$ given below:

$$spec(\mu_R(\alpha_i, \alpha_j)) = \{-0.20962, 0.4000, 0.50481 + 0.3561i, 0.50481 - 0.3561i\}$$

$$spec(\eta_R(\alpha_i, \alpha_j)) = \{-0.023607, 0.42361, 0.5 - 0.1i, 0.5 + 0.1i\}$$

$$spec(\nu_R(\alpha_i, \alpha_j)) = \{-0.027074, 0.4000, 0.61354 - 0.25845i, 0.61354 + 0.25845i\}$$

Theorem 22 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices and $L(G)$ be its Laplacian matrix. If $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_m$, $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$ and $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_m$ are the absolute eigenvalues of $L(\mu_R(\alpha_i, \alpha_j))$, $L(\eta_R(\alpha_i, \alpha_j))$ and $L(\nu_R(\alpha_i, \alpha_j)))$ respectively, then*

(i) $$\sum_{i=1,\delta_i\in\Delta}^{m} \delta_i = 2 \sum_{1\leq i<j\leq m} \mu_R(\alpha_i, \alpha_j);$$
$$\sum_{i=1,\upsilon_i\in\Upsilon}^{m} \upsilon_i = 2 \sum_{1\leq i<j\leq m} \eta_R(\alpha_i, \alpha_j); \text{ and}$$
$$\sum_{i=1,\omega_i\in\Omega}^{m} \omega_i = 2 \sum_{1\leq i<j\leq m} \nu_R(\alpha_i, \alpha_j).$$

(ii) $$\sum_{i=1,\delta_i\in\Delta}^{m} \delta_i^2 = 2 \sum_{1\leq i<j\leq m} (\mu_R(\alpha_i, \alpha_j))^2 + \sum_{i=1}^{m} d^2_{\mu_R(\alpha_i,\alpha_j)}(\alpha_i);$$
$$\sum_{i=1,\upsilon_i\in\Upsilon}^{m} \upsilon_i^2 = 2 \sum_{1\leq i<j\leq m} (\eta_R(\alpha_i, \alpha_j))^2 + \sum_{i=1}^{m} d^2_{\eta_R(\alpha_i,\alpha_j)}(\alpha_i); \text{ and}$$
$$\sum_{i=1,\omega_i\in\Omega}^{m} \omega_i^2 = 2 \sum_{1\leq i<j\leq m} (\nu_R(\alpha_i, \alpha_j))^2 + \sum_{i=1}^{m} d^2_{\nu_R(\alpha_i,\alpha_j)}(\alpha_i).$$

Proof : The theorem can be proved on the similar lines as the proof of Theorem 18.

Definition 52 *The **Laplacian energy** of the picture fuzzy graph $G = (S, R)$, denoted by $LE(G)$, is defined as*

$$LE(G) = (LE(\mu_R(\alpha_i, \alpha_j)), LE(\eta_R(\alpha_i, \alpha_j)), LE(\eta_R(\alpha_i, \alpha_j))) = \left(\sum_{i=1}^{m} |\rho_i|, \sum_{i=1}^{m} |\xi_i|, \sum_{i=1}^{m} |\varsigma_i|\right);$$

where $\rho_i = \delta_i - \frac{2 \sum_{1\leq i<j\leq m} \mu_R(\alpha_i,\alpha_j)}{m}$; $\xi_i = \upsilon_i - \frac{2 \sum_{1\leq i<j\leq m} \eta_R(\alpha_i,\alpha_j)}{m}$; $\varsigma_i = \omega_i - \frac{2 \sum_{1\leq i<j\leq m} \nu_R(\alpha_i,\alpha_j)}{m}$.

Example: Again we will be using the example used in definition 52 to find the energy of the particulr matrix which is given as:

$$LE(\mu_R(\alpha_i, \alpha_j)) = 1.845103, LE(\eta_R(\alpha_i, \alpha_j)) = 1.467021, LE(\nu_R(\alpha_i, \alpha_j)) = 1.758581$$

Theorem 23 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices and $L(G)$ be its Laplacian matrix. If $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_m$, $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$ and $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_m$ are the absolute eigenvalues of $L(\mu_R(\alpha_i, \alpha_j))$, $L(\eta_R(\alpha_i, \alpha_j))$ and $L(\nu_R(\alpha_i, \alpha_j)))$, respectively, then*

(i) $\sum\limits_{i=1}^{m} \rho_i = 0, \sum\limits_{i=1}^{m} \xi_i = 0$ *and* $\sum\limits_{i=1}^{m} \varsigma_i = 0$;

(ii) $\sum\limits_{i=1}^{m} \rho_i^2 = 2N_\mu,\ \sum\limits_{i=1}^{m} \xi_i^2 = 2N_\eta,\ \sum\limits_{i=1}^{m} \varsigma_i^2 = 2N_\nu$; *where*

$$N_\mu = M_\mu + \frac{1}{2}\sum_{i=1}^{m}\left(d_{\mu(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\mu_R(\alpha_i)}{m}\right);$$

$$N_\eta = M_\eta + \frac{1}{2}\sum_{i=1}^{m}\left(d_{\eta(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\eta_R(\alpha_i)}{m}\right) \text{ and}$$

$$N_\nu = M_\nu + \frac{1}{2}\sum_{i=1}^{m}\left(d_{\nu(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\nu_R(\alpha_i)}{m}\right).$$

Proof : The proof of the theorem is obvious.

Theorem 24 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices and $L(G)$ be its Laplacian matrix. Then*

(i) $$LE(\mu_R(\alpha_i, \alpha_j)) \leq \sqrt{2mM_\mu + m\sum_{i=1}^{m}\left(d_{R(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\mu_R(\alpha_i,\alpha_j)}{m}\right)^2};$$

(ii) $$LE(\eta_R(\alpha_i, \alpha_j)) \leq \sqrt{2mM_\eta + m\sum_{i=1}^{m}\left(d_{R(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\eta_R(\alpha_i,\alpha_j)}{m}\right)^2};$$

(iii) $$LE(\nu_R(\alpha_i, \alpha_j)) \leq \sqrt{2mM_\nu + m\sum_{i=1}^{m}\left(d_{R(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\nu_R(\alpha_i,\alpha_j)}{m}\right)^2}.$$

Proof : The proof can be given on the similar lines as the proof of the Theorem 19.

Theorem 25 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices and $L(G)$ be its Laplacian matrix. Then*

(i) $$LE(\mu_R(\alpha_i, \alpha_j)) \geq 2\sqrt{M_\mu + \frac{1}{2}\sum_{i=1}^{m}\left(d_{R(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\mu_R(\alpha_i,\alpha_j)}{m}\right)^2};$$

(ii) $$LE(\eta_R(\alpha_i, \alpha_j)) \geq 2\sqrt{M_\eta + \frac{1}{2}\sum_{i=1}^{m}\left(d_{R(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum\limits_{1\leq i<j\leq m}\eta_R(\alpha_i,\alpha_j)}{m}\right)^2};$$

(iii) $LE(\nu_R(\alpha_i,\alpha_j)) \geq 2\sqrt{M_\nu + \frac{1}{2}\sum_{i=1}^{m}\left(d_{R(\alpha_i,\alpha_j)}(\alpha_i) - \frac{2\sum_{1\leq i<j\leq m}\nu_R(\alpha_i,\alpha_j)}{m}\right)^2}.$

Proof : The proof can be given on the similar lines as the proof of the Theorem 19.

The results obtained in the above theorems provide us the upper bound and lower bound of the Laplacian energy of the picture fuzzy graph G.

Theorem 26 *Let $G = (S, R)$ be a picture fuzzy graph on m vertices and $L(G)$ be its Laplacian matrix. Then*

(i) $LE(\mu_R(\alpha_i,\alpha_j)) \leq |\rho_1| + \sqrt{(m-1)(2N_\mu - \rho_1^2)}$;

(ii) $LE(\eta_R(\alpha_i,\alpha_j)) \leq |\xi_1| + \sqrt{(m-1)(2N_\eta - \xi_1^2)}$;

(iii) $LE(\nu_R(\alpha_i,\alpha_j)) \leq |\varsigma_1| + \sqrt{(m-1)(2N_\nu - \varsigma_1^2)}$.

Proof : The proof can be given on the similar lines as the proof of the Theorem 20.

5.2 Energy and Laplacian Energy of Picture Fuzzy Directed Graph

In case of the directed graph, the adjacency matrix $A(G)$ of a picture fuzzy directed graph is not necessarily symmetric. Therefore, the eigenvalues of the adjacency matrix may be complex numbers. This section generalizes the concept of energy and Laplacian energy for picture fuzzy directed graphs.

Definition 53 *The **spectrum** of the adjacency matrix $A(G)$ of the picture fuzzy directed graph $G = (S, \overrightarrow{R})$ is given by $\{(\Theta, \Phi, \Psi)\}$, where Θ, Φ and Ψ are the set of the eigenvalues of $A(\mu_{\overrightarrow{R}}(\alpha_i,\alpha_j))$, $A(\eta_{\overrightarrow{R}}(\alpha_i,\alpha_j))$ and $A(\nu_{\overrightarrow{R}}(\alpha_i,\alpha_j)))$, respectively.*

Definition 54 *The **energy of the picture fuzzy directed graph** G is given as:*

$$E(G) = (E(\mu_{\overrightarrow{R}}(\alpha_i,\alpha_j)), E(\eta_{\overrightarrow{R}}(\alpha_i,\alpha_j)), E(\nu_{\overrightarrow{R}}(\alpha_i,\alpha_j))) = \Big(\sum_{i=1,\theta_i\in\Theta}^{m}|Re(\theta_i)|, \sum_{i=1,\phi_i\in\Phi}^{m}|Re(\phi_i)|, \sum_{i=1,\psi_i\in\Psi}^{m}|Re(\psi_i)|\Big);$$

where $Re(\theta_i)$, $Re(\phi_i)$ and $Re(\psi_i)$ represents the real part of the eigenvalues θ_i, ϕ_i and ψ_i, respect -ively.

Theorem 27 *Let $G = (S, \overrightarrow{R})$ be a picture fuzzy directed graph and $A(G)$ be its adjacency matrix. If $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_m$, $\phi_1 \geq \phi_2 \geq \ldots \geq \phi_m$ and $\psi_1 \geq \psi_2 \geq \ldots \geq \psi_m$ are the absolute eigenvalues of $A(\mu_R(\alpha_i, \alpha_j)$, $A(\eta_R(\alpha_i, \alpha_j)$ and $A(\nu_R(\alpha_i, \alpha_j)$, respectively, then $\sum\limits_{i=1, \theta_i \in \Theta}^{m} Re(\theta_i) = 0$, $\sum\limits_{i=1, \phi_i \in \Phi}^{m} Re(\phi_i) = 0$ and $\sum\limits_{i=1, \psi_i \in \Psi}^{m} Re(\psi_i) = 0$.*

Definition 55 *Let $G = (S, \overrightarrow{R})$ be a picture fuzzy graph on m vertices. The out-degree matrix $D^{out}(G) = [d_{ij}]$, of G is a $m \times m$ diagonal matrix defined as:*

$$d_{ij} = \begin{cases} d_G^{out}(\alpha_i) & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 56 *Let $G = (S, \overrightarrow{R})$ be a picture fuzzy directed graph on m vertices. The Laplacian matrix of a picture fuzzy directed G, denoted by $L(G)$ is defined as*

$$L(G) = D^{out}(G) - A(G);$$

where $D^{out}(G)$ & $A(G)$ are the out degree matrix and adjacency matrix of the picture fuzzy direc -ted graph G, respectively.

Definition 57 *The spectrum of the Laplacian matrix $L(G)$ of the picture fuzzy directed graph $G = (S, \overrightarrow{R})$ is given by $\{(\Delta, \Upsilon, \Omega)\}$, where Δ, Υ and Ω are the set of the eigenvalues of $L(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j))$, $L(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j))$ and $L(\nu_{\overrightarrow{R}}(\alpha_i, \alpha_j))$, respectively.*

Theorem 28 *Let $G = (S, \overrightarrow{R})$ be a picture fuzzy graph on m vertices and $L(G)$ be its Laplacian matrix. If $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_m$, $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$ and $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_m$ are the absolute eigenvalues of $L(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j))$, $L(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j))$ and $L(\nu_{\overrightarrow{R}}(\alpha_i, \alpha_j))$ respectively, then*

$$\sum_{i=1, \delta_i \in \Delta}^{m} Re(\delta_i) = tr(L(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j)));$$

$$\sum_{i=1, \upsilon_i \in \Upsilon}^{m} Re(\upsilon_i) = tr(L(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j)));$$

and

$$\sum_{i=1, \omega_i \in \Omega}^{m} Re(\omega_i) = tr(L(\nu_{\overrightarrow{R}}(\alpha_i, \alpha_j))).$$

Proof: The proof can be given on the similar lines as the proof of the Theorem 18.

Definition 58 *The* ***Laplacian energy*** *of the picture fuzzy directed graph* $G = (S, \overrightarrow{R})$ *denoted by* $LE(G)$ *is defined as*

$$LE(G) = \big(LE(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j)), LE(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j)), LE(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j))\big) = \left(\sum_{i=1}^{m} |\rho_i|, \sum_{i=1}^{m} |\xi_i|, \sum_{i=1}^{m} |\varsigma_i|\right);$$

where $\rho_i = Re(\delta_i) - \dfrac{\sum_{i=1,\delta_i \in \Delta}^{m} Re(\delta_i)}{m}$; $\xi_i = Re(\upsilon_i) - \dfrac{\sum_{i=1,\upsilon_i \in \Upsilon}^{m} Re(\upsilon_i)}{m}$; $\varsigma_i = Re(\omega_i) - \dfrac{\sum_{i=1,\omega_i \in \Omega}^{m} Re(\omega_i)}{m}$.

Theorem 29 *Let* $G = (S, \overrightarrow{R})$ *be a picture fuzzy directed graph on* m *vertices and* $L(G)$ *be its Laplacian matrix. If* $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_m$, $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$ *and* $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_m$ *are the absolute eigenvalues of* $L(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j))$, $L(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j))$ *and* $L(\nu_{\overrightarrow{R}}(\alpha_i, \alpha_j)))$*,respectively, then* $\sum_{i=1}^{m} \rho_i = 0$, $\sum_{i=1}^{m} \xi_i = 0$ *and* $\sum_{i=1}^{m} \varsigma_i = 0$.

Proof: The proof of the theorem is obvious.

For illustrating the proposed definitions, we consider the following example of a picture direct -ed fuzzy graph:

Example: Suppose $G = (S, R)$ be a picture fuzzy graph as given in Figure 5.2.

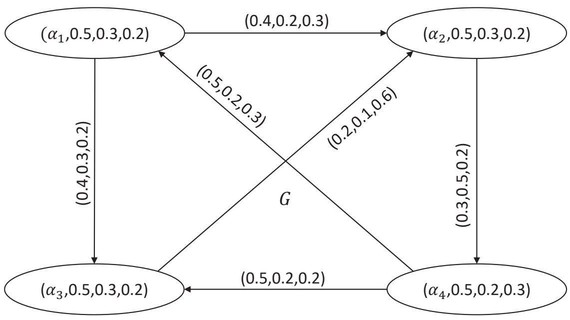

Figure 5.2: Graph $G = (S, R)$ for Laplacian Energy

In view of the above definitions and Figure 5.2, the adjacency matrix of picture fuzzy direct -ed graph can be given by

$$A(G) = \begin{pmatrix} (0.0, 0.0, 0.0) & (0.4, 0.2, 0.3) & (0.4, 0.3, 0.2) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.3, 0.5, 0.2) \\ (0.0, 0.0, 0.0) & (0.2, 0.1, 0.6) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.5, 0.2, 0.3) & (0.0, 0.0, 0.0) & (0.5, 0.2, 0.2) & (0.0, 0.0, 0.0) \end{pmatrix}.$$

The spectrum of the picture fuzzy directed graph G can be computed as

$$Spec(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = \{0.48586, -0.17786 + 0.3976i, -0.17786 - 0.3976i, -0.13014\};$$
$$Spec(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = \{0.3387, -0.1208 + 0.27689i, -0.1208 - 0.27689i, -0.0970\};$$
$$Spec(\nu_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = \{0.39225, -0.11764 + 0.32108i, -0.11764 - 0.32108i, -0.156972\}.$$

Hence, the spectrum of picture fuzzy directed graph G may be presented as

$$\begin{aligned} Spec(G) = \big\{&(0.48556, 0.3387, 0.39225), (-0.17786 + 0.3976i, -0.1208 + 0.27689i, -0.11764 + 0.32108i),\\ &(-0.17786 - 0.3976i, -0.1208 - 0.27689i, -0.11764 - 0.32108i), (-0.13014, -0.0970, -0.156972), \big\}. \end{aligned}$$

The calculation of the components for energy of picture fuzzy directed graph G has been listed below:

$$E(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = -0.13014;\ E(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = -0.0970;\ E(\nu_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = -0.156972.$$

Hence, the energy of picture fuzzy directed graph G is

$$E(G) = (-0.13014, -0.0970, -03156972).$$

Next, the out-degree matrix $D(G)$ and the Laplacian matrix $L(G)$ of the picture fuzzy directed graph G are given by

$$D^{out}(G) = \begin{pmatrix} (0.8, 0.5, 0.5) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.2, 0.1, 0.6) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.5, 0.2, 0.3) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (0.0, 0.0, 0.0) & (1.0, 0.4, 0.5) \end{pmatrix}$$

and

$$L(G) = \begin{pmatrix} (0.8, 0.5, 0.5) & (-0.4, -0.2, -0.3) & (-0.4, -0.3, -0.2) & (0.0, 0.0, 0.0) \\ (0.0, 0.0, 0.0) & (0.2, 0.1, 0.6) & (0.0, 0.0, 0.0) & (-0.3, -0.5, -0.2) \\ (0.0, 0.0, 0.0) & (-0.2, -0.1, -0.6) & (0.5, 0.2, 0.3) & (0.0, 0.0, 0.0) \\ (-0.5, -0.2, -0.3) & (0.0, 0.0, 0.0) & (-0.5, -0.2, -0.2) & (0.1, 0.4, 0.5) \end{pmatrix},$$

respectively. The Laplacian spectrum of the picture fuzzy directed graph G can be computed as

$$Spec(\mu_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = \{0.02329, 0.8883 + 0.2635i, 0.8883 - 0.2635i, 0.7000\};$$
$$Spec(\eta_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = \{-0.04585, 0.4824 + 0.2599i, 0.48244 - 0.2599i, 0.5809\};$$
$$Spec(\nu_{\overrightarrow{R}}(\alpha_i, \alpha_j)) = \{0.61958 + 0.28912i, 0.61958 - 0.28912i, 0.5864, 0.0744\}.$$

Hence, the Laplacian spectrum of picture fuzzy directed graph G may be written as

$$\begin{aligned} Spec(G) = \big\{&(0.02329, -0.04585, 0.61958 + 0.028912i), (0.8883 + 0.2635i, 0.4824 + 0.2599i, 0.61958 - 0.28912i),\\ &(0.8883 - 0.2635i, 0.48244 - 0.2599i, 0.5864), (0.7000, 0.5809, 0.0744)\big\}. \end{aligned}$$

The calculation of the components for Laplacian energy of picture fuzzy directed graph G has been listed below:

$$LE(\mu_{\overrightarrow{R}}(\alpha_i,\alpha_j)) = 2.49989;\ LE(\eta_{\overrightarrow{R}}(\alpha_i,\alpha_j)) = 1.59163;\ LE(\nu_{\overrightarrow{R}}(\alpha_i,\alpha_j)) = 1.89996$$

Hence, the Laplacian energy of picture fuzzy directed graph G is

$$LE(G) = (2.49989, 1.59163, 1.89996).$$

5.3 Algorithm for Selection Process Using Picture Fuzzy Graph Energy

In this section, we focus on the application of the proposed energy/Laplacian energy of pictu re fuzzy directed graph in a real world problem related to the site selection process. For ens -uring the sustainable development, the use of natural resources in an environmental consci -ousness framework has got a remarkable popularity in recent decades. Establishment of hyd -ropower plants certainly provides high usability, better reliability and clean source of energy. The problem of site selection for the hydropower plants also comprise of political, social, environmental and cultural aspects in addition to the technical requirements.

Decision making methods are often used in various selection processes where the final task is to select the best one out of the given set of alternatives. While drawing some concluding remarks in applicable fields, the experts mainly focus on different correlated factors with their prior perception and expertise. The preference relation is supposed to be the best and fruitful tool to achieve the actual sorting of the given set of alternatives among which the experts put forward their preference over other alternatives. In order to implement the preference relation concept, we would consider the information in the shape of picture fuzzy numbers as follows:

Definition 59 *A **picture fuzzy preference relation** $(PFPR)$ on the universe of discourse $U = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m\}$ is represented by a matrix $R = (\widetilde{r_{ij}})_{m\times m}$, where $\widetilde{r_{ij}} = \big((\alpha_i,\alpha_j), \mu(\alpha_i,\alpha_j), \eta(\alpha_i,\alpha_j), \nu(\alpha_i,\alpha_j)\big)$ $\forall\ i,\ j = 1, 2, \ldots, m$.*

For the sake of simplicity, suppose $\widetilde{r_{ij}} = (\mu_{ij}, \eta_{ij}, \nu_{ij})$, where μ_{ij} is the degree to which the object α_i has been preferred over the object α_j, η_{ij} is the degree to which the expert is in dilemma whether to prefer the object α_i or α_j. In addition to this, ν_{ij} gives the *degree to which* α_i *is not preferred to* α_j *and*

$$r_{ij} = 1 - (\mu_{ij}(\alpha) + \eta_{ij}(\alpha) + \nu_{ij}(\alpha))$$

gives the amount of refusal with the following constraint:

$$0 \leq \mu_{ij}(\alpha) + \eta_{ij}(\alpha) + \nu_{ij}(\alpha) \leq 1,\ \mu_{ij} = \nu_{ji},\ \eta_{ij} = \eta_{ji},\ \nu_{ij} = \mu_{ji}\ \textit{ and } \mu_{ii} = 1, \eta_{ii} = \nu_{ii} = 0;$$

$$\forall\ i, j = 1, 2, \ldots, m.$$

Suppose that the issue of site selection for the establishment of hydropower plant is formulated as:

- Based on the comprehensive survey conducted by the government agencies, let there are four possible locations/sites ($\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$) for the hydropower plant to be establis -hed.
- The survey has a detailed database and reports for all the four possible sites in context with the various deterministic features such as ecological safety, plant safety, social factor, economical factors, maximum efficiency, hydrological factors, environmental factor etc.
- For conducting the evaluation process based on the survey database report, suppose there are three experts ($e_k;\ k = 1, 2, 3$) who have been independently deputed. Based on the their experience, the expert's comparative opinions have been marked in the form picture fuzzy numbers.
- Further, picture fuzzy preference relations in the form of matrices have been constructed as the initial step for the site selection process.

In view of the proposed energy/Laplacian energy of picture fuzzy directed graphs with preference relations, an algorithm for accomplishing the computing task of site selection along with a flow chart is being presented in Figure 5.3.

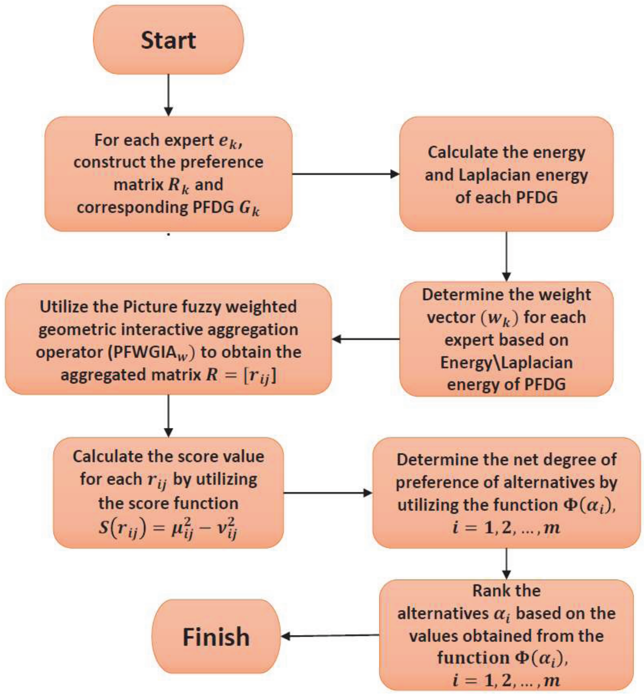

Figure 5.3: Flow chart of algorithm for alternatives selection process

Working Methodology of Proposed Algorithm

- **Step 1:** The experts compare the involved factors and present some initial inputs for the computing process in the shape of picture fuzzy preference relations, represented in the form of matrices $R_k = (\widetilde{r_{ij}}^{(k)})_{4\times 4}$ $(k = 1, 2, 3)$.

- **Step 2:** Consider a suitable picture fuzzy directed graph G_k corresponding to the $PFPRs$ given by $R_k(k = 1, 2, 3)$.

- **Step 3:** Compute the energy of each picture fuzzy directed graph as per the definition of the energy of PFDG.

- **Step 4:** The weight vector for each expert can be calculated by using

$$w_k = (w_\mu^k, w_\eta^k, w_\nu^k) = \left(\frac{E(G_\mu)_k}{\sum_{l=1}^{k} E(G_\mu)_l}, \frac{E(G_\eta)_k}{\sum_{l=1}^{k} E(G_\eta)_l}, \frac{E(G_\nu)_k}{\sum_{l=1}^{k} E(G_\nu)_l} \right); k = 1, 2, 3.$$

- **Step 5:** In this step, we use picture fuzzy weighted average or picture fuzzy ordered weighted average or picture fuzzy hybrid average aggregation operator recently given by Garg [39].

In this way, we aggregate the three picture fuzzy preference relations R_1, R_2 and R_3 given in step 1 into a single preference relation R.

- **Step 6:** We compute the score values by utilizing the score function

$$S(r_{ij}) = \mu_{ij}^2 - \nu_{ij}^2;$$

and tabulate them in the form of a matrix $S(R) = [r_{ij}]$.

- **Step 7:** Next, we determine the net degree of preference of alternatives by utilizing the function $\phi(\alpha_i)$ given by Wang and Fan [154] as follows:

$$\phi(\alpha_i) = \sum_{j=1, j\neq i}^{m} (r_{ij} - r_{ji}), i = 1, 2, 3, \ldots m.$$

- **Step 8:** On the basis of the highest value of the net degree, finally we choose the optimal alternative by ranking all the $\alpha_i' s$, i.e,

$$\alpha_1 > \alpha_2 > \alpha_4 > \alpha_3.$$

Hence, we conclude that the site α_1 is the best site for the establishment of hydropower plant based on our proposed methodology and algorithm.

Remark: In step 4, we can replace the concept of energy by the concept of Laplacian energy for the evaluation of weights. In this case, we will be using the following formula for the calculation of weights:

$$w_k = (w_\mu^k, w_\eta^k, w_\nu^k) = \left(\frac{LE(G_\mu)_k}{\sum\limits_{l=1}^{k} LE(G_\mu)_l}, \frac{LE(G_\eta)_k}{\sum\limits_{l=1}^{k} LE(G_\eta)_l}, \frac{LE(G_\nu)_k}{\sum\limits_{l=1}^{k} LE(G_\nu)_l} \right); k = 1, 2, 3.$$

All the computations can similarly by performed for the evaluation process.

Important Comparative Remarks:

For the sake of justification in connection with the proposed technique, we consider two examples of problems of site selection which have been solved recently by different researchers. Gundogdu et al.[34] proposed Picture Fuzzy Linear Assignment Method and Jovicic et.al [122] proposed Picture Fuzzy ARAS Method to solve the site selection problems.

- In both the approaches, the decision matrix has been constructed by considering the available alternatives and the laid down criteria according to the respective needs.

- In step 4, we can replace the concept of energy by the concept of Laplacian energy for the evaluation of weights. In this case, we will be using the following formula for the calculation of weights:

$$w_k = (w_\mu^k, w_\eta^k, w_\nu^k) = \left(\frac{LE(G_\mu)_k}{\sum\limits_{l=1}^{k} LE(G_\mu)_l}, \frac{LE(G_\eta)_k}{\sum\limits_{l=1}^{k} LE(G_\eta)_l}, \frac{LE(G_\nu)_k}{\sum\limits_{l=1}^{k} LE(G_\nu)_l} \right); k = 1, 2, 3.$$

  All the computations can similarly by performed for the evaluation process.

- The difference is in the process of considering the vertices and edges where the vertices of the particular individuals and connecting edges would represent the mutual relationships and makes the situation easier to understand and interpret. The data of the problem under consideration fits for the picture fuzzy graph-theoretic approach.

- The advantage of our method over other existing methods are explained by following remarks.

PF Linear Assignment Method [34]	Initially the alternatives and the criteria-based table have been designed with the help of experts. Utilizing Picture Fuzzy Weighted Averaging operator (PFWA), the individual decision matrix has been accumulated and the elements have been computed with the picture fuzzy score function. Further, the rank frequency of the positive valued matrix/weighted matrix π is calculated. After solving these two matrices the solutions are obtained. The weight vector is dependent on the expert view.
PF ARAS Method [122]	The linguistic criteria weight matrices are considered with the help of decision makers. The picture fuzzy evaluation matrices have been evaluated and normalized to determine the picture fuzzy decision matrix. Further, in this process the picture fuzzy weighted decision matrix is calculated. In the end, the author calculated the picture fuzzy optimality function which leads to the defuzzified value of the optimality function and ultimately the utility degree of each alternative is obtained (for ranking).
Laplacian Energy of Picture Fuzzy Graph	But in our case the data is collected only for the graph after the graph is made according to the various criteria. We calculated the weight vector of the graph which makes it more reliable. The mutual relationships among the criteria and alternatives have been translated with the concept of Laplacian energy for the evaluation of the necessary weights.

5.4 Comparative Remarks, Advantages and Limitations of the Proposed Methodology

On the basis of the work proposed in the manuscript, we present some important comparative remarks & advantageous features behind the implementation of picture fuzzy graphs and their operations:

- As mentioned earlier, the incorporation of intuitionistic fuzzy sets and Pythagorean fu -zzy sets has some limitations and not able to capture the full information specification of the situation. Therefore, the additional components of the degrees of membership, neutral membership, non-membership and degree of refusal in case of the picture fuzzy sets certainly provide a wider coverage and wider geometrical span.

- In this way we find that the proposed graphs & operations are sufficiently capable to address the connected dependance due to the incompleteness of the information hav -ing the refusal factor in a more reliable way.

- The drawback in the existing literature of the intuitionistic fuzzy graphs and Pythag -orean fuzzy graphs is that the condition does not allow the experts/decision makers to allocate the membership values of their own choice (Refer Table 5.1). Somehow, this makes the experts bounded for giving their input in a particular defined domain. However, the proposed picture fuzzy graphs provide a generalization feature which make a strong impact.

Table 5.1: Concerns Raised in IFSs and PyFSs

R	C_1	C_2	C_3	C_4
C_1	$(1.0 + 0.0 + 0.0 = 1)$	$(0.40 + 0.20 + 0.69 > 1)$	$(0.36 + 0.19 + 0.79 > 1)$	$(0.56 + 0.17 + 0.62 > 1)$
C_2	$(0.68 + 0.20 + 0.44 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$	$(0.40 + 0.24 + 0.56 > 1)$	$(0.51 + 0.29 + 0.61 > 1)$
C_3	$(0.76 + 0.20 + 0.42 > 1)$	$(0.54 + 0.24 + 0.42 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$	$(0.48 + 0.14 + 0.77 > 1)$
C_2	$(0.49 + 0.17 + 0.68 > 1)$	$(0.59 + 0.29 + 0.53 > 1)$	$(0.77 + 0.17 + 0.38 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$

- The weights are being evaluated using the energy and the Laplacian energy which play a key role in the evaluation process. This is because the utility factors of the available alternatives are directly translated into the weights with the help of the energy.

5.5 Conclusions

The Pythagorean fuzzy graph-theoretic model and concepts are however sufficient to discuss the issues related with uncertainty, impreciseness & inconsistency of the information, but the containment of the refusal degree has not been considered. We have successfully put forward the novel notion of energy and Laplacian energy for the picture fuzzy graph along with the bounds on them. Through the proposed approach, we are certainly able to model accordingly and deal with the refusal component for providing a better geometrical span. The proposed concepts are well composed and clearly discussed with illustrative fuzzy graph examples. The implementation of the proposed algorithm has been successfully presented by taking a hydropower plant site selection problem into account.

Chapter 6

Conclusions & Future Work

In the present thesis, we have studied and proposed some new extension of fuzzy sets with various results and applications. The findings of the work carried out in the various chapters are being listed along with possible scope of future work:

- A novel concept of cohesive fuzzy set (CHFS) has been successfully proposed which has the dual benefits of complex fuzzy set with coverage of hesitant fuzzy set. Various properties and identities have also been proposed to increase the understanding of the concept.
- The applicability of cohesive fuzzy set has been explained in detail in case of reference signals using the concept of Inverse/Discrete Fourier transformation. The numerical example has also been presented using the proposed methodologies.
- Second application of cohesive fuzzy set has been presented in case of solar activity in which the proposed concept is utilized to obtain the interval contains maximum amount of sunspots.
- The proposed concept of cohesive has been proved to be very reliable and therefore will be of great help in solving various uncertainties problems in future.
- The novel concept of complex neutrosophic matrix has been proposed and the fundamentals of the concept have also been explained with the help of various operations for better clarity.
- The matrix norm and power convergence of complex neutrosophic matrix have been studied and discussed thoroughly.

- Further, new similarity measures have been presented and the property of positive definiteness of presented similarity measures have also been discussed for better clarity.
- The applicability of the proposed similarity measure have been presented in case of medical diagnosis.
- Various operators in case of complex fuzzy matrix have been discussed in detail for better understanding and a similarity measure has also been proposed.
- The applicability of the complex fuzzy matrix has been studied in case of signal identification and this also validates the use of proposed theory.
- The four exponential similarity measures have been proposed for the case of single valued neutrosophic set and in addition a classification problem has been presented.
- The applicability of proposed exponential measures has been presented in case of decision making problem.
- The 'useful' information measure with various other information measures have been presented and validated with the help of various theorems.
- The future work of the proposed concept can also been presented for the case of constrained optimization with a suitable applicability for the decision models.
- Laplacian energy for the case of picture fuzzy graph is calculated and through the proposed approach we are able to deal with the refusal component for providing a better geometrical span.
- The applicability of the proposed concepts has been applied in case of hydrogen power plant. Further, the concept of isomorphic graphs, planar graphs, dual graphs, regular graphs, etc., can analogously defined and applied in various application fields of engineering design, system science, networking etc. Also, these definitions can further be applicably enhanced to "*hesitant picture fuzzy graph*" and "*picture fuzzy soft graph*".

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