STUDY OF THE APPLICATIONAS OF DISCRETE LINEAR CHIRP TRANSFORM

Dissertation submitted in fulfillment of the requirements for the Degree of

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By

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DECLARATION

I hereby declare that the work reported in the M-Tech. dissertation entitled **“Study of the applications of Discrete Linear Chirp Transform”** submitted at **Jaypee University of Information Technology, Waknaghat, India,** is an authentic record of my work carried out under the supervision of **Dr. Sunil Datt Sharma**. I have not submitted this work elsewhere for any other degree or diploma.

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CERTIFICATE

This is to certify that the work reported in the M-Tech. dissertation entitled "**Study of the Applications of Discrete Linear Chirp Transform**", submitted by **Vivek Thakur** at **Jaypee University of Information Technology, Waknaghat**, **India,** is a bonafide record of his original work carried out under my supervision. This work has not been submitted elsewhere for any other degree or diploma.

Dr. Sunil Datt Sharma

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ABSTRACT

The discrete Fourier transform (DFT) plays a significant role in analyzing characteristics of stationary signals in the frequency domain in signal processing. The DFT can be implemented in a very efficient way using the fast Fourier transform (FFT) algorithm. However, many actual signals by their nature are non-stationary signals which make the choice of the DFT to deal with such signals not appropriate. Alternative tools for analyzing non-stationary signals come with the development of time-frequency distributions (TFD). The Wigner-Ville distribution is a time-frequency distribution that represents linear chirps in an ideal way, but it suffers from the problem of cross-terms which makes the analysis of such tools unacceptable for multi-component signals. Consequentially, for the analysis of the chirp signal Fractional Fourier Transform (FrFT) has been reported. The FrFT converts a chirp signal from time domain to another domain corresponding to fractional order α. It provides us with an additional degree of freedom (order of the transform α). Later the DLCT is discussed in the literature which is not a time- frequency transform but rather a frequency chirp-rate transform. It converts a non-sparse signal into sparse one using its property of modulation and duality.

In this dissertation, we have studied the applications of discrete linear chirp transform (DLCT) and its performance analysis with various other transforms like the fractional Fourier transform (FrFT) and the discrete cosine transform (DCT). The discrete linear chirp transform (DLCT) can be considered a generalization of the DFT to analyze non-stationary signals. The DLCT is a joint frequency chirp-rate transformation, capable of locally representing signals in terms of linear chirps. Important properties of this transform are discussed and explored. The efficient implementation of the DLCT is given by taking advantage of the FFT algorithm. Since DLCT transform can be implemented in a fast and efficient way, this would make a candidate to use it for many applications, including elimination of the cross-terms in the Wigner-Ville distribution, signal compression, filtering, signal separation, communication systems, and in chirp rate estimation. In this dissertation work we have studied the sparsity of DLCT and the application of DLCT in data compression. The simulation results in Matlab shows that the discrete linear chirp transform (DLCT) has higher sparsity and thus higher resolution, this property is used in data compression.

LIST OF ABBREVIATIONS

AF	Ambiguity Function
B	Frequency Spread
CLCT	Continuous Linear Chirp Transform
Cr	Compression Ratio
CS	Compressive Sensing
DCCT	Discrete Cosine Chirp Transform
DCFT	Discrete Cosine Fourier Transform
DCT	Discrete Cosine Transform
DFrFT	Discrete Fractional Fourier Transform
DFT	Discrete Fourier Transform
DLCT	Discrete Linear Chirp Transform
DSP	Digital Signal Processing
EMD	Empirical Mode Decomposition
FFT	Fast Fourier Transform
FM	Frequency Modulation
FrFT	Fractional Fourier Transform
FT	Fourier Transform
ICLCT	Inverse Continuous Linear Chirp Transform
IDCCT	Inverse Discrete Cosine Chirp Transform
IDCFT	Inverse Discrete Cosine Fourier Transform
IDFrFT	Inverse Discrete Fractional Fourier Transform
IDLCT	Inverse Discrete Linear Chirp Transform
IFrFT	Inverse Fractional Fourier Transform
IMF	Intrinsic Mode Function
ISAR	Inverse Synthetic Aperture Radar
LFM	Linear Frequency Modulator
MSE	Mean Square Error
PCT	Polynomial Chirp Transform
SAR	Synthetic Aperture Radar

SNR	Signal to Noise Ratio
STFT	Short Time Fourier Transform
TB	Time Bandwidth Product
TF	Time Frequency
TFD	Time Frequency Distribution
TFM	Time Frequency Method
WT	Wavelet Transform
WVD	Wigner Ville Distribution

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 OVERVIEW

The term chirp is a short pulse, high-pitched sound. This pulse is called a chirped pulse. Scientifically, the term chirp means a wave whose instantaneous frequency fluctuates over time. Often chirps arise in nature as a result of the Doppler effect, the phenomenon by which the perceived frequency of a wave is changed whenever the wave is coming from or reflecting-off a moving body. Chirps come in many frequency sweep such as : linear chirp, quadratic chirp, logarithmic-chirp, etc.

Application of chirp signal :

- Radar
- Sonar and
- Spread spectrum communication

Over the past several decades, the field of digital signal processing has been significantly contributing to the different areas of human endeavours in one way or the other. While conventional signal processing by and large expects stationary behaviour of the signal during the window of observation, it is worthwhile to note that, most of the man-made and natural signals are non-stationary in nature and hence time-frequency methods are more suitable than conventional Fourier based signal processing techniques.

1.2 HISTORY OF FOURIER TRANSFORM

Fourier Transform (FT) is one of most common spectral analysis technique. Fourier transformation maps one-dimensional time domain signal into a one dimensional frequency domain signal, i.e., the signal spectrum. Although, the Fourier transform provides the signal's spectral content, it fails to indicate the time location of the spectral components, which is important, for example, when we consider non-stationary or time-varying signals. In order to describe these signals, time-frequency representations are used. A time-frequency representation maps one-dimensional time domain signal into a two-dimensional function of time and frequency. Prime reason for the failure of FT is "the analyzing function'' which are complex sinusoid, are spread throughout the time. So, if we want to capture local feature in time, we need basis function that are highly localized in time.

The Fourier transform is the basis for some of the time-frequency representations. The Fourier transform decomposes a signal (a function of time) into the frequencies that make it up. The Fourier transform of a function of time itself is a complex-valued function of frequency, whose absolute value represents the amount of that frequency present in the original function, and whose complex argument is the phase offset of the basic sinusoid in that frequency. The Fourier transform is called the frequency domain representation of the original signal. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time. A time signal is decomposed into its different frequency components by calculating the Fourier integral. Mathematically, Fourier transform of a function $x(t)$ is as given below

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{1.1}$$

In above equation $x(t)$ is a time domain signal, $X(f)$ is the Fourier transform of an integrable function, f is the value of the angular frequency, j is the imaginary number.

To compute the $X(f)$, it is needed to integrate $x(t)$ overall time. Mathematically, due to both sine waves and cosine waves are significant in the whole time domain, so Fourier transform is available at any given time. This means that during the whole intervals, Fourier transform cannot provide simultaneous time, frequency localization and the Fourier coefficients (amplitude) which are depended on the behaviour of the function.

Inverse Fourier transform is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \tag{1.2}$$

1.3 TIME-FREQUENCY (TF) ANALYSIS

There are three ways to explore the information about any signal in literature.

- *Time domain representation*: Any signal can be described naturally as a function of time. It gives the information about amplitude variation with respect to time. But it tends to obscure information about frequency, because it assumes that the two variables time and frequency are mutually exclusive and orthogonal.

- *Frequency domain representation*: Any practical signal can be represented in the frequency domain by its Fourier transform. The Fourier transform is in general a complex quantity. Its magnitude is called the magnitude spectrum and phase is defined as the phase spectrum [51]. The square of the magnitude spectrum is the energy spectrum and shows signal energy distribution over the frequency domain. But the magnitude spectrum tells about frequencies that are present in the signal, the "time of arrival" of those frequencies. Therefore, frequency domain representation hides the information about timing, as FT of a signal does not mention the variable time.
- *Time-frequency representation*: As time and frequency domain representations are inadequate to give all the information possess by the signal, an obvious solution is to seek a representation of the signal as a 'two-variable' function or distribution whose domain is two-dimensional time-frequency space. Its constant-time cross section shows the frequencies present at any time and constant-frequency cross-section shows the times at which those frequencies are present [52], [53]. Such a representation is called time-frequency distribution (TFD). Similarly, the plane in which signal is analyzed is defined as time-frequency plane.

1.4 TIME FREQUECY ANALYSIS METHODS (TFM)

TFMs are used to analyze a signal in time and frequency domains simultaneously. A straight forward extension of the conventional Fourier transform, called Short-Time Fourier transform (STFT) attempts to bring out the evolutionary nature of the signals, both in time and frequency. Other than STFT, TFMs have been largely limited to academic research because of the complexity of the algorithms and the limitations in computing power. TFMs are mainly of two categories:

(i) Linear TFMs such as STFT, WT.

(ii) Quadratic TFMs, also called Energy Distributions such as WVD.

In contrast with the Linear TFMs, which decompose the signal on elementary components, the purpose of the Quadratic TFMs is to deal out the energy of the signal over the two variables viz. time and frequency. Among the Quadratic TFMs, WVD is the simplest and the most powerful, in representation and characterization.

1.4.1 SHORT-TIME FOURIER TRANSFORM (STFT)

Short-Time Fourier transform (STFT) is known to be the first TFM that was applied in practical applications like speech processing systems, ISAR imaging, order tracking etc.

Fourier analysis becomes inadequate when the signal contains non-stationary or transitory characteristics like transients, trends etc. In an effort to correct this, Dennis Gabor [56] adapted the Fourier transform to analyze small sections of the signal at a time. In order to introduce time-dependency in the Fourier transform, a simple and intuitive solution consists in pre-windowing the signal to be analyzed x(t) around a particular time t, calculating its Fourier transform, and doing that for each time instant t. The resulting transform called the Short-Time Fourier transform, is therefore defined as:

$$STFT(\tau, f) = \int_{-\infty}^{\infty} x(t) g_{t,f^*}(t) dt = \int_{-\infty}^{\infty} x(t) g(t-\tau) e^{-j2\pi ft} dt \qquad (1.3)$$

where, $g(t)$ is a short time analysis window, localized around $t = 0$ and $f = 0$. Because multiplication by the relatively short window $g(t-\tau)$ effectively suppresses the signal outside a neighbourhood around the analysis time point $t = \tau$, the STFT is a local spectrum of the signal $x(t)$. This relation expresses that the total signal can be decomposed as a weighted sum of elementary waveforms $g_{t,f}(t) = g(t-\tau)e^{j2\pi ft}$. These waveforms are obtained from the window $g(t)$ by a translation in time and a translation in frequency. The corresponding group of translation in both time and frequency is called Weyl-Heisenberg group.

A time-localized Fourier transform performed on the signal within the window as shown in Figure 1.1. Subsequently, the window is removed along the time, and another transform is performed. The signal segment within the window function is assumed to be stationary. As a result, the STFT decomposes a time signal into a 2D time-frequency domain, and variations of the frequency within the window function are revealed While the STFT's compromise between time and frequency information can be useful, the drawback is that once a particular size is chosen for the time window, it remains the same for all frequencies. The time resolution of the STFT is proportional to the effective duration of the analysis window $g(t)$. Similarly, the frequency resolution of the STFT is proportional to the effective bandwidth of the analysis window $g(t)$.

Consequently, for the STFT, we have a trade-off between the time and frequency resolutions. On one hand, a good time resolution requires a short window $g(t)$. On the other hand, a good frequency resolution requires a narrow-band filter i.e. a long window $g(t)$. This is the major drawback of STFT.

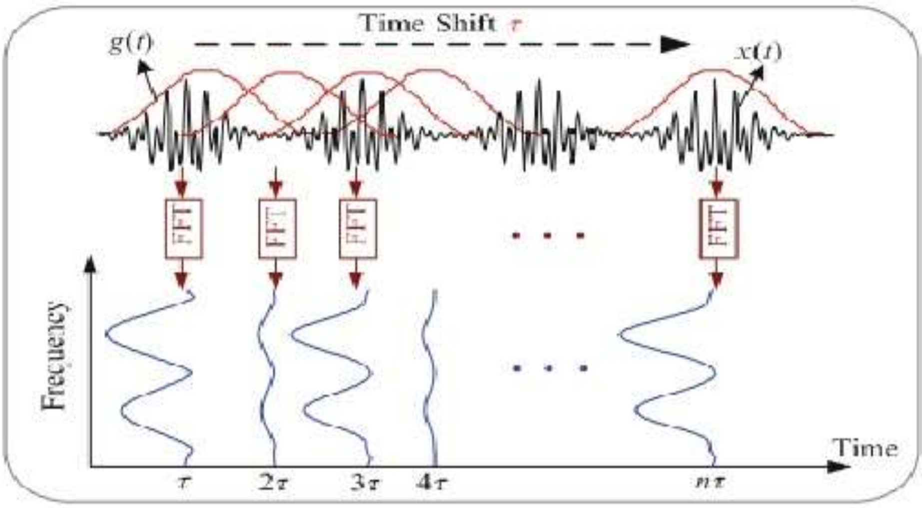


Figure 1.1 : Time localized Fourier transform.

1.4.2 WAVELET TRANSFORM (WT)

The wavelet transform is similar to the Fourier transform (or much more to the windowed Fourier transform) with a completely different merit function. The main difference is that: Fourier transform decomposes the signal into sines and cosines, i.e. the functions localized in Fourier space; in contrary the wavelet transform uses functions that are localized in both the real and Fourier space. Generally, the wavelet transform can be expressed by the following equation:

$$WT(s,\tau) = \langle x, \Psi_{s,\tau} \rangle = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} x(t)\Psi^*\left(\frac{t-\tau}{s}\right)dt \qquad (1.4)$$

where τ shifts time, s modulates the width (not frequency), and $\Psi(t)$ is mother wavelet.

By comparing the signal with a set of functions obtained from the scaling and shift of a base wavelet, it is realized as shown in Figure 1.2.

Continuous Wavelet Transform is a transform by which signals can be modeled as a linear combination of translations and dilations of a simple oscillatory function of finite duration called a mother wavelet $\Psi(t)$. It provides very good spectral resolution at low frequencies at the expense of temporal resolution and very good temporal resolution at high frequencies at the expense of spectral resolution. This distinct feature of the Wavelet Transform makes it suitable for analyzing non-stationary acoustic signals. Wavelet transforms have been widely applied to the problem of transient detection and processing, primarily because the transform basis functions provide good time localization and it involves the tracking of local transform maxima across analysis scales.

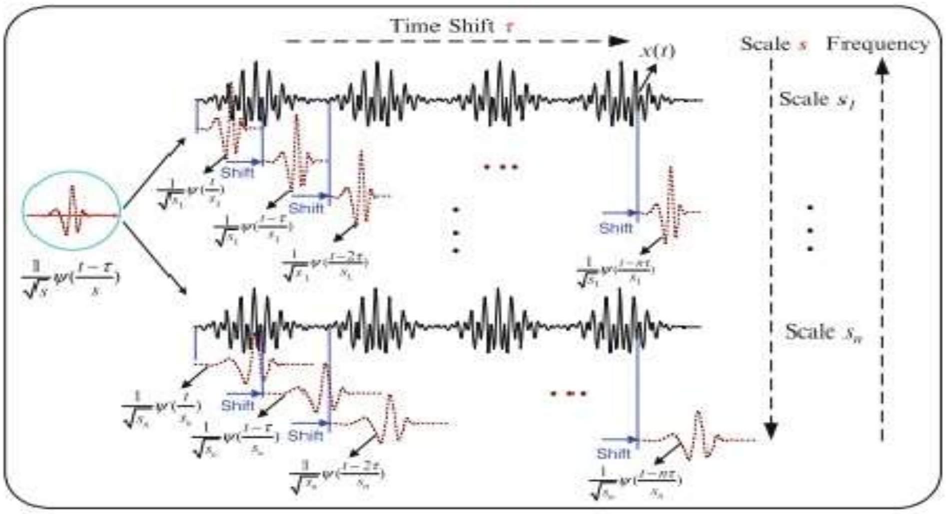


Figure 1.2 : Wavelet transform with different scaling shift of basic wavelet.

To overcome the problems of redundancy and computational load, Mallat's filter bank implementation called discrete Wavelet transform is now widely used. According to multi scale filtering structure, Wave packet transform can divide the entire time-frequency plane into subtle tilings, while the classical WT can only find its finer analysis for lower-band only. Hence Discrete Wave packet transform is more competent to handle wide-band and high-frequency narrow band signals like transients. As a tool to process data from multiple channels, even this transform is computationally intensive. However, Win Sweldon's Lifting based implementation is a practical solution for the fast implementation of Wavelet and Wave packet transforms.

1.4.3 WIGNER VILLE DISTRIBUTION

The Wigner-Ville distribution (WVD) is one member of the Cohen class which is a simple yet powerful tool to analyze the Doppler history of SAR signals [60]. Wigner originally developed the distribution for use in quantum mechanics in 1932, and it was introduced for signal analysis by Ville sixteen years later. To obtain the Wigner-Ville distribution at a particular time, we add up pieces made from the product of the signal at a past time multiplied by the signal at a future time. The continuous WVD of a signal is derived as [60]:

$$WVD(t, f) = \int_{-\infty}^{\infty} s\left(t - \frac{\tau}{2}\right) s^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \qquad (1.5)$$

The WVD can be regarded as the TF distribution offering the best resolution in the form of delta-pulses along the instantaneous frequency of a signal [60]. Additionally, the lack of smoothing maximally conserves the information content of the signal. The WVD is always real-valued, preserves time and frequency shifts, and satisfies the marginal properties. A more thorough description of the properties of the WVD is offered in [54], [55], [60].

One disadvantage is that problems arise in using the WVD for signals consisting of multiple components. Since it is a non-linear transformation, the WVD signal is not simply the sum of the WVD of each part. For instance, given a signal composed of two parts s_1 and s_2 such that

$$s(t) = s_1(t) + s_2(t) \qquad (1.6)$$

the spectrum of s is the sum of the Fourier transforms of each component:

$$s = s_1 + s_2 \qquad (1.7)$$

However, the energy density (which is related to the WVD of the signal) is not the sum of the energy densities of each part [54]:

$$|s|^2 = (|s_1| + |s_2|)^2 \qquad (1.8)$$

$$= |s_1|^2 + |s_2|^2 + 2R\{|s_1| * |s_2|\}$$

$$\neq |s_1|^2 + |s_2|^2 \qquad (1.9)$$

where the $R\{\cdot\}$ operation retains the real component of its argument.

The non-linearity of the WVD emphasizes the need to remove all clutter contributions to the signal prior to computing the TF transform. If clutter is not removed, even if the signal occupies a bandwidth well-separated from the clutter, the WVD cross-terms may obscure the target signal [55]. If the clutter is removed but the processed signal data contains multiple moving targets, cross-terms between these signals will still be present in the WVD. Generally, detection and tracking of the instantaneous frequency for multiple targets is completed by combining the WVD with the Hough transform. The Hough transform is typically used for detecting straight lines in noisy imagery, although it may also be used to find higher-order polynomials (such as parabolas) traced out by accelerating targets in the time-frequency domain.

1.5 DISSERTATION ORGANIZATION

This dissertation includes five chapters. An outline of each chapter is given below:

Chapter 1st gives an overview of various time-frequency analysis techniques and the need of time-frequency analysis.

Chapter 2nd is dedicated to the literature survey. The research papers which are relevant to this dissertation are discussed here.

Chapter 3rd presents a study of linear chirp signal and various sparsity analysis tools. It also discussed the performance of DLCT, DFrFT and DCT.

Chapter 4th presents the application of DLCT for data compression. Data compression is typically done by transforming the signal into frequency and thresholding it to keep the more significant components. Sparseness of the signal, in either time or frequency, is required for the convex optimization in compressive sensing to perform well.

Chapter 5th concludes this dissertation, summarizing the major results and offering suggestions for future work on this topic.

CHAPTER 2

LITERATURE SERVEY

In order to start the dissertation, the first step is to study the research papers that have been published by other researchers. The papers that are related to this title are chosen and studied. With the help of this literature review, it gives more clear understanding to write this dissertation.

The topic of time-frequency methods is one of the modern DSP tools for nonstationary signal processing. Like all fields and particularly emerging ones, it has a plethora of different motivations. Many applications are reported in the fields of speech and image processing, communications, radar etc.

The application of Fractional Fourier Transform (FrFT) in parameters estimation of radar echo is the latest topic of interest. Many Time–Frequency methods are proposed including FrFT in the field of radar signal processing.

Namias *et al.* introduced Fractional Fourier Transform in the field of quantum mechanics for solving some classes of differential equations efficiently [57]. Since then, a number of applications of Fractional Fourier Transform have been developed, mostly in the field of optics. The FrFT has the ability to process chirp signals better than the conventional Fourier Transform. FrFT is basically a time-frequency distribution, a parameterized transform with parameter α, related to the chirp-rate. It provides us with an additional degree of freedom (order of the transform), which in most cases results in significant gains over the classical Fourier transform. It is well known that in sonar systems, chirp processing can be applied in a number of areas. Some FrFT applications are reported in radars.

Ozaktas *et al*. [58], [64] have come up with a discrete implementation of Fractional Fourier Transform. Like Cooley-Tukey's FFT, this efficient algorithm computes FrFT in O(NlogN) time which is about the same time as the ordinary FFT. Hence, in applications where FrFT replaces ordinary Fourier transform for performance improvement, no additional implementation cost will occur.

Candan *et al*. [64] gives a satisfactory definition of the discrete FrFT that is fully consistent with the continuous transform. This definition has the same relation with the DFT as the continuous FrFT has with the ordinary continuous Fourier Transform.

Almeida *et al.* [65] has interpreted FrFT as a rotation in the time-frequency plane. This paper describes its relationship with other TFMs such as WVD, AF, STFT and spectrogram, which support's the FrFT's interpretation as a rotational operator.

Ozaktas *et al.* [52] has interpreted filtering method in fractional Fourier domains, which may enable significant reduction of MSE compared to ordinary Fourier domain filtering. This reduction comes at essentially no additional computational cost because of the availability of the efficient algorithm for computing FrFT.

Capus *et al.* [61], [62] gives the short-time implementation of FrFT. STFT variants of FrFT can be implemented in two ways, depending on how the optimum alpha is chosen. The optimum alpha can be selected for the whole data block, or one for each processing block length. This shows improvements in time-frequency resolutions with bat signals, linear and non-linear chirps. Individual chirps in a mixture of chirps can be extracted using FrFT by a filtering and reconstruction technique. Both linear as well as non-linear chirps can be extracted by this method.

Song *et al.* [66] gives two iterative interpolation algorithms for the parameter estimation of linear frequency modulation (LFM) signal using fractional Fourier transform (FrFT). The estimated parameter of an LFM signal can be obtained by locating the peak of the periodogram in the FrFT domain. These algorithms improve the accuracy of parameter estimation by employing the FrFT coefficients relative to the true parameters and applying interpolation algorithms iteratively to refine the parameter estimation approach. The proposed algorithms can utilize more information from FrFT results, thereby achieving improvements in either accuracy or efficiency.

Alkishriwo *et al*. [43] we have studied Compressive sensing to simplify the frequency transformation and thresholding steps, commonly done in data compression, into one. Sparseness of the signal, in either time or frequency, is required for the convex optimization in compressive sensing to perform well. Although sparseness of certain signals, in either time or frequency, is guaranteed by the uncertainty principle signals composed of chirps are not however sparse in either domain. This paper proposes an orthogonal linear-chirp transform, the discrete linear chirp transform (DLCT), to represent any signal in terms of linear chirps, with modulation and dual properties. Using the DLCT the sparseness of the signal in either time or frequency can be assessed, and if not sparse in neither of these domains, the modulation and dual properties of the DLCT provide a way to transform the signal into a sparse signal.

Alkishriwo *et al*. [44] gives the discrete linear chirp transform (DLCT) for decomposing a non-stationary signal into intrinsic mode chirp functions. The decomposition of a signal into a finite number of intrinsic mode functions (IMFs) was introduced by the empirical mode decomposition (EMD). It exploits the local time-scale signal characteristics of the signal and provides spectral estimates obtained via the Hilbert transform. Although efficient, the EMD does not provide an analytic representation of the IMFs and is susceptible to noise and to closeness or overlap of the frequency of the IMFs. Using linear chirps as IMFs, the DLCT, a joint frequency instantaneous frequency procedure, provides a parsimonious local orthogonal representation of nonstationary signals. Moreover, the DLCT allows a parametric estimation of the instantaneous frequency of the signal that is robust to noise and to closeness or overlap in the instantaneous frequency of the modes. More importantly, the DLCT can be used to represent and process signals that are sparse in a joint time–frequency sense.

Hari *et al*. [46] we have studied a method for non–stationary decomposition using the Discrete Linear Chirp Transform (DLCT) for FM Demodulation. Non–stationary signal decomposition can be done using either the empirical mode decomposition (EMD) or the Discrete Linear Chirp Decomposition (DLCT) methods. These methods decompose non-stationary signals using local time-scale signal characteristics. While the EMD decomposes the signal into a number of intrinsic mode functions (IMFs), the DLCT obtains a parametric model based on a local linear chirp model. Analytically the DLCT considers localized zero–mean linear chirps as special IMFs. The DLCT is a joint frequency instantaneous–frequency orthogonal transformation that extends the discrete Fourier transform (DFT) for processing of non–stationary signals. FM demodulation is commonly done by computing the signal derivative to convert it into an amplitude demodulation. This paper shows that the demodulation can be approached with the EMD and the DLCT and that the second method provides better results.

Alkishriwo *et al*. [6] have studied an algorithm based on the fractional Fourier transform to separate the different components of a signal in the Wigner time-frequency domain. Its target is to obtain a compressed representation for such a signal containing a minimal number of parameters. The procedure gets rid of the noise and the cross-terms after separating the signal components. For the signals under consideration having chirps and sinusoids, the fractional Fourier transform is used to rotate the components to obtain a sinusoidal or impulsive sparse representation. The procedure relies on filtering or windowing after obtaining the order of the fractional Fourier transform for each of the components. This approach is very effective in extracting the linear chirps and sinusoids from the noise and in eliminating the cross-terms from the Wigner distribution.

Alkishriwo *et al.* [45] studied a signal compression technique to decrease transmission rate (increase storage capacity) by reducing the amount of data necessary to be transmitted. The discrete linear chirp transform (DLCT) is a joint frequency instantaneous-frequency transform that decomposes the signal in terms of linear chirps. The DLCT can be used to transform signals that are not sparse in either time or frequency, such as linear chirps, into sparse signals. On the basis of the reviewed literature following objectives for the study have been selected. These objectives are :

- Study the performance of discrete linear chirp transform (DLCT) for sparsity analysis.
- Performance analysis of DLCT for data compression.

CHAPTER 3

SIGNAL PROCESSING TOOLS FOR SPARSITY ANALYSIS

Sparsity or compressibility reflects the fact that information carried by certain signal is much smaller than their bandwidth. Most signals are not sparse in the time domain, so linear transformations are used to make them sparse in either time or frequency using certain basis. In this chapter the signal processing tool FrFT and DLCT have been studied for sparsity analysis.

3.1 FRACTIONAL FOURIER TRANSFORM (FrFT)

Chirps are signals which exhibit a change in instantaneous frequency with time (either linear or non-linear) and are of particular interest in sonar, radars, acoustic communications, seismic surveying, ultrasonic applications, etc. The potential of FrFT lies in its ability of FrFT to process chirp signals better than the conventional Fourier Transform. The transform absorbs the chirp parameters in its kernel by a parameter α.

Namias *et al.* introduced Fractional Fourier Transform [57] in the field of quantum mechanics for solving some classes of differential equations efficiently. Later, Ozaktas *et al* [58] came up with the discrete implementation of FrFT. Since then, a number of applications of FrFT have been developed, mostly in the field of optics. However, it remains relatively unknown in signal processing. As a generalization of the ordinary Fourier transform, the FrFT is only richer in theory and more flexible in applications at low cost. Therefore, the transform is likely to have something to offer in every area in which Fourier transforms and related concepts are used. The FrFT is basically a time- frequency distribution. It provides us with an additional degree of freedom (order of the transform), In most of the cases fractional Fourier transform perform better than the classical Fourier transform. With the development of FrFT and related concepts, we see that the ordinary frequency domain is merely a special case of a continuum of fractional Fourier domains. So in every area in which Fourier transforms and frequency domain concepts are used, there exists the potential for improvement by using the FrFT.

3.1.1 BASIC CONCEPT OF FRACTIONAL TRANSFORM

Before formally defining the Fractional Fourier Transform, we want to know that "What is a fractional transform?" and "How can we make a transformation to be fractional?" First we see a transformation T, we can describe the transformation as following:

$$T\{f(x)\} = F(u) \tag{3.1}$$

where, f and F are two functions with variables x and u respectively. As seen, we can say that F is a T transform of f. Now, another new transform can be defined as below:

$$T^{\alpha}\{f(x)\} = F_{\alpha}(u) \tag{3.2}$$

We call T^{α} here the "α-order fractional T transform" and the parameter α is called the "fractional order". This kind of transform is called "fractional transform".

Which satisfy following constraints:

1. Boundary condition:

$$\begin{aligned} T^{0}\{f(x)\} &= f(u) \\ T^{1}\{f(x)\} &= F(u) \end{aligned} \tag{3.3}$$

2. Additive property:

$$T^{\alpha}\{T^{\beta}\{f(x)\}\} = T^{\alpha+\beta}\{f(x)\} \tag{3.4}$$

3.1.2 LINEAR CHIRP SIGNAL

A linear chirp signal, its phase and its instantaneous frequency are given by the following equations. Two parameters completely define a chirp namely the start frequency f_0 and slop of the chirp.

$$\text{Chirp signal} = e^{j\left(kt^2 + f_0 t + c\right)}$$

$$\text{Phase} = kt^2 + f_0 t + c$$

$$\text{Instantaneous frequency} = 2kt + f_0$$

where, f_0, c and $2k$ are the starting frequency, initial phase and chirp rate or slope respectively.

3.1.3 DEFINATION OF FrFT

The continuous fractional Fourier transform (FrFT) is defined as [51], [52], [56]

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\, dt \tag{3.5}$$

Where $-\pi/2 < \alpha < \pi/2$ is called the fractional order and $K_\alpha(t,u)$ is the kernel of the transformation which is defined as :

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{1 - j\cot(\alpha)}{2\pi}} \exp\left(j\dfrac{t^2 + u^2}{2}\cot\alpha - jut\csc(\alpha) \right), \alpha \neq n\pi \\ \delta(t-\tau), \alpha = 2n\pi \\ \delta(t+\tau), \alpha = (2n+1)\pi \end{cases} \tag{3.6}$$

When $\alpha = 0$, the FrFT of the signal $x(t)$ is the signal itself, and if $\alpha = \pm\pi/2$, the FrFT becomes the Fourier transform of the signal. That is why it is considered a generalization of the Fourier transform.

The signal $x(t)$ can be obtained by the Inverse Fractional Fourier Transform (IFrFT) as :

$$x(t) = \int_{-\infty}^{\infty} X_\alpha(u) K_\alpha^*(t,u)\, du \tag{3.7}$$

Where "*" stands for complex conjugate.

FrFT computation can be interpreted as a sequence of steps viz. a multiplication by a chirp in one domain followed by a Fourier transform, then multiplication by a chirp in the transform domain and finally a complex scaling. So, chirps form the basis functions of FrFT. There are various other definitions of the FrFT. Of all these, the definition given above is particularly desirable because of its many properties and the relation to the classical Fourier transform.

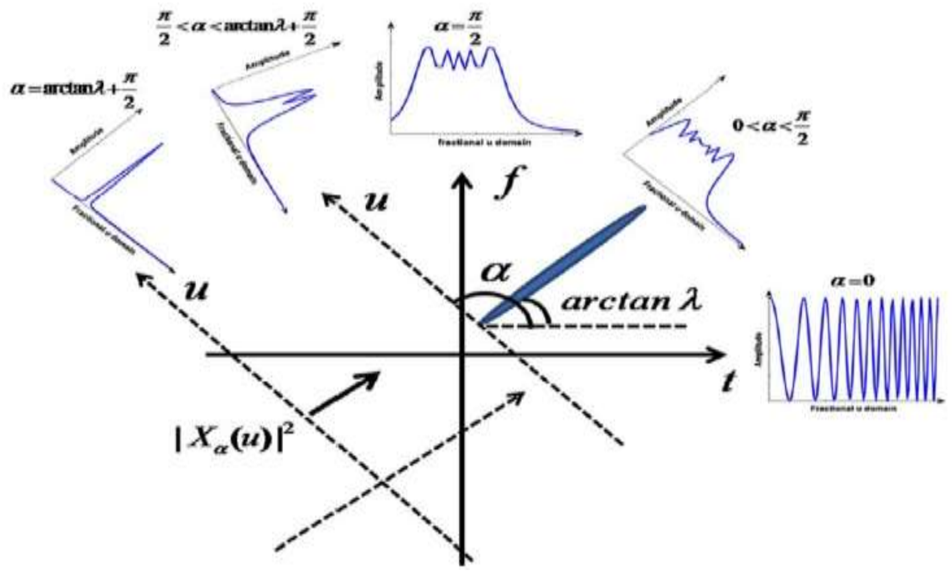


Figure 3.1 : FrFT of the chirp signal with different order of transform.

3.1.4 PROPERTIES OF FRACTIONAL FOURIER TRANSFORM

In this section, we list some fundamental properties of FrFT :

(1) Linearity

Let F^{α} denotes the αth order fractional operator. Then

$$F^{\alpha}\left[\sum_l b_l f_l(u)\right] = \sum_l b_l F^{\alpha}\left[f_l(u)\right] \tag{3.8}$$

(2) Integer order

When α is equal to an integer l, the α^{th} order fractional Fourier transform is equivalent to the l^{th} integer power of the ordinary Fourier transform. It means that

$$F^{\alpha} = (F)^{\alpha} \tag{3.9}$$

Moreover, it has following relation

$$F^2 = P \qquad \text{(parity operator)} \tag{3.10}$$

$$F^3 = F^{-1} = (F)^{-1} \qquad \text{(inverse transform operator)} \tag{3.11}$$

$$F^4 = F^0 = I \qquad \text{(identity operator)} \tag{3.12}$$

$$F^{j} = F^{j \bmod 4} \tag{3.14}$$

(3) Inverse

$$\left(F^{\alpha}\right)^{-1} = F^{-\alpha} \tag{3.15}$$

(4) Index additivity

$$F^{\alpha_1} F^{\alpha_2} = F^{\alpha_1 + \alpha_2} \tag{3.16}$$

(5) Commutativity

$$F^{\alpha_1} F^{\alpha_2} = F^{\alpha_2} F^{\alpha_1} \tag{3.17}$$

(6) Associativity

$$\left(F^{\alpha_1} F^{\alpha_2}\right) F^{\alpha_3} = F^{\alpha_1}\left(F^{\alpha_2} F^{\alpha_3}\right) \tag{3.18}$$

(7) Time reversal

Let P denotes the parity operator. $P[f(u)] = f(-u)$, then

$$F^{\alpha} P = P F^{\alpha} \tag{3.19}$$

$$F^{\alpha}[f(-u)] = f_{\alpha}(-u) \tag{3.20}$$

3.1.5 DISCRETE FEACTIONAL FOURIER TRANSFORM (DFrFT)

Many authors have proposed a discrete fractional Fourier transform (DFrFT) [12], [13]. Different approaches are available to evaluate DFrFT, which can be divided into four different classes on the basis of their methodology of evaluation. These classes are :

i. Eigenvector based method
ii. Sampling based method
iii. Linear combination method
iv. Weighted summation based method

The discrete fractional Fourier transform (DFrFT) is defined in terms of a particular set of eigenvectors

$$X_{\alpha}(\rho)=\sum_{n=0}^{N-1}K_{\alpha}(n,\rho)x(n) \tag{3.21}$$

where the kernel $K_{\alpha}(n,\rho)$ of the transformation has the following spectral expression

$$K_{\alpha}(n,\rho)=\sum_{k\in M}v_k(\rho)e^{-j\alpha k}v_k(n) \tag{3.22}$$

Where $v_k(n)$ is the kth discrete Hermite-Gaussian function as defined in [12] and $M=(0,\cdots\cdots,N-2,N-N\,\mathrm{mod}\,2)$.

The signal $x(n)$ can be recovered using the inverse discrete fractional Fourier transform (IDFrFT) as :

$$x(n)=\sum_{\rho=0}^{N-1}K_{\alpha}^{*}(n,\rho)X_{\alpha}(\rho) \tag{3.23}$$

The most important property of FrFT is the rotation property [14], [15]. It can be used to rotate a linear chirp in the time-frequency plane to become a sinusoid or an impulse by setting the fractional order (α) to an appropriate value-which is the fractional order that corresponds to the chirp rate of the signal. Now, we have to find the connection between the chirp-rate γ and the fractional order α of the FrFT.

For a discrete signal $x(n)$, we can define the relation between the fractional order (α)and the chirp rate (γ) as :

$$\alpha=-\tan^{-1}\left(\frac{1}{2\gamma}\right) \tag{3.24}$$

If x(t) is a continuous linear chirp given by

$$x(t)=\exp(j(\gamma t^2+\Omega t))$$

Substitute $x(t)$ into equation (3.5) we get

$$X_{\alpha}(u)=\sqrt{\frac{1-j\cot(\alpha)}{2\pi}}e^{j\frac{u^2}{2}\cot\alpha}\int_{-\infty}^{\infty}e^{j(\cot\alpha+2\gamma)\frac{t^2}{2}}\times e^{-j(u\csc\alpha-\Omega)t}dt$$

$$= \sqrt{\frac{1 - j\cot\alpha}{a}} \exp(j\frac{u^2}{2}\cot\alpha)\exp(\frac{b^2}{2a})$$

Where $a = -j\cot\alpha - j2\gamma$ and $b = ju\csc\alpha - j\Omega$. $|X_\alpha(u)| \to \infty$, when

$$\cot\alpha + 2\gamma = 0 \tag{3.25}$$

From the above equation, we can write the relation between α and γ as

$$\alpha = -\tan^{-1}(\frac{1}{2\gamma}) \tag{3.26}$$

The signal $x(n)$ can be defined in discrete form as

$$x(n) = \exp(j\frac{2\pi}{N}(\beta n^2 + kn))$$

So, we can write the relation between the discrete chirp rate β and the fractional order α as

$$\alpha = -\tan^{-1}(\frac{1}{2\beta}) \tag{3.27}$$

The relation between α and γ was shown geometrically in [16], [17]. Figure 3.2 illustrates the plot of equation (3.27).

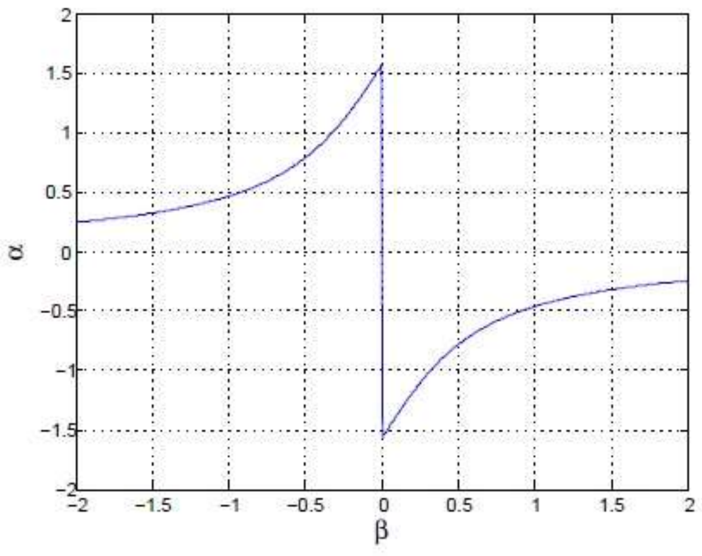


Figure 3.2 : Relation between the fractional order α and the discrete chirp rate β

3.2 DISCRETE CHIRP FOURIER TRANSFORM

The discrete chirp-Fourier transform (DCFT) was defined in [12]. Assumed a signal $x(n)$ of length N, the discrete chirp-Fourier transform (DCFT) of this signal is

$$X_c(k,r) = \frac{1}{\sqrt{N}} \sum x(n) \exp(-j\frac{2\pi}{N}(rn^2 + kn)) \quad (3.28)$$

$$0 \le r,k \le N-1$$

where k represents the frequencies and r is an arbitrarily fixed integer that represents the chirp rates. The DCFT is the same as the DFT when $r = 0$. The inverse discrete chirp transform (IDCFT) is given as

$$x(n) = \exp(j\frac{2\pi}{N}rn^2)\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} X_c(k,r)\exp(j\frac{2\pi}{N}kn) \quad (3.29)$$

Where $0 \le n \le N-1$

The DCFT approximates the chirp rate by integer numbers r . So, when using the DCFT to detect a chirp signal, the discrete chirp rate r_0 of the signal should be an integer to assure that the parameter can be matched and that the peak will not be lost. This restriction affects the practical applications of the DCFT.

3.3 DISCRETE COSINE TRANSFORM

Discrete cosine transform (DCT) of a signal $x(n)$ is given by

$$Y(k) = w(k)\sum_{n=1}^{N} x(n)\cos\left(\frac{\pi}{2N}(2n-1)(k-1)\right) \quad (3.30)$$

Where $k = 1,2,\cdots\cdots N$, N is the length of signal $x(n)$ and $w(k)$ is given by

$$w(k) = \begin{cases} \frac{1}{\sqrt{N}}, k = 1 \\ \sqrt{\frac{2}{N}}, 2 \le k \le N \end{cases}$$

DCT is clearly related to the DFT. We can often reconstruct a sequence very accurately only from a few coefficients, a useful property for the applications requiring data reduction.

3.4 DISCRETE LINEAR CHIRP TRANSFORM

3.4.1 LINEAR CHIRP BASIS

The term "chirp" derives from the bird chirp or cricket sounds- a short pulse, high-pitched sound. This pulse is called a chirped pulse. Scientifically, the term chirp means a wave whose instantaneous frequency fluctuates over time. Chirps come in many frequency sweep forms: linear chirp, quadratic chirp, logarithmic-chirp, etc.

A linear chirp is a function whose frequency changes linearly with time. For example, while a wave function of the form $\exp(j\Omega_0 t)$ has constant frequency Ω_0, the chirp $\exp(j(\gamma_0 t^2 + \Omega_0 t))$ has an instantaneous frequency $\Omega_0 + 2\gamma_0 t$ at time $t \in R$. Often chirps arise in nature as a result of the Doppler effect, the phenomenon by which the perceived frequency of a wave is changed whenever the wave is coming from or reflecting off a moving body. Chirps have historically been of great interest in applications such as sonar and radar. Therefore, we need to use linear chirp bases instead of the classical Fourier bases because they are more suitable for representing the frequency changes of Non-stationary signals.

3.4.2 CONTINUOUS LINEAR CHIRPS

Let the space $L^2(R)$ be a Hilbert space of complex functions such that

$$\|x\| = \int_{-\infty}^{\infty} |x(t)|^2 dt < +\infty$$

The inner product of $\langle x, y\rangle \in L^2(R)$ is defined by

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

where $y^*(t)$ is the complex conjugate of $y(t)$.

The continuous linear chirp transform (CLCT) of $x(t) \in L^2(R)$ is defined as

$$X(\Omega,\gamma) = \int_{-\infty}^{\infty} x(t) e^{-j(\Omega t + \gamma t^2)} dt \tag{3.31}$$

The inverse continuous linear chirp transform (ICLCT) is given by

$$x(t) = \int_{-\infty}^{\infty} X(\Omega,\gamma) e^{j(\Omega t + \gamma^2 t)} d\Omega \tag{3.32}$$

Where $-\infty < \gamma < \infty$. We can prove that $x(t)$ is the inverse continuous linear chirp transform (ICLCT) of $X(\Omega,\gamma)$ as follows. We have

$$x(t) = \int_{-\infty}^{\infty} X(\Omega,\gamma) e^{j(\Omega t + \gamma^2 t)} d\Omega$$

Using equation (3.32), we get

$$\iint x(\tau) e^{j\Omega(t-\tau)} e^{j\gamma(t^2-\tau^2)} d\Omega d\tau$$

Using the following integral

$$\int_{-\infty}^{\infty} e^{j\Omega(t-\tau)} d\Omega = \delta(t-\tau)$$

We have

$$\int_{-\infty}^{\infty} x(\tau) e^{j\gamma(t^2-\tau^2)} \delta(t-\tau) d\tau = x(t)$$

The CLCT is the generalization of the conventional Fourier transform. The CLCT can remove the effect of the chirp rate on the channel bandwidth of chirp communication systems if we filter the signal at the corresponding chirp rate. Therefore, we present the CLCT to overcome the broadness of the channel bandwidth.

3.4.3 DISCRETE LINEAR CHIRPS

In this section, we develop an orthogonal representation using linear chirps for a discrete signal $x(n)$ of finite length $0 \le n \le N-1$. A discrete-time linear chirp

$$\phi_{\beta,k}(n) = \exp(j\frac{2\pi}{N}(\beta n^2 + kn)) \tag{3.33}$$

is characterized by the discrete frequency $2\pi k/N$ and by its chirp rate β, a continuous variable connected with the instantaneous frequency of the chirp

$$IF(n) = \frac{2\pi}{N}(2\beta n + k) \tag{3.34}$$

Assuming a finite support for β, i.e., $-\Gamma \le \beta < \Gamma$, we can construct an orthonormal basis $\{\phi_{\beta,k}(n)\}$ with respect to k in the supports of β and n as

$$\int_{-\Gamma}^{\Gamma} \sum_{n=0}^{N-1} \phi_{\beta,k}(n)\phi^{*}_{\beta,l}(n) d\beta = \int_{-\Gamma}^{\Gamma} N\delta(k-m) d\beta$$

$$= 2\Gamma N\delta(k-l) \tag{3.35}$$

Thus, we have the linear-chirp representation for a discrete signal $x(n)$, $0 \le n \le N-1$, to be

$$x(n) = \int_{-\Gamma}^{\Gamma} \sum_{k=0}^{N-1} \frac{X(k,\beta)}{N} \exp(j\frac{2\pi}{N}(\beta n^2 + kn)) d\beta \tag{3.36}$$

where the coefficients $X(k,\beta)$ are obtained by using the orthogonality of the basis as

$$X(k,\beta) = \sum_{n=0}^{N-1} x(n) \exp(-j\frac{2\pi}{N}(\beta n^2 + kn)) \tag{3.37}$$

To get the discrete transformation, we approximate the chirp rate as

$\beta = Cm$, where $C = \frac{2\Gamma}{L}$

So that $-\frac{L}{2} \le m \le \frac{L}{2} - 1$ is integer

So we have the discrete linear-chirp transform (DLCT) [43], [45]

$$X(k,m) = \sum_{n=0}^{N-1} x(n) \exp(-j\frac{2\pi}{N}(Cmn^2 + kn)) \tag{3.38}$$

Where $0 \le k \le N-1$ and $-\frac{L}{2} \le m \le \frac{L}{2} - 1$

Inverse discrete linear-chirp transform (IDLCT) is given by

$$x(n)=\sum_{m=-L/2}^{L/2-1}\sum_{k=0}^{N-1}\frac{X(k,m)}{LN}\exp(j\frac{2\pi}{N}(Cmn^2+kn)) \qquad (3.39)$$

Where $0\le n\le N-1$ and $C=\frac{2\Gamma}{L}$

The DLCT is not a time-frequency transformation, but a frequency chirp-rate transformation. One could think of the DLCT as a generalization of the discrete Fourier transform (DFT).Actually,

$$X(k,m)=\frac{1}{N}X(k)\Theta DFT\left\{\exp(-j\frac{2\pi}{N}Cm)\right\} \qquad (3.40)$$

Where "Θ"is the circular convolution. If $m=0$, then $X(k,0)$ is the DFT of $x(n)$ or the representation using chirp bases with zero chirp rates.

3.4.4 PROPERTIES OF DLCT

The properties of the DLCT are very much similar to those of the DFT. We are mainly interested in the modulation and the duality properties which will be useful in time-frequency shifts and in representing time-impulses and functions of them which cannot be represented when the chirp rate is assumed to be finite.

(1) Modulation property : if $X(k,m)$ is the DLCT of $x(n)$ then the linear-chirp modulated signal

$$y(n)=x(n)\exp(j\frac{2\pi}{N}(C_0m_0n^2+k_0n)) \qquad (3.41)$$

Where $C_0=\xi C$, has a DLCT

$$Y(k,m)=X(k-k_0,m-\xi m_0) \qquad (3.42)$$

Where ξ should be an integer to preserve the discrete nature of the transform. This shifting property allows the transformation of one chirp representation into another, and in precise, the transformation of chirp representations into sinusoidal representations.

(2) Duality property : Though the finite chirp rate assumption made before permits a large range of values for the chirp rate it cannot be used to represent signals that are impulses and functions of impulses in time. To include them we consider a duality property for the DLCT.

Interchanging the time and frequency variables and using equation (3.38) and (3.39)

$$X(n,-\bar{m}) = \sum_{k=0}^{N-1} x(-k)\exp(j\frac{2\pi}{N}(C\bar{m}k^2 + nk)) \tag{3.43}$$

Where $0 \le n \le N-1$ and $-\frac{L}{2} \le \bar{m} \le \frac{L}{2} - 1$

Then we have

$$x(-k) = \sum_{\bar{m}=-L/2}^{L/2-1} \sum_{n=0}^{N-1} \frac{X(k,-\bar{m})}{LN} \exp(-j\frac{2\pi}{N}(C\bar{m}k^2 + nk)) \tag{3.44}$$

Where $0 \le k \le N-1$ and $C = \frac{2\Gamma}{L}$

Using the same procedure, we can show that

$$x_{\bar{m}}(-k) = \sum_{n=0}^{N-1} \frac{X(n,-\bar{m})}{N} \exp(-j\frac{2\pi}{N}(C\bar{m}k^2 + nk)) \tag{3.45}$$

Which is also equal to $x(-k)$. So, if $x(n)$ is an impulse or a function of impulses, then its DFT would be a constant or a sinusoid of zero frequency, and its DLCT can be calculated.

We can find the relation between m and $\bar{m}$ or (β and $\bar{\beta}$) from the time-frequency distribution of a linear chirp. The IF of a linear chirp has a slope of 2 β from the time axis and a slope of 2 $\bar{\beta}$ from the frequency axis. Given a linear chirp

$h(t) = \exp(-j\gamma t^2)$ for $-\infty < t < \infty$, its Fourier transform is

$$H(\Omega) = \frac{1}{2\sqrt{\pi\gamma}} e^{-j\frac{\pi}{4}} \exp(j\frac{\Omega^2}{4\gamma})$$

If we calculate the dual of $H(\Omega)$, we get

$$H(t) = \frac{1}{2\sqrt{\pi \bar{\gamma}}} e^{-j\frac{3\pi}{4}} \exp(-j\frac{t^2}{4\bar{\gamma}})$$

As,

$$IF_{h(t)}(t) = IF_{H(t)}(t) = -2\gamma t = -\frac{2}{4\bar{\gamma}} t$$

Hence

$$\bar{\gamma} = \frac{1}{4\gamma}$$

In the discrete form, we have

$$C\bar{m} = \frac{1}{4Cm} \text{ or } \bar{\beta} = \frac{1}{4\beta}$$

If $\beta = \bar{\beta} = 0.5$, then the slope of the IF is equal to 45° which separates the time-frequency space into two symmetric halves.

(3) Linearity property : a signal $x(n) = a_1 x_1(n) + a_2 x_2(n), 0 \le n \le N-1$ has the DLCT transform

$$X(k,m) = a_1 X_1(k,m) + a_2 X_2(k,m) \tag{3.46}$$

3.5 DISCRETE COSINE CHIRP TRANSFORM

In the last section we consider the local representation of signals in terms of complex linear chirps, and therefore develop the discrete linear chirp transform (DLCT). This is a joint chirp-rate frequency transform, that generalizes the discrete Fourier transform (DFT). The presented discrete cosine chirp transform (DCCT) is more applicable to signal compression application.

For a discrete real-valued signal $x(n)$ of finite support $0 \le n \le N-1$, we define its DCCT as

$$X(k,m) = \sum_{n=0}^{N-1} x(n) \cos(\frac{Cm\pi n^2 + k\pi(2n+1)}{2N}) \tag{3.47}$$

Where $0 \le k \le N-1$ and $-\frac{L}{2} \le m \le \frac{L}{2}-1$

A representation in terms of cosines with instantaneous frequency

$IF(n,k) = \beta\pi n / N + k\pi / N$

The assumptions made for the DLCT associated to the chirp rate β are still valid for the DCCT. That is, we consider its support finite, $-\Gamma \le \beta \le \Gamma$, and that $\beta = mC$, and $C = 2\Lambda / L$. We can think of the DCCT as a generalization of the discrete cosine transform as $X(k,0)$ is equal to the DCT of $x(n)$. The DCCT decomposes a signal using real linear chirp as

$$\psi_{\beta,k}(n) = \cos\left(\frac{\beta\pi n^2}{2N} + \frac{k\pi(2n+1)}{2N}\right)$$

The inverse discrete cosine chirp transform (IDCCT) for reconstructing the original signal is given by

$$x(n) = \sum_{m=-L/2}^{L/2-1} \sum_{k=0}^{N-1} \frac{2X(k,m)}{LN} \cos\left(\frac{Cm\pi n^2 + k\pi(2n+1)}{2N}\right) \tag{3.48}$$

Where $0 \le n \le N-1$

Linearity property : the DCCT is a linear transform, as for any $x_1(n)$ and $x_2(n)$ with a DCCT transform $X_1(k,m)$ and $X_2(k,m)$ respectively then the signal

$x(n) = ax_1(n) + bx_2(n)$

has the DCCT transform given by

$$X(k,m) = aX_1(k,m) + bX_2(k,m) \tag{3.49}$$

Where a and b are constants. This property can be easily proved because the summation is a linear operator.

If $x_1(n)$ and $x_2(n)$ are real finite signals in the time support $n = 0,\cdots,N-1$ and their discrete cosine chirp transforms respectively are

$$X_1(k,m) = \sum_{n=0}^{N-1} x_1(n) \cos(\frac{Cm\pi n^2 + k\pi(2n+1)}{2N})$$

and

$$X_2(k,m) = \sum_{n=0}^{N-1} x_2(n) \cos(\frac{Cm\pi n^2 + k\pi(2n+1)}{2N})$$

that is

$$x_1(n) \Leftrightarrow X_1(k,m)$$

$$x_2(n) \Leftrightarrow X_2(k,m)$$

Then for $x(n) = ax_1(n) + bx_2(n)$, DCCT can be written as

$$X(k,m) = \sum_{n=0}^{N-1} x(n) \cos(\frac{Cm\pi n^2 + k\pi(2n+1)}{2N})$$

$$= a\sum_{n=0}^{N-1} x_1(n) \cos\left(\frac{Cm\pi n^2 + k\pi(2n+1)}{2N}\right) + b\sum_{n=0}^{N-1} x_2(n) \cos\left(\frac{Cm\pi n^2 + k\pi(2N+1)}{2N}\right)$$

which gives

$$X(k,m) = aX_1(k,m) + bX_2(k,m)$$

Linearity is very important property for the DCCT since it can be used in many applications such as signal modelling, compressive sensing, signal separation and other applications. Therefore, we can decompose a real signal $x(n)$ in terms of real chirps as

$$x(n) = \sum_{i=1}^{p} d_i \cos\left(\frac{\beta_i n^2 + k_i(2n+1)}{2N}\right)$$

where d_i, β_i, and k_i are amplitudes, chirp rates, and frequencies of p real linear chirps.

3.6 SPARSITY COMPARISON OF DLCT, DFRFT AND DCT

Sparsity or compressibility reflects the fact that information carried by certain signal is much smaller than their bandwidth. Most signals are not sparse in the time domain, so linear transformations are used to make them sparse in either time or frequency using certain basis [18]. The Discrete Cosine Transform (DCT) can be used to obtain a sparse representation in frequency for such signals. However, non-stationary signals, such as chirps may not be sparse in either time or frequency, but rather in an intermediate domain.

Sparseness is an essential signal characteristic in many applications such as compressive sensing, compression, and de-noising. It can be defined as a concentration of a signal energy on a few coefficients and the rest of them have low energy so that they can be neglected. Therefore, the transform that can give higher sparsity (few coefficients) is considered better than the one that gives low sparsity (too many coefficients). For mono-component signals, we can measure the sparsity of a signal analytically by measuring the broadness of its support in the transformed domain (could be time or frequency).

The frequency spread (B) can be defined in the discrete form as [19]

$$B = \sqrt{\frac{\sum_k \left(\omega_k - \langle \omega_k \rangle\right)^2 \left|X(e^{j\omega_k})\right|^2}{\sum_k \left|X(e^{j\omega_k})\right|^2}} \tag{3.50}$$

Where $\omega_k = 2\pi k / N$ for $k = 0,1,\cdots,N-1$ and $\langle \omega_k \rangle$ is the expected value given by

$$\langle \omega_k \rangle = \frac{\sum_k \omega_k \left|X(e^{jw_k})\right|^2}{\sum_k \left|X(e^{j\omega_k})\right|^2}$$

Similarly, the time spread (T) can be defined as

$$T = \sqrt{\frac{\sum_n \left(n - \langle n \rangle\right)^2 \left|x(n)\right|^2}{\sum_n \left|x(n)\right|^2}} \tag{3.51}$$

Where $\langle n \rangle$ is the expected value by

$$\langle n \rangle = \frac{\sum_n n|x(n)|^2}{\sum_n |x(n)|^2}$$

In equation (3.52) and (3.53), for finite energy signals and without loss of generality we can assume the energy of the signal is normalized

$$\sum_n |x(n)|^2 = \frac{1}{N}\sum_k \left|X(e^{j\omega_k})\right|^2 = 1 \tag{3.52}$$

The idea of measuring the sparsity by determining the broadness of the time spread or the frequency spread for mono-component signals can be generalized to multi-component signals. Since the DLCT can separate the components of the signal, we can define the sparsity measure in the frequency domain for multi-component signals as

$$B = \sum_{i=1}^{p} B_i \tag{3.53}$$

and the time spread as

$$T = \sum_{i=1}^{p} T_i \tag{3.54}$$

where T_i and B_i are the time and frequency spread for each component of a signal which has p components. We can also define the sparsity metric for multi-component signals in the form of time-bandwidth product (TB) as

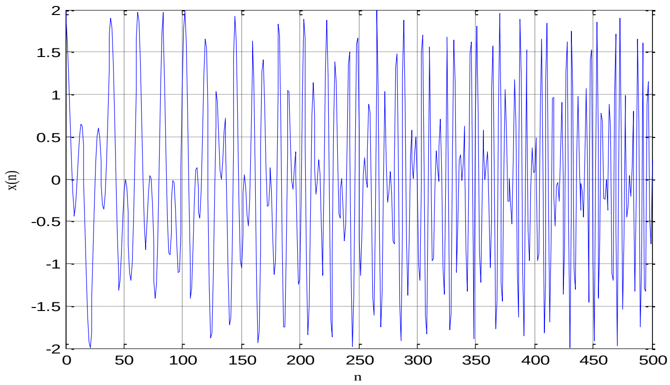
$$TB = \sum_{i=1}^{p} T_i B_i \tag{3.55}$$

3.7 SIMULATION RESULTS

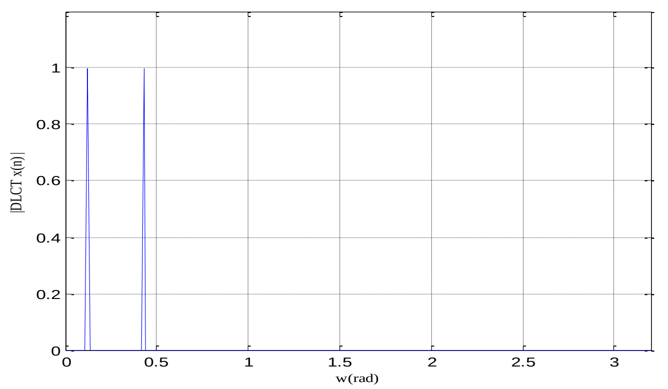
To evaluate the sparsity of the DLCT and the DFrFT, we use a synthetic signal $x_1(n)$ which is generated as follows

$$x_1(n) = \exp\left(j\frac{2\pi}{512}\left(0.1n^2 + 10n\right)\right) + \exp\left(j\frac{2\pi}{512}\left(0.1n^2 + 35n\right)\right)$$

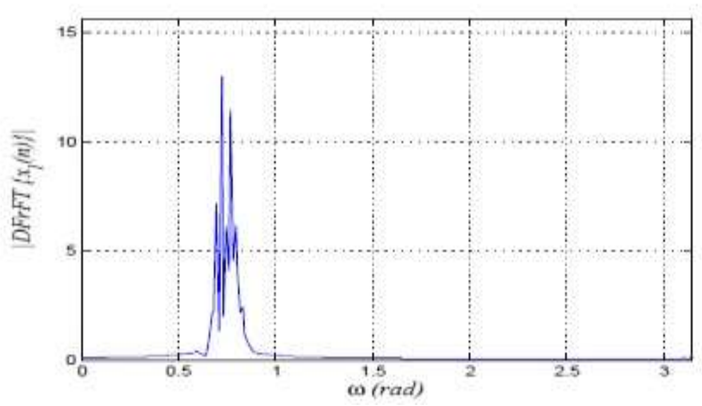
The signal $x_1(n)$ having chirp rate $\beta = 0.1$ and frequencies $(k_1, k_2) = (10,35)$ is shown in Figure 3.3(a) and the discrete linear chirp transform of $x_1(n)$ with $\beta = 0.1$ is shown in Figure 3.3(b) while Figure 3.3(c) shows the discrete fractional Fourier transform of $x_1(n)$ with $\beta = 0.1$ $(\alpha \approx -0.44\pi)$ and Figure 3.3(d) shows the discrete cosine transform of $x_1(n)$.



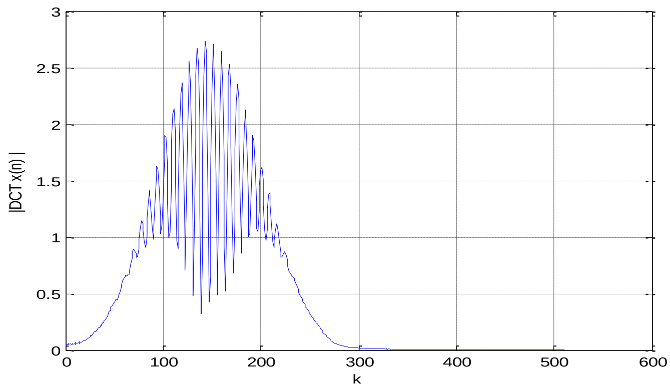
(a)



(b)



(c)



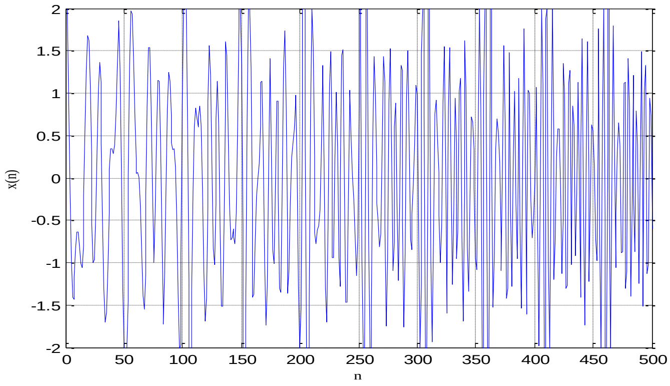
(d)

Figure 3.3 : (a) Signal $x_1(n)$ in time domain; (b) The DLCT of $x_1(n)$ with $\beta = 0.1$; (c) The $|DFrFT\{x_1(n)\}|$ with $\alpha = -0.44\pi$ and (d) The $|DCT\{x_1(n)\}|$.

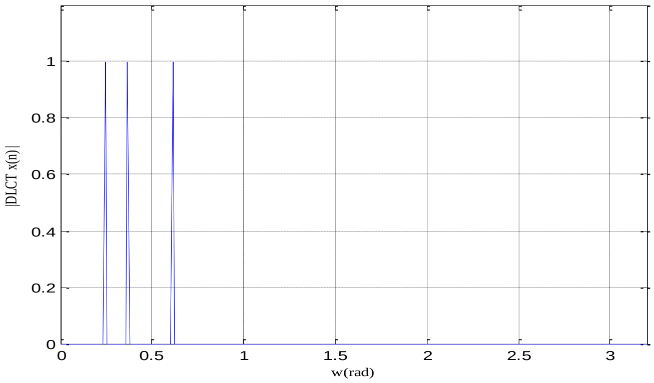
And signal $x_2(n)$ which is generated as follows

$$x_2(n) = \exp\left(j\frac{2\pi}{512}\left(0.1n^2 + 20n\right)\right) + \exp\left(j\frac{2\pi}{512}\left(0.1n^2 + 30n\right)\right) + \exp\left(j\frac{2\pi}{512}\left(0.1n^2 + 50n\right)\right)$$

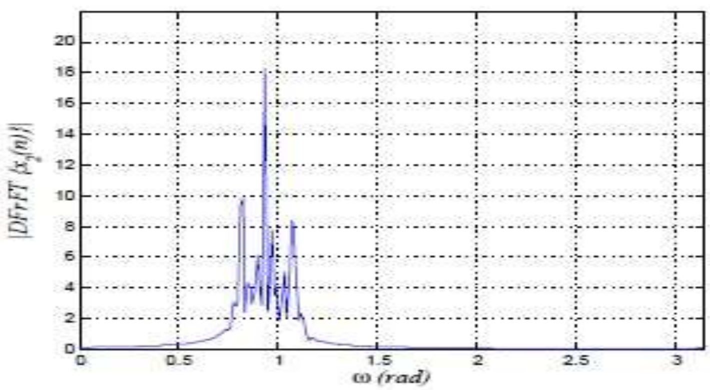
The signal $x_2(n)$ having chirp rate $\beta = 0.1$ and frequencies $(k_1, k_2, k_3) = (20,30,50)$ is shown in Figure 3.4(a) and the discrete linear chirp transform of $x_2(n)$ with $\beta = 0.1$ is shown in Figure 3.4(b) while Figure 3.4(c) shows the discrete fractional Fourier transform of $x_2(n)$ with $\beta = 0.1$ $(\alpha \approx -0.44\pi)$ and Figure 3.4(d) shows the discrete cosine transform of $x_2(n)$



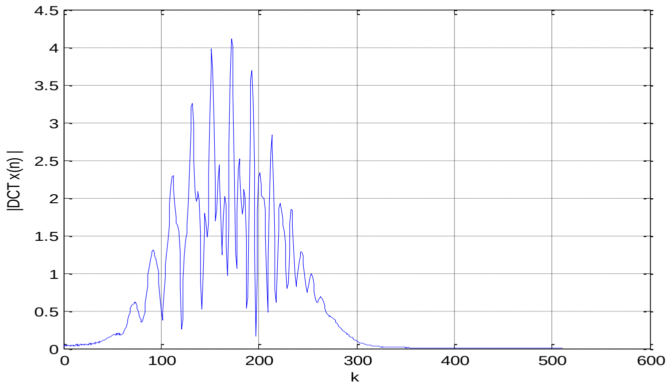
(a)



(b)



(c)



(d)

Figure 3.4 : (a) Signal $x_2(n)$ in time domain; (b) The DLCT of $x_2(n)$ with $\beta = 0.1$; (c) The $|DFrFT\{x_2(n)\}|$ with $\alpha = -0.44\pi$ and (d) The $|DCT\{x_2(n)\}|$.

From Figure 3.3 and Figure 3.4, It is clear that, the DLCT gives a transformed signal that is sparser than the transformed signal that we obtain using the DFrFT and DCT.

We find the frequency spread of the signals in Figure 3.3 and Figure 3.4 for DLCT is less as compared to that of DFrFT. Since $B_{DLCT} << B_{DFrFT} << B_{DCT}$, this implies that the transformed signal that we obtain using the DLCT is much sparser than the transformed signal using the DFrFT and DCT.

In general for any combination of ideal linear chirps, the resolution of the DLCT is finer than the resolution of the DFrFT and the DCT. All the algorithms that use the DLCT or the DFrFT or the DCT for parametric characterization of chirps depend on searching for peaks for all possible chirp rates or fractional orders to obtain the optimal chirp rate or the optimal fractional order that maximizes the $|DLCT\{x(n)\}|$ or equivalently $|DFrFT\{x(n)\}|$.

When the chirp rate of the DLCT matches the optimal chirp rate or (the fractional order of the DFrFT matches the optimal fractional order), then the transformed signal at the optimal chirp rate (optimal fractional order) will be sparser than for any other chirp rate or fractional order because the time-bandwidth product will be minimum.

3.8 SUMMARY

Sparsity or compressibility reflects the fact that information carried by certain signal is much smaller than their bandwidth. Most signals are not sparse in the time domain, so linear transformations are used to make them sparse in either time or frequency using certain basis. Non-stationary signals, such as chirps may not be sparse in either time or frequency, but rather in an intermediate domain. DLCT is used to convert such signal into sparse one with the help of modulation and duality property. The simulation result shows that the sparsity of DLCT is much better than that of FrFT and DCT.

CHAPTER 4

APPLICATION OF DLCT FOR DATA COMPRESSION

4.1 SIGNAL COMPRESSION

With the growth of communication systems and information technology, and their ability to serve image, video and voice, requires more data to be transmitted or stored. Signal compression transforms a signal into an efficient squeezed form, for transmission or storage, that can be decompressed back to produce a close approximation of the original data. The aim of signal compression is to minimize data rate to preserve bandwidth, while keeping the quality and intelligibility of the original signal. Unfortunately, the compression ratio is inversely proportional to the quality of the signal. Hence, there is always a trade-off between compression ratio and quality [47].

The performance of compression algorithms is measured by the signal to noise ratio SNR and the compression ratio Cr :

$$SNR = 10\log\left(\sigma_o^2 / \sigma_d^2\right) \tag{4.1}$$

$$Cr = \frac{\text{Length of original signal}}{\text{Length of compressed signal}} \tag{4.2}$$

where σ_o^2 is the variance of the original signal and σ_d^2 is the variance of the difference between original and rebuilt signals. Another factor that plays an important role in compression is the threshold value. After calculating the DLCT of a signal, many of the coefficients of the resulted signal are close to or equal to zero. Thus, we can amend those coefficients to produce more zeros by zeroing them out using certain threshold.

4.2 COMPRESSIVE SENSING

The conventional standard in digital signal processing for reconstructing signals from measured data follows Shannon sampling theorem. This approach promises the preservation of the information that is in the signal being sampled, but the cost is reflected in the number of samples that are required to represent the signal. Recently, the new theory of compressive sensing, also known as compressive sampling or sparse recovery has emerged [33] as an substitute to the traditional sampling theory. Compressive sensing states that we can reconstruct certain signals using fewer samples than those required by the sampling theory if we satisfy two conditions: sparsity and incoherence which means the sensing vectors are as different as possible from the sparsity basis. If we satisfy those conditions, signal reconstruction can be attained from cardinally smaller measurements by using $l_1 -$ minimization [48].

Compressive sensing (CS) [48] goals to take advantage of the signal's sparser representation dictated by the uncertainty principle. Although CS offers very good results for signals that are sparse in either time or frequency, it does not for signals that are not significantly sparse in either time or frequency domains such as the case of chirp signals [49], [50]. Time-frequency analysis is needed to get an intermediate domain where the signal is sparser than in time or in frequency. The Fractional Fourier Transform or the polynomial time-frequency transforms can be used, to obtain a sparse representation of a signal that is not sparse in time or frequency, or sparse in either of these domains a joint frequency instantaneous-frequency and its dual joint time and instantaneous-frequency transform.

Consider a finite support real signal with values given by a vector $x \in R^n$, and that is expressed in terms of the basis $\psi = [\psi_1 ....\psi_n]$ [48] as

$$x = \sum_{j=1}^{N} s_j \psi_j \quad \text{or} \quad x = \psi s \tag{4.3}$$

where ψ is an $N \times N$ matrix, and s is a vector of size $N \times 1$. The basis that transforms x into a sparse signal s can be, for instance, the one for the discrete cosine transform for a certain class of signals.

When the signal x is sparse it can be represented with $K << N$ nonzero coefficients. Compressive sensing assumes that the K nonzero coefficients are not extracted directly, but we project the vector x onto a matrix ϕ of size $M \times N$ where $M < N$. The matrix ϕ is called the measurement matrix and it satisfies the condition that the columns of the sparsity basis ψ cannot sparsely represent the rows of the measurement matrix ϕ (incoherence condition).

We can represent the measurement signal z as follows

$$z = \phi x = \phi \psi s = \theta s \tag{4.4}$$

where z is a vector of size $M \times 1$. Reconstruction of the signal is a convex optimization aimed at recovering the signal via l_1-minimization as shown in

$$\hat{s} = \arg\min \|s\|_1 \quad \text{subject to} \quad z = \Theta s \tag{4.5}$$

from which we can recover s, and then we use the inverse basis to obtain the original signal x.

4.3 SIGNAL COMPRESSION USING DLCT

Signal compression goals to decrease transmission rate (or increase storage capacity) by reducing the amount of data needed to be transmitted. DLCT is a joint frequency instantaneous-frequency transform that decomposes the signal in terms of linear chirps. The DLCT can be used to transform signals that are not sparse in either time or frequency, such as linear chirps, into sparse signals [45].

Compressive sensing attempts to simplify the frequency transformation and thresholding steps, commonly done in data compression, into one. Sparseness of the signal, in either time or frequency, is required for the convex optimization in compressive sensing to perform well. Although sparseness of certain signals, in either time or frequency, is assured by the uncertainty principle signals composed of chirps are not however sparse in either domain. The discrete linear chirp transform (DLCT) is an orthogonal linear-chirp transform, used to represent any signal in terms of linear chirps, with the help of modulation and dual properties. Using the DLCT the sparseness of the signal in either time or frequency can be achieved, and if not sparse in neither of these domains, the modulation and dual properties of the DLCT provide a way to transform the signal into a sparse signal.

The main goal of signal compression is to reduce the amount of data that we want to transmit or store. The direct and the dual DLCT are used to represent signals that can be better represented by one of them locally [45]. Considering that a sinusoid has a chirp rate $\beta = 0$, while an impulse has as chirp rate $\beta \to \infty$, we separate signals into two groups: one having $0 \le |\beta| \le 0.5$, corresponding to a linear chirp with a slope with an angle in $[-45^\circ, 45^\circ]$, and the other for $0.5 < |\beta| < \infty$ corresponding to a linear chirp with a slope with an angle in $[45^\circ, 90^\circ]$ or $[-45^\circ, -90^\circ]$. The value of $\beta = 0.5$ is not arbitrarily chosen since it relates to the slope of the instantaneous frequency such that

$$\text{Slope} = \tan(\theta) = 2\beta \tag{4.6}$$

If $\beta = 0.5$, then $\theta = \pi/4$ which is the angle that separates the time-frequency space into two symmetric halves.

4.3.1 COMPRESSION ALGORITH

Algorithm for signal compression using DLCT is shown is Figure 4.1. [45],

Consider the local representation of a signal $x(n)$, $0 \le n \le N-1$, as a superposition of p linear chirps

$$x(n) = \sum_{j=0}^{P-1} a_j \exp\left(i \frac{2\pi}{N} \left(\beta_j n^2 + k_j n\right) + i\psi_j \right) \tag{4.7}$$

$$= x_{(|\beta_j| \le 0.5)}(n) + x_{(|\beta_j| > 0.5)}(n)$$

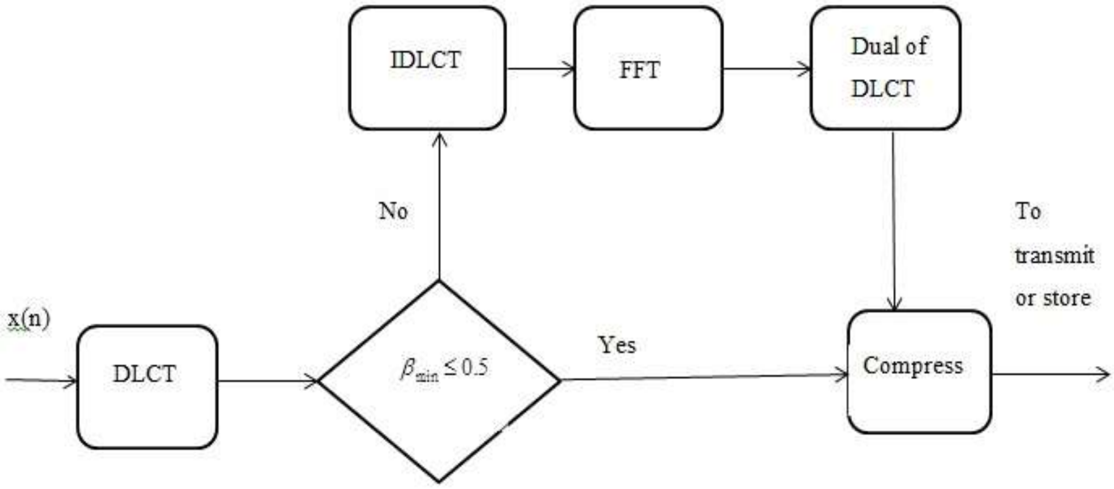


Figure 4.1 : Compression algorithm.

where $\{\alpha_j, \psi_j, k_j, \beta_j\}$ are the amplitude, phase, frequency, and chirp rate of the i^{th} linear chirp. The algorithm has two paths for the signal, the upper which is the dual path and the lower which is the direct path. Depending on the minimum value of the extracted βs for certain segment of the signal, we can do the compression either by the dual path or by the direct path. The coefficients $\{a_j, \psi_j, k_j, \beta_j\}$ are extracted and from theses coefficients we can reconstruct an approximation for the signal $x(n)$, where the arrangement of these coefficients is done according to following the data structure.

4.3.2 STRUCTURE OF THE COEFFICIENTS

Structure of the coefficients for sending or storing the extracted parameters is shown in Fig.4.2 [45], we choose P chirp rates that correspond to the peaks of chirps which forms the signal and P is the order of the chirp model. Then, from each vector which corresponds to the chosen chirp rates from the chirp transform $X(k,\beta)$ or $\hat{x}(n,\beta)$ matrix, we select M_j amplitudes, phases and frequencies or samples that have more power of the signal concentrated upon them.

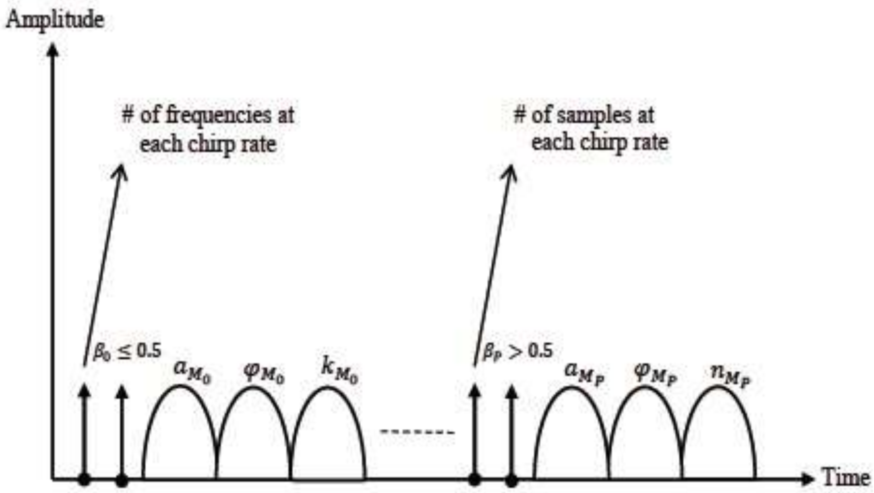


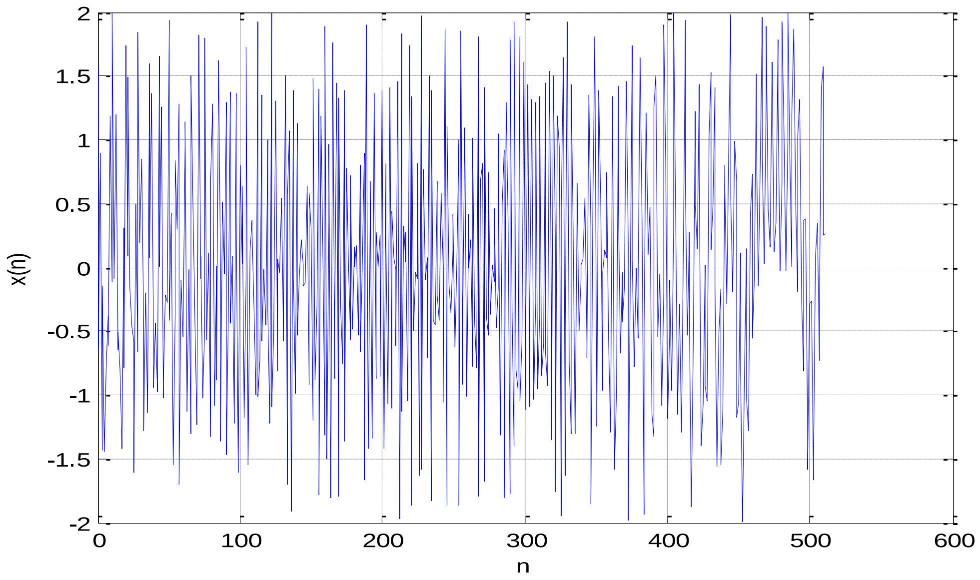
Figure 4.2 : Structure of the coefficients .

4.4 SIMULATION RESULTS

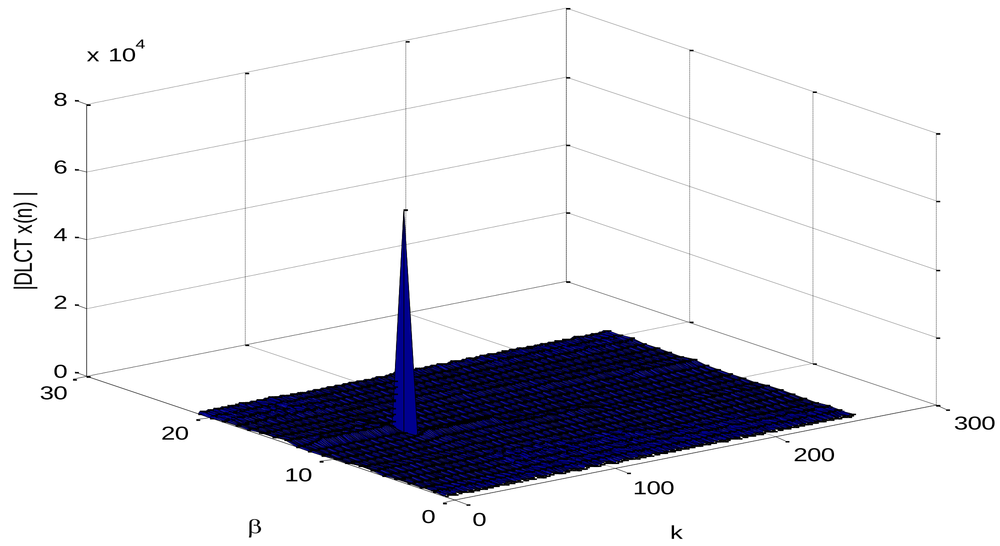
Consider a signal

$$x_3(n) = \exp\left(j\frac{2\pi}{256}\left(0.1n^2 + 60n\right)\right)$$

Figure 4.3(a) shows the time domain representation of original signal $x_3(n)$ and figure 4.3(b) shows the 3-D plot of $|X(k,\beta)|$. At location $(k,\beta)=(60,0.1)$ the transformation shows a peak corresponding to the given chirp with the given frequency and rate.



(a)

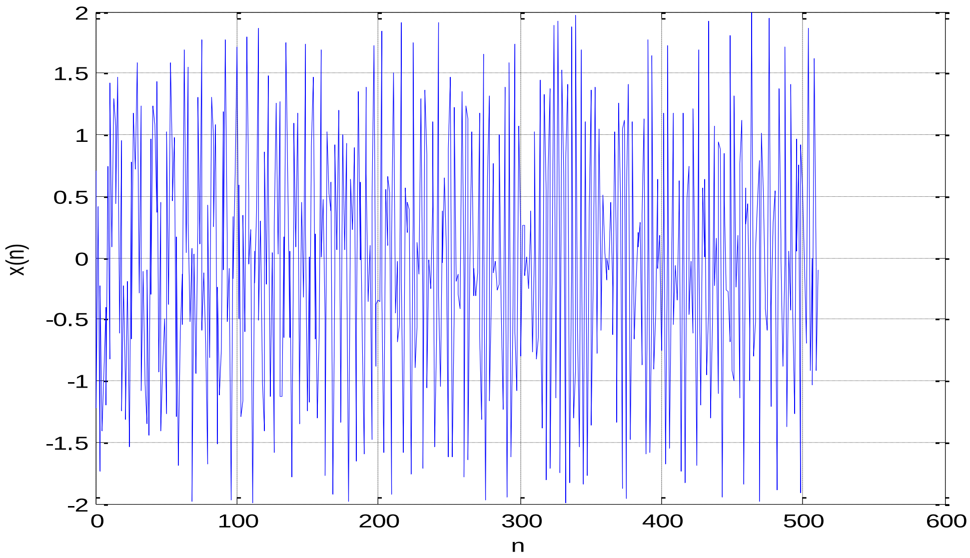


(b)

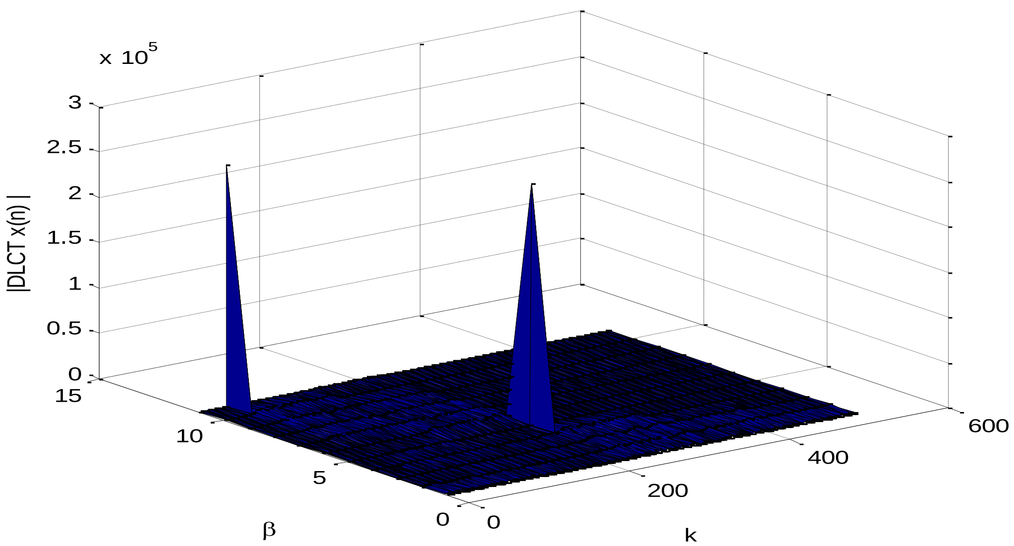
Figure 4.3 : (a) signal $x_3(n)$ in time domain; (b) the DLCT of signal $|DLCT\{x_3(n)\}|$.

Consider another signal

$$x_4(n)=\exp\left(j\frac{\pi}{4}\right)\cdot\exp\left[j\frac{2\pi}{512}\left(0.5n^2+35n\right)\right]+\exp\left(j\frac{\pi}{6}\right)\cdot\exp\left[j\frac{2\pi}{512}\left(-0.1n^2+230n\right)\right]$$



(a)



(b)

Figure 4.4 : (a) signal $x_4(n)$ in time domain; (b) the DLCT of signal $\left|DLCT\{x_4(n)\}\right|$.

Figure 4.4(a) shows the time domain representation of original signal $x_4(n)$ and figure 4.4(b) shows the 3-D plot of $|X(k,\beta)|$. At locations $(k,\beta)=(35,0.5)$ and $(k,\beta)=(230,-0.1)$ the transformation shows two peaks corresponding to the given chirp with the given frequencies and rates.

The compression of signal using DLCT based on the above algorithm achieves better performance than DCCT because the sparsity is the main criteria in the compression which is better for DLCT.

4.5 SUMMARY

Compressive sensing (CS) aims to take advantage of the signal's sparser representation dictated by the uncertainty principle. Although CS offers very good results for signals that are sparse in either time or frequency, it does not for signals that are not significantly sparse in either time or frequency domains such as the case of chirp signals. The DLCT is used to transform signals that are not sparse in either time or frequency, such as linear chirps, into sparse signals. Hence, the simulation shows that the CS using DLCT provides better result as compared to other existing CS techniques.

CHAPTER 5

CONCLUSION AND FUTURE WORK

5.1 CONCLUSION

In this dissertation, we have compared performance of DLCT in terms of sparsity with DFrFT and DCT. The discrete linear chirp transform (DLCT) is based on discrete complex linear chirps. It is not a time-frequency transformation, but rather a frequency chirp-rate transformation that generalizes the discrete Fourier transform and can be implemented with the fast Fourier transform algorithm. The parameters of a chirp or combination of chirps can be clearly determined with this transform. It also provides a modulation property that allows shifting of chirps into other chirps or sinusoids. The representation of impulses or functions of impulses is possible via a duality property of the transform. We have analyzed and compared the results of the discrete linear chirp transform (DLCT) with the discrete fractional Fourier transform (DFrFT) and discrete cosine transform (DCT) in terms of sparsity and resolution. Simulations result shows that the DLCT outperforms the DFrFT and DCT over these important aspects. Also the Compressive sensing using DLCT achieves better performance than other tools used for data compression.

5.2 FUTURE WORK

In future DLCT may be applied for the following applications :

- One dimensional and two dimensional signal (image) compression.
- In image processing, the DLCT decomposition can be used to analyze images. Hence, we can apply image watermarking to the analyzed images, explore its robustness against attacks, and compare its performance with similar existed techniques such an EMD and wavelet.
- For efficient utilization of bandwidth in communication
- Signal de-noiseing
- Speech processing etc.

REFERENCES

[1] S. C. Pei and J. J. Ding, "*Relations between fractional operations and time-frequency distributions, and their applications*," IEEE Trans. Signal Process., vol. 49, no. 8, pp. 1638–1655, Aug. 2001.

[2] S. C. Pei and J. J. Ding, "*Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing*", IEEE Trans. Signal Process., vol. 55, no. 10, pp. 4839–4850, Oct. 2007.

[3] S. C. Pei and J. J. Ding, "*Fractional Fourier Transform, Wigner Distribution, and Filter Design for Stationary and Nonstationary Random Processes*", IEEE Trans. Signal Process.,

vol. 58, no. 8, pp. 4079–4092 , August 2010.

[4] L. Cohen, "*Time-frequency analysis"*, Englewood Cliffs, NJ: Prentice-Hall, 1995.

[5] S. Qazi, A. Georgakis, L. Stergioulas, and M. Shikh-Bahaei, "*Interference suppression in the Wigner distribution using fractional Fourier transformation and signal synthesis*", IEEE Transactions on Signal Processing, vol. 55, no. 6, pp. 3150-3154, Jun.2007.

[6] O. A. Alkishriwo, L. F. Chaparro, and A. Akan, "*Signal separation in the Wigner distribution using fractional Fourier transform*", European Signal Processing Conference, EUSIPCO, Barcelona, Spain, pp. 1879-1883, Sep. 2011.

[7] V. Namias, "*The fractional order Fourier transform and its application to quantum mechanics*", IMA Journal of Applied Mathematics, vol. 25, no. 3, pp. 241-265, Apr. 1980.

[8] L. B. Almeida, "*The fractional Fourier transform and time-frequency representations*", IEEE Transactions on Signal Processing, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

[9] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, "*The Fractional Fourier Transform with Applications in Optics and Signal Processing*", John Wiley, New York, USA, 2001.

[10] X.-G. Xia, "*Discrete chirp-Fourier transform and its application to chirp rate estimation*", IEEE Transactions on Signal Processing, vol. 48, no. 11, pp. 3122-3133, Nov.2000.

[11] S. Mann and S. Haykin, “*The chirplet transform: Physical considerations*", IEEE Transactions on Signal Processing, vol. 43, no. 11, pp. 2745-2761, Nov. 1995.

[12] C. Candan, M. Kutay, and H. Ozaktas, “*The discrete fractional Fourier transform*", IEEE Trans. on Signal Processing, vol. 48, no. 5, pp. 1329-1337, May 2000.

[13] S. C. Pei and M. H. Yeh, “*Improved discrete fractional Fourier transform*", Optics Letters, vol. 22, no. 14, pp. 1047-1049, Jul. 1997.

[14] C. Capus, Y. Rzhanov, and L. Linnett, “*The analysis of multiple linear chirp signals*", in Proc. of IEE Seminar on Time-scale and Time-Frequency Analysis and Applications, pp. 4/1-4/7, Feb. 2000.

[15] L. Durak and O. Arikan, “*Short-time Fourier transform: Two fundamental properties and an optimal implementation*", IEEE Transactions on Signal Processing, vol. 51, no. 5, pp. 1231-1242, May 2003.

[16] C. Capus and K. Brown, “*Short-time fractional Fourier methods for the time-frequency representation of chirp signals*", Journal of Acoustical Society of America, vol. 113, no. 6, pp. 3253-3263, Jun. 2003.

[17] D. Cowell and S. Freear, “*Separation of overlapping linear frequency modulated (LFM) signals using the fractional Fourier transform*", IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, vol. 57, no. 10, pp. 2324-2333, Oct. 2010.

[18] J. Starck, F. Murtagh, and J. Fadili, “*Sparse Image and Signal Processing*”, Cambridge University Press, Sep. 2010.

[19] L. Claude and P. Vilbe, “*On the uncertainty principle in discrete signals*", IEEE Transactions on Circuits and Systems-II: Analog and Digital Signal Processing, vol. 39, no. 6, pp. 394-395, Jun. 1992.

[20] S. Qian and D. Chen, “*Adaptive Chirplet based signal approximation*”, in Proc. Of IEEE International Conference on Acoustics, Speech, and Signal Processing, ICASSP, vol. 3, pp. 1781-1784, May 1998.

[21] H. Ozaktas, Z. Zalevsky, and M. Kutay, “*The Fractional Fourier Transform: with Applications in Optics and Signal Processing*”, John Wiley & Sons, 2001.

[22] M. Martone, “*A multicarrier system based on the fractional Fourier transform for time-frequency-selective channels*", IEEE Transactions on Communications, vol. 49, no. 6, pp. 1011-1020, Jun. 2001.

[23] E. Sejdic, I. Djurovic, and L. Stankovic, “*Fractional Fourier transform as a signal processing tool: an overview of recent developments*", Signal Processing, vol. 91, no. 6, pp. 1351-1369, Jun. 2011.

[24] Q. Yin, S. Qian, and A. Fing, “*A fast refinement for adaptive Gaussian chirplet decomposition*", IEEE Transactions on Signal Processing, vol. 50, no. 6, pp. 1298-1306, Jun. 2002.

[25] Y. Lu, R. Demirli, G. Cardoso, and J. Saniie, “*A successive parameter estimation algorithm for chirplet signal decomposition*", IEEE Transactions on Signal Processing, vol. 53, no. 11, pp. 2121-2131, Nov. 2006.

[26] Z. K. Peng, G. Meng, F. L. Chu, Z. Q. Lang, W. M. Zhang, and Y. Yang, “*Polynomial chirplet transform with application to instantaneous frequency estimation*", IEEE Transactions on Instrumentation and Measurement, vol. 60, no. 9, pp. 3222-3229, Sep. 2011.

[27] N. Ahmed, T. Natarajan, and K. R. Rao, “*Discrete cosine transform*", IEEE Transactions on Computers, vol. C-23, no. 1, pp. 90-93, Jan. 1974.

[28] M. Adjrad, A. Belouchrani, and A. Ouldali, “*Estimation of chirp signal parameters using state space modelization by incorporating spatial information*", in Proc. of IEEE Seventh International Symposium on Signal Processing and Its Applications, France, vol. 2, pp. 531-534, Jul. 2003.

[29] H. Li and P. M. Djuric, “*MMSE estimation of nonlinear parameters of multiple linear/ quadratic chirps*", IEEE Transactions on Signal Processing, vol. 46, no. 3, pp. 796-800, Mar. 1998.

[30] R. G. Baraniuk, “*Compressive sensing*", IEEE Signal Processing Magazine, vol. 24, no. 4, pp. 118-121, Jul. 2007.

[31] E. J. Candes, “*Compressive sampling,*" in Proc. of the International Congress of Mathematicians, Madrid, Spain, pp. 1-20, 2006.

[32] G. Peyre, “*Best basis compressed sensing*”, IEEE Transactions on Signal Processing, vol. 58, no. 5, pp. 2613-2622, May 2010.

[33] D. Donoho, “*Compressed sensing*," IEEE Transactions on Information Theory, vol. 52, no. 4, pp. 1289-1306, Apr. 2006.

[34] P. Flandrin, G. Rilling, and P. Goncalves, “*Emperical mode decomposition as a filter bank*," IEEE Signal Processing Letter, vol. 11, no. 2, pp. 112-114, Feb. 2004.

[35] M. Molla, M. A. Shaikh, and K. Hirose, “*Time-frequency representation of audio signals using Hilbert spectrum with effective frequency scaling*", in Proc. of the IEEE International conference on Computer and Information Technology, Khulna, Bangladesh, pp. 335-340, Dec. 2008.

[36] C. Park, D. Looney, P. Kidmose, M. Ungstrup, and D. P. Mandic, “*Time-Frequency analysis of EEG asymmetry using bivariate empirical mode decomposition*", IEEE Transactions on Neural Systems and Rehabilitation Engineering, vol. 19, no. 4, pp. 366-373, Aug. 2011.

[37] T. Huang, C Chou, W. Fang, A. Li, Y. Chang, B. Huang, and Y. Shau, “*Time-frequency analysis of heart sound signals based on Hilbert-Huang transformation*", in Proc. of the IEEE International Symposium on Consumer Electronics, Harrisburg, PA,USA, pp. 1-3, Jun. 2012.

[38] E. Sejdic, L. Stankovic, M. Dakovic, and J. Jiang, “*Instantaneous frequency estimation using S-transform*", IEEE Signal Processing Letters, vol. 15, pp. 309-312, Feb. 2008.

[39] L. Stankovic, I. Djurovic, and R. Lakovic, “*Instantaneous frequency estimation by using the Wigner distribution and linear interpolation*", Signal Processing, vol. 83, no. 3, pp. 483-491, Mar. 2003.

[40] I. Djurovic, “*Viterbi algorithm for chirp-rate and instantaneous frequency estimation*", Signal Processing, vol. 91, no. 5, pp. 1308-1314, May 2011.

[41] Z. Hussain and B. Boashash, “*Adaptive instantaneous frequency estimation of multicomponent signals using quadratic time-frequency distributions*," IEEE Trans. On Signal Processing, vol. 50, no. 8, pp. 1866-1876, Aug. 2002.

[42] J. Lerga, V. Sucic, and B. Boashash, "*An efficient algorithm for instantaneous frequency estimation of nonstationary multicomponent signals in low SNR*", EURASIP Journal on Advances in Signal Processing, vol. 2011, pp. 1-16, Jan 2011.

[43] Osama A. Alkishriwo and Luis F. Chaparro, *"A Discrete linear chirp transform (DLCT) for data compression*", 11th International Conference on Information Science, Signal Processing and their Applications (ISSPA), Montreal, QC, pp. 1283 – 1288, July 2012.

[44] Osama A. Alkishriwo, Aydin Akan, Luis F. Chaparro, "*Intrinsic mode chirp decomposition of non-stationary signals*", IET Signal Processing, vol. 8, no. 3, pp. 267 – 276, May 2014.

[45] Osama A. Alkishriwo and Luis F. Chaparro, "*Signal compression using the Discrete linear chirp transform (DLCT)*", 20th European Signal Processing Conference (EUSIPCO), Bucharest, pp. 2128 – 2132, Aug. 2012.

[46] A. Hari, O.A. Alkishriwo, L.F. Chaparro, Aydin Akan, "*Non-stationary decomposition using the Discrete linear chirp transform (DLCT) for FM demodulation*", 21st European Signal Processing Conference (EUSIPCO), Marrakech, pp. 1-5, Sept. 2013.

[47] A. Gersho, "*Advances in speech and audio compression*", in Proc. of IEEE, vol. 82, no. 6, pp. 900-918, Jun. 1994.

[48] R. G. Baraniuk, "*Compressive sensing*", IEEE Signal Processing Magazine, vol. 24, no. 4, pp. 118-121, Jul. 2007.

[49] E. J. Candes, "*Compressive sampling*", in Proc. of the International Congress of Mathematicians, Madrid, Spain, pp. 1-20, 2006.

[50] G. Peyre, "*Best basis compressed sensing*", IEEE Transactions on Signal Processing, vol. 58, no. 5, pp. 2613-2622, May 2010.

[51] I. Daubechies and W. Sweldons, "*Factoring wavelet transforms into lifting steps*," The Journal of Fourier Analysis and Application, vol. 4, no. 3, pp. 247-261, 1998.

[52] A. Kutay, H. Ozaktas, L. Onural and O. Arikan, *"Optimal filtering in fractional fourier domains*," International Conference on Acoustic, Speech, and Signal Processing, vol. 2, pp. 937-940, 1995.

[53] I. Samil and A. Nehrai, "*Beamforming using fractional fourier transform*", IEEE Transactions on Signal Processing, vol.51, no.6, pp. 1663-1668, 2003.

[54] T.Thayaparan, "*Linear and quadratic time-frequency representations"*, Technical Memorandum DREO-TM-2000-080, 2000.

[55] S. Barbarossa and A. Farina, "*Space-Time-Frequency Processing of Synthetic Aperture Radar Signals*", IEEE Transactions on Aerospace and Electronic Systems, vol. 30, no. 2, pp. 341–358, 1992.

[56] D. Gabor, "*Theory of Communication*", Joint Information Exchange Environment, vol.93, no.26, pp 429-457, 1946.

[57] V. Namias, "*The Fractional Order Fourier Transform and its Application to Quantum Mechanics*", Journal of the Institute of mathematics, vol. 25, pp. 241-265, 1980.

[58] H. Ozaktas, Z. Zalevsky and A Kutay, "*FrFT with Applications in Optics and Signal Processing"*, John Wiley and Sons Publication, 1st edition, 2001.

[59] H. Ozaktas, O. Arikan, A. Kutay and G. Bozdagi, "*Digital Computation of the Fractional Fourier Transform*", IEEE Transactions on Signal Processing, vol.44, no.9, pp. 234-239, 1996.

[60] W. Rieck, "Time-Frequency Distribution of Multichannel SAR-Data for Autofocusing of Moving Targets," In Radar Conference Publication, no. 449, pp. 224–228, 1997.

[61] C. Capus and K. Brown,"*Short-Time Fractional Fourier Methods for Time- Frequency Representation of Chirp Signals",* Journal of Acoustical Science of America, vol.113, no.6, pp. 234.240, 2003.

[62] C. Capus, Y. Rzhanov and Laurie Linnett, "*Analysis of Multiple Linear Chirp Signals*", IEE Seminar on Time-Frequency Analysis and Application, pp. 1-7, 2000.

[63] L. Yanping, X. Mengdao, B. Zheng, "*SAR multiple targets detection and parameter estimation based on filtering in fractional Fourier domain*", IEEE Transactions on Signal Processing, vol. 26, pp. 1-6, 2005.

[64] C. Candan, A. Kutay and H. Ozaktas, "*The Discrete Fractional Fourier Transform*", IEEE Transactions on Signal Processing, vol.48, no.5, pp. 1329-1337, 2000.

[65] L. Almeida, "*The FrFT and Time-Frequency Representations*", IEEE Transactions on Signal Processing, vol.42, no.11, pp. 3084-3091,1994.

[66] J. Song and Y. Lin, "*Iterative Interpolation for Parameter Estimation of LFM Signal Based on Fractional Fourier Transform*", Journal of Circuits, Systems and signal Processing, vol. 32, no. 3, pp. 1489-1499, 2013.