INVESTIGATION OF CERTAIN CURVES AND METALLIC STRUCTURES ON MANIFOLDS

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By

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DECLARATION BY THE SCHOLAR

I hereby declare that the work reported in the Ph.D. thesis entitled "**Investigation of certain curves and metallic structures on manifolds**" at **Jaypee University of Information Technology, Waknaghat, Solan (H.P.) India**, is an authentic record of my work carried out under the supervision of **Dr. Pradeep Kumar Pandey**. I have not submitted this work elsewhere for any other degree or diploma. I am fully responsible for the contents of my Ph.D. Thesis.

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SUPERVISOR'S CERTIFICATE

This is to certify that the work reported in the Ph.D. thesis entitled "**Investigation of certain curves and metallic structures on manifolds**" submitted by **Sameer** at **Jaypee University of Information Technology, Waknaghat,** is a bonafide record of his original work carried out under my supervision. This work has not been submitted elsewhere for any other degree or diploma.

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(SAMEER)

Abstract

The title of the present thesis is ***"Investigation of certain curves and metallic structures on manifolds"***. The main objective of this thesis is to investigate the magnetic curves, slant curves, contact CR-submanifold of a Kenmotsu manifold with Killing tensor field, metallic structures, adapted connections, and differential equations for indicatrices, spacelike and timelike curves. There are two types of renowned structures on manifolds, namely, almost contact metric structures and almost Hermitian structures. The Kenmotsu, Sasakian and trans-Sasakian are the classes of almost contact metric structures. On the other side, the complex space forms, Kaehler, nearly Kaehler manifolds are particular cases of almost Hermitian manifolds.
The thesis is divided into seven chapters and each chapter is further divided into various sections and subsections.

In Chapter 1, firstly, we give a historical background of differential Geometry. Secondly, we have discussed two important structures on manifolds, *i.e.*, contact structure and complex structure. Moreover, we give some basic definitions and results: manifold, differentiable manifold, tangent space, vector field, Lie bracket, affine connection, torsion tensor, Riemannian manifold, Levi-Civita connection, curvature tensors, complex manifold, Hermitian manifold, contact manifold, almost contact metric manifold, Cosymplectic manifold, Kenmotsu manifold, and Sasakian Manifold.

Chapter 2, deals with the study of magnetic curves and slant curves in Kenmotsu manifolds. In this chapter, the magnetic trajectories associated with contact magnetic fields have been investigated and classification theorem is proved for the normal magnetic curves. Moreover, a characterization result is obtained for the Frenet curve to be a slant curve. Also, we gave some results on the curvature and torsion.

In Chapter 3, we investigate the properties of the contact CR-submanifold with Killing tensor field and obtained some results in Kenmotsu manifolds. Furthermore, we gave some examples in a Kenmotsu manifold that satisfies the condition of Killing

tensor field.

Chapter 4 is devoted to the study of metallic structures. We have investigated some structure on manifolds, namely the Bronze structure and the Copper structure. We obtained some examples of the Bronze structures and the Copper structures on manifolds. Moreover, we studied connections and integrability of the Bronze and the Copper structures. Finally, the Bronze and Copper Riemannian manifolds are investigated.

In chapter 5, we explore almost complex Norden Silver manifolds as well as Kaehler-Norden Silver manifolds. We introduce the adapted connections of first type, second type, and third type to an almost complex Norden Silver manifold. Further, the necessary and sufficient conditions for the integrability have been established for an almost complex Norden Silver structure. Besides, we investigate that a complex Norden Silver map is a harmonic map between Kaehler-Norden Silver manifolds.

In Chapter 6, we investigate the distance function which satisfies the 4th-order differential equation of the Frenet curve in Euclidean 3-space. We show that Tangent, Binormal, and Principal Normal indicatrices do not form non-trivial differential equations. Finally, we obtain the 4th-order differential equations for spacelike and timelike curves.

List of Publications

Journal Articles

- Sameer, P. K. Pandey, Adapted connections on Kaehler-Norden Silver manifolds and harmonicity, Honam Mathematical Journal, Vol. 43(4), 701-715, 2021. **(Indexed: ESCI)**

- Sameer, P. K. Pandey, On Contact CR-submanifold of a Kenmotsu manifold with Killing tensor field, *Kragujevac Journal of Mathematics*, Vol. 47(1), 95-104, Published online: 19-08-2020. **(Indexed: Scopus, ESCI (Web of science))**

- P. K. Pandey, S. Mohammad, Magnetic and slant curves in Kenmotsu manifolds, *Surveys in Mathematics and its Applications*, Vol. 15, 139-151, 2020. **(Indexed: Scopus)**

- P. K. Pandey, Sameer, Bronze differential geometry, *Journal of Science and Arts*, Vol. 45(4), 973-980, 2018. **(Indexed: ESCI (Web of science))**

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Conference Papers

- Contact CR-submanifold of a Kenmotsu manifold with Killing tensor field, *International Conference on Emergent Research in Mathematics and Engineering (ICERME 2019)*, organized by the Department of Mathematics, National Institute of Technology (NIT) Agartala, Tripura, India, held on 17-18 May, 2019.

- Copper Differential Geometry, *4th International Conference on Recent Advances in Mathematical Sciences and its Applications (RAMSA 2020)*, organized by the Department of Mathematics, Jaypee Institute of Information Technology (JIIT), Noida, Uttar Pradesh, India, held on 9-11 Jan, 2020.

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Chapter 1

Introduction and Literature Survey

1.1 A Glimpse of History

The word Geometry arises from ancient Greek word, geo-Earth; metry-measurement. Earth's measurement deals with the size, shape and properties of surrounding space. Euclid, a Greek mathematician is referred to as the founder of geometry.

Differential geometry is a deep, interesting and beautiful mathematical subject. It deals with the geometric properties of curves and surfaces in space. Differential geometry is among the oldest discipline of Mathematics. Its significance can be understood by the writing at the ancient Plato's Academy in Athens that "*let no one ignorant of geometry enter*". Leonhard Euler (1707-1783), a Swiss mathematician investigated the theory of surfaces and curvature. Gaspard Monge (1746 - 1818), a French mathematician is known as the father of differential geometry.

During the span of the twentieth century, two major manifolds, namely, even dimensional manifold and odd dimensional manifold played a very important role in the development of differential geometry. Schouten *et al.* [35], introduced the notion of complex structure and Hermitian metric on a differentiable manifold. Ehresmann (1905-1979) [22], gave the concept of an almost complex structure on an even dimensional differentiable manifold in 1950. Hodge [116], Calabi, and Spencer [33], Goldberg [98], Mishra [93], Yano [45], and many more geometers studied different properties of complex manifolds and almost complex manifolds.

In 1958, the notion of an odd dimensional manifold was introduced by Boothby, and Wang [115]. Gray [43], investigated the odd-dimensional manifold from the topological point of view in 1959. The structure presented by the Gray is known as a contact structure.

In 1960, Sasaki [105, 106] investigated the manifolds with the help of tensor analysis and these manifolds are known as contact manifolds and almost contact manifolds. Okumura [64], Sasaki, and Hatakeyama [107], Hatakeyama [118], Hatakeyama *et al.* [119], Kobayashi, and Nomizu [101], Mishra [91, 92], Blair [30], and several other authors studied various properties of almost contact metric manifolds.

K. Kenmotsu [44], introduced a class of almost contact manifold in 1972. Janssen *et al.* [32], named this structure as Kenmotsu structure and the differentiable manifold associated with this structure is said to be a Kenmotsu manifold.

In 1978, the notion of CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu [2], that generalizes the complex and totally real submanifolds. B.Y. Chen [16], studied CR-submanifold and obtained some fundamental properties of CR-submanifolds of a Kaehler manifold. From the last four decades, CR-submanifolds are an active field of research and play an important role in several areas of differential geometry.

The role of celebrity numbers π, e and the Golden proportion has always fascinated the mathematicians. The Golden proportion is also known as Golden mean, Divine ratio, Golden ratio and Golden number. Nevertheless, the Silver proportion and the Bronze proportion are also well known and much-sought-after numbers by mathematicians due to their own elegance and use in design, architecture and physics. The Golden proportion is a significant member of the $metallic\, means\, family$ (MMF). The metallic means family contains all the quadratic irrational numbers that are real root of the algebraic equation $x^2 - nx - 1 = 0$, where n is a positive integer.
In 1999, Spinadel [114] introduced the generalization of metallic proportion and stud-

ied some of its interesting properties. We can obtain the Golden proportion ($n = 1$), the Silver proportion ($n = 2$), the Bronze proportion ($n = 3$), the Copper proportion ($n = 4$), the Nickel proportion ($n = 5$) and so on, by obtaining the positive solutions of the algebraic equation $x^2 - nx - 1 = 0$. The convergence of metallic means was found to be faster as the value of n increases in the equation $x^2 - nx - 1 = 0$, and the irrationality among the metallic means decreases. The Golden proportion is most irrational among all the irrational numbers signifying that the convergence of the Golden proportion is slower than that of all the other metallic means [114].

1.2 Manifold

An n-dimensional manifold $\overline{M}$ is locally homeomorphic to an open set in a Euclidean space of dimension n. In other words, a topological space $\overline{M}$ is called an n-dimensional manifold if neighborhood of each point of $\overline{M}$ is homeomorphic to an open set in $\mathbb{R}^n$.

Suppose f be a real valued function defined on an open set $O \subset \mathbb{R}^n$. A mapping f from O into $\mathbb{R}^n$, *i.e.*, $f : O \to \mathbb{R}^n$ is called C^r, if it possesses continuous partial derivatives up to r on O. If a function f is merely continuous, then f is called a C^0-function on O. If f is a C^r-function for every non-negative integer r, then the function f is known as C^∞ or smooth function on O. If f is analytic on O, then $f \in C^\omega(O)$.

Suppose O denote an open set of an n-dimensional topological manifold and $p \in \overline{M}$. Let ϕ be a homeomorphism from O onto an open set E of $\mathbb{R}^n$, *i.e.*, $\phi : O \to E$. If p is a point in O_i and $\phi_i(p) = (u^1(p), u^2(p), \dots u^n(p))$, then the set O_i is known as a co-ordinate neighbourhood, the numbers $u^i(p), i \in \Lambda$ are called local coordinates on $\overline{M}$ at the point p, and the pair (O_i, ϕ_i) is said to be a local chart on $\overline{M}$ [72].

The charts (O_1, ϕ_1), and (O_2, ϕ_2) are said to be C^r-related if $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ are C^r-functions or O_1 and O_2 are disjoint.

The collection of all C^r-related charts, (O_i, ϕ_i), $i \in \Lambda$ (index set), is called an atlas or C^r-atlas, i.e., $\overline{M} = \bigcup_i O_i$.

Let (O_i, ϕ_i) and (O_j, ϕ_j) be two C^r-atlases on $\overline{M}$, where $i, j \in \Lambda$. The union of these two C^r-atlases need not be a C^r-atlas. Two C^r-atlases are said to be equivalent if union of two C^r-related charts is a C^r-atlas on $\overline{M}$.

A differentiable manifold $\overline{M}$ of dimension n is a collection of open charts (O_i, ϕ_i), $i \in \Lambda$ on $\overline{M}$, where $O_i(\phi_i)$ is an open subset of $\mathbb{R}^n$, such that the following conditions are satisfied:

(i) $\overline{M} = \bigcup_i O_i, i \in \Lambda$,

(ii) for any pair i and j in Λ, the mapping $\phi_j \circ \phi_i^{-1}$ is a differentiable mapping of $\phi_i(O_i \cap O_j)$ onto $\phi_j(O_i \cap O_j)$,

(iii) the collection (O_i, ϕ_i) is maximal family of open charts for which (i) and (ii) hold [72].

1.3 Tangent Space

Suppose $\overline{M}$ is a differentiable manifold of dimension n. Suppose $C^\infty(p)$ represents the set all real valued functions defined on some neighbourhood of a point p. Let $\mathcal{U}_p$ denote the tangent vector at a point $p \in \overline{M}$. A tangent vector $\mathcal{U}_p$ is a mapping $\mathcal{U}_p : C^\infty(p) \to \mathbb{R}$, satisfying the following properties

(i) $\mathcal{U}_p(f) \in \mathbb{R}$, for all $f \in C^\infty(p)$.

(ii) The linearity property

$\mathcal{U}_p(af + bg) = a(\mathcal{U}_p f) + b(\mathcal{U}_p g)$, where $f, g \in C^\infty(p)$, and $a, b \in \mathbb{R}$.

(iii) The Leibniz property

$\mathcal{U}_p(fg) = g(\mathcal{U}_p f) + f(\mathcal{U}_p g)$, where $f, g \in C^\infty(p)$.

The tangent space of a manifold $\overline{M}$ at a point p is represented by $T_p\overline{M}$. The set of all tangent vectors at a point p forms a vector space over $\mathbb{R}$ under the operation of addition and scalar multiplication, *i.e.*,

(i) $(\mathcal{U}_p + \mathcal{V}_p)f = \mathcal{U}_p f + \mathcal{V}_p f$,

(ii) $(\lambda\mathcal{U}_p)f = \lambda(\mathcal{U}_p f)$, for all $\lambda \in \mathbb{R}$ and f in $C^\infty(p)$.

In other words, a tangent space at a point p, is the set of all tangent vectors to manifold $\overline{M}$. The collection of all tangent vectors, along with the information of the point to which they are tangent is said to be a tangent bundle and it is represented by $\Gamma(T\overline{M})$ [75].

1.4 Vector Field

Suppose $\overline{M}$ is a differentiable manifold. A vector field $\mathcal{U}$ on $\overline{M}$ is a function that assigns to each point p of $\overline{M}$ a tangent vector $\mathcal{U}_p \in T_p\overline{M}$, where $T_p\overline{M}$ be a tangent space at point p. If f is a differentiable function on $\overline{M}$, *i.e.*, $f \in C^\infty(p)$ and $\mathcal{U}$ be a vector field, then $\mathcal{U}f$ is a function on $\overline{M}$, such that $(\mathcal{U}f)(p) = \mathcal{U}_p f$. If $\mathcal{U}f$ is differentiable for every function f, then a vector field $\mathcal{U}$ is called differentiable [75].

1.5 Affine Connection

Suppose $\overline{M}$ represents a differentiable manifold. An affine connection $\overline{\nabla}$ on $\overline{M}$ is a mapping

$$\overline{\nabla} : \Gamma(T\overline{M}) \times \Gamma(T\overline{M}) \to \Gamma(T\overline{M}),$$

denoted by $\overline{\nabla}(\mathcal{U}, \mathcal{V}) = \overline{\nabla}_{\mathcal{U}}\mathcal{V}$, satisfying the following properties:

(i) $\overline{\nabla}_{f\mathcal{U}+g\mathcal{V}}\mathcal{W} = f\overline{\nabla}_{\mathcal{U}}\mathcal{W} + g\overline{\nabla}_{\mathcal{V}}\mathcal{W}$,

(ii) $\overline{\nabla}_{\mathcal{U}}(\mathcal{V} + \mathcal{W}) = \overline{\nabla}_{\mathcal{U}}\mathcal{V} + \overline{\nabla}_{\mathcal{U}}\mathcal{W}$,

(iii) $\overline{\nabla}_{\mathcal{U}}(f\mathcal{V}) = f\overline{\nabla}_{\mathcal{U}}\mathcal{V} + (\mathcal{U}f)\mathcal{V}$,

where f, g are $C^\infty(\overline{M})$, and $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \Gamma(T\overline{M})$. The operator $\overline{\nabla}_{\mathcal{U}}$ is known as a covariant differentiation with respect to vector field $\mathcal{U}$, and $\overline{\nabla}_{\mathcal{U}}\mathcal{V}$ is said to be a covariant derivative of $\mathcal{V}$ with respect to $\mathcal{U}$ [72].

1.6 Lie Bracket

Suppose $\mathcal{U}$ and $\mathcal{V}$ are any two vector fields. The Lie derivative of $\mathcal{V}$ with respect to $\mathcal{U}$ is known as Poisson bracket or Lie bracket of the vector fields $\mathcal{U}, \mathcal{V}$ at a point p on $\overline{M}$, and is given by

$$[\mathcal{U}, \mathcal{V}]_p f = \mathcal{U}_p(\mathcal{V} f) - \mathcal{V}_p(\mathcal{U} f),$$

$\forall f \in C^\infty(\overline{M})$, and $p \in \overline{M}$.

The Lie bracket is also given as

$$[\mathcal{U}, \mathcal{V}] = \overline{\nabla}_{\mathcal{U}} \mathcal{V} - \overline{\nabla}_{\mathcal{V}} \mathcal{U}.$$

"The Lie bracket satisfies the following properties:

(i) $[\mathcal{U}, \mathcal{V}] = -[\mathcal{V}, \mathcal{U}]$

(ii) $[\mathcal{U}, \mathcal{U}] = 0$

(iii) $[\mathcal{U}, \mathcal{V} + \mathcal{W}] = [\mathcal{U}, \mathcal{V}] + [\mathcal{U}, \mathcal{W}]$

(iv) $[\mathcal{U}, \mathcal{V}](f + g) = [\mathcal{U}, \mathcal{V}]f + [\mathcal{U}, \mathcal{V}]g$

(v) $[\mathcal{U}, \mathcal{V}](fg) = f[\mathcal{U}, \mathcal{V}]g + g[\mathcal{U}, \mathcal{V}]f$

(vi) $[f\mathcal{U}, g\mathcal{V}] = fg[\mathcal{V}, \mathcal{U}] + f(\mathcal{U} g)\mathcal{V} - g(\mathcal{V} f)\mathcal{U}$

(vii) $[[\mathcal{U}, \mathcal{V}], \mathcal{W}] + [[\mathcal{V}, \mathcal{W}], \mathcal{U}] + [[\mathcal{W}, \mathcal{U}], \mathcal{V}] = 0,$

where $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are the vector fields in $\Gamma(T\overline{M})$, and $f, g \in C^\infty(\overline{M})$" [72].

1.7 Torsion Tensor

"Suppose $\overline{M}$ denote a differentiable manifold of n-dimension with affine connection $\overline{\nabla}$. The torsion tensor field T of type $(1, 2)$ of the connection $\overline{\nabla}$ is defined by the mapping

$$T : \Gamma(T\overline{M}) \times \Gamma(T\overline{M}) \to \Gamma(T\overline{M})$$

such that

$$T(\mathcal{U}, \mathcal{V}) = \overline{\nabla}_{\mathcal{U}}\mathcal{V} - \overline{\nabla}_{\mathcal{V}}\mathcal{U} - [\mathcal{U}, \mathcal{V}],$$

for any vector fields $\mathcal{U}, \mathcal{V} \in \Gamma(T\overline{M})$, and $[\mathcal{U}, \mathcal{V}]$ is the Lie bracket of $\mathcal{U}$ and $\mathcal{V}$" [72].

From above equation, it is clear that torsion tensor T is skew symmetric, *i.e.*,

$$T(\mathcal{U}, \mathcal{V}) = -T(\mathcal{V}, \mathcal{U}).$$

The torsion tensor satisfies the following properties:

(i) $T(\mathcal{U}, \mathcal{V})$ is $\mathbb{R}$-bilinear,

(ii) $T(\mathcal{U}, \mathcal{V})$ is $C^\infty(\overline{M})$-bilinear.

1.8 Riemannian Manifolds

Let $\overline{M}$ be a differentiable manifold of dimension n. A Riemannian metric g on $\overline{M}$ is a 2-tensor field, and satisfying the following conditions:

(i) $g(\mathcal{U}, \mathcal{U}) = 0$ if and only if $\mathcal{U} = 0$,

(ii) g is positive definite, *i.e.*, $g(\mathcal{U}, \mathcal{U}) > 0$ if $\mathcal{U} \neq 0$,

(iii) g is symmetric, *i.e.*, $g(\mathcal{U}, \mathcal{V}) = g(\mathcal{V}, \mathcal{U})$,

(iv) g is bilinear, *i.e.*, $g(a\mathcal{U} + b\mathcal{V}, \mathcal{W}) = ag(\mathcal{U}, \mathcal{W}) + bg(\mathcal{V}, \mathcal{W})$, $a, b \in \mathbb{R}$.

A Riemannian metric g with the differentiable manifold $\overline{M}$ is called a Riemannian manifold, and is denoted by $(\overline{M}, g)$.

If $g(\mathcal{U}, \mathcal{V}) = 0$ for all $\mathcal{V} \neq 0$ and $\mathcal{U} = 0$, then g is said to be a semi-Riemannian metric. The manifold $\overline{M}$ with semi-Riemannian metric g is called Semi-Riemannian manifold or pseudo-Riemannian manifold [72].

1.9 Riemannian Connection or Levi-Civita Connection

Suppose $(\overline{M}, g)$ denotes the Riemannian manifold of n-dimension and $\overline{\nabla}$ represents the affine connection. Now, the affine connection $\overline{\nabla}$ on $\overline{M}$ is said to be a Riemannian connection, if it satisfies the following:

(i) the connection $\overline{\nabla}$ is torsion free, *i.e.*, $T(\mathcal{U}, \mathcal{V}) = 0$,

or

$$[\mathcal{U}, \mathcal{V}] = \overline{\nabla}_{\mathcal{U}}\mathcal{V} - \overline{\nabla}_{\mathcal{V}}\mathcal{U},$$

(ii) the connection $\overline{\nabla}$ is metric compatible, *i.e.*,

$$(\overline{\nabla}_{\mathcal{U}} g)(\mathcal{V}, \mathcal{W}) \ vanishes,$$

where $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are vector fields in $\Gamma(T\overline{M})$ [72].

1.10 Curvature Tensors

"The curvature tensor $\mathcal{R}$ of a Riemannian manifold $(\overline{M}, g)$ is a correspondence that associates to each pair $\mathcal{U}, \mathcal{V} \in \Gamma(T\overline{M})$, a mapping $\mathcal{R}(\mathcal{U}, \mathcal{V}) : \Gamma(T\overline{M}) \to \Gamma(T\overline{M})$, defined by

$$\mathcal{R}(\mathcal{U}, \mathcal{V})\mathcal{W} = \overline{\nabla}_{\mathcal{U}}\overline{\nabla}_{\mathcal{V}}\mathcal{W} - \overline{\nabla}_{\mathcal{V}}\overline{\nabla}_{\mathcal{U}}\mathcal{W} - \overline{\nabla}_{[\mathcal{U},\mathcal{V}]}\mathcal{W},$$

where $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ and $\mathcal{W} \in \Gamma(T\overline{M})$" [72].

For $\overline{M} = \mathbb{R}^n$, the curvature tensor is always zero, *i.e.*, $\mathcal{R}(\mathcal{U}, \mathcal{V})\mathcal{W} = 0$.

The curvature tensor $\mathcal{R}$ of $(\overline{M}, g)$ satisfies the following properties:

(i) $\mathcal{R}$ is bilinear, *i.e.*,

$$\mathcal{R}(f\mathcal{U}_1 + g\mathcal{U}_2, \mathcal{V}_1) = f\mathcal{R}(\mathcal{U}_1, \mathcal{V}_1) + g\mathcal{R}(\mathcal{U}_2, \mathcal{V}_1),$$

$$\mathcal{R}(\mathcal{U}_1, f\mathcal{V}_1 + g\mathcal{V}_2) = f\mathcal{R}(\mathcal{U}_1, \mathcal{V}_1) + g\mathcal{R}(\mathcal{U}_1, \mathcal{V}_2),$$

f, g belongs to $C^\infty(\overline{M})$, and $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1, \mathcal{V}_2 \in \Gamma(T\overline{M})$.

(ii) The curvature operator $\mathcal{R}(\mathcal{U},\mathcal{V}):\Gamma(T\overline{M})\to\Gamma(T\overline{M})$ is linear, *i.e.*,

$$\mathcal{R}(\mathcal{U},\mathcal{V})(\mathcal{W},X)=\mathcal{R}(\mathcal{U},\mathcal{V})\mathcal{W}+\mathcal{R}(\mathcal{U},\mathcal{V})X,$$

$$\mathcal{R}(\mathcal{U},\mathcal{V})f\mathcal{W}=f\mathcal{R}(\mathcal{U},\mathcal{V})\mathcal{W},$$

f belongs to $C^\infty(\overline{M})$, and $\mathcal{U},\mathcal{V},\mathcal{W},X\in\Gamma(T\overline{M})$.

(iii) $\mathcal{R}(\mathcal{U},\mathcal{V})\mathcal{W}+\mathcal{R}(\mathcal{V},\mathcal{W})\mathcal{U}+\mathcal{R}(\mathcal{W},\mathcal{U})\mathcal{V}=0.$

1.11 Complex Manifolds

Let $\overline{M}$ denote a differentiable manifold of $2n$-dimension. A tensor field J of type $(1,1)$ on $\overline{M}$ is called an almost complex structure, if J is an endomorphism on the tangent space $T_p\overline{M}$, at point $p\in\overline{M}$ such that

$$J^2=-I,$$

where I represents the identity tensor field. A differentiable manifold $\overline{M}$ with a fixed almost complex structure J is known as an almost complex manifold.

The Nijenhuis tensor with respect to tensor field J is a vector valued bilinear function N or $[J,J]$, given by

$$N(\mathcal{U},\mathcal{V})=[J,J](\mathcal{U},\mathcal{V})=J^2[\mathcal{U},\mathcal{V}]+[J\mathcal{U},J\mathcal{V}]-J[J\mathcal{U},\mathcal{V}]-J[\mathcal{U},J\mathcal{V}],$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on $\overline{M}$.

Suppose J represents an almost complex structure of the almost complex manifold $\overline{M}$. Then J is known as a complex structure if the Nijenhuis tensor $[J,J]$ of J vanishes, *i.e.*,

$$[J,J](\mathcal{U},\mathcal{V})=0,$$

A manifold $\overline{M}$ equipped with J is called a complex manifold [47].

1.12 Hermitian Manifolds

Suppose $\overline{M}$ is an almost complex manifold associated with almost complex structure J. If a Riemannian metric g satisfies

$$g(J\mathcal{U}, J\mathcal{V}) = g(\mathcal{U}, \mathcal{V}),$$

for any vector fields $\mathcal{U}$ and $\mathcal{V}$ on $\overline{M}$, then g is called a Hermitian metric on $\overline{M}$. An almost complex manifold endowed with a Hermitian metric g is known as an almost Hermitian manifold. A complex manifold with a Hermitian metric is said to be a Hermitian manifold.

Suppose $\overline{M}$ denotes an almost complex manifold with J, and Hermitian metric g. The fundamental 2-form Ω of $\overline{M}$ is given by

$$\Omega\,(\mathcal{U}, \mathcal{V}) = g(\mathcal{U}, J\mathcal{V}),\;\; \mathcal{U}, \mathcal{V} \in \Gamma(T\overline{M}).$$

The fundamental 2-form Ω have the following properties:

(i) $\Omega(\mathcal{U}, \mathcal{V}) = -\Omega(\mathcal{V}, \mathcal{U})$

(ii) $\Omega(\mathcal{U}, \mathcal{V}) = \Omega(J\mathcal{U}, J\mathcal{V})$

(iii) $\Omega(J\mathcal{U}, \mathcal{V}) = -\Omega(\mathcal{U}, J\mathcal{V})$

If $d\Omega = 0$, then Ω is said to be closed. If Ω is closed, then a Hermitian metric g with an almost complex manifold $\overline{M}$ is known as a Kaehler metric. An almost complex manifold $\overline{M}$ equipped with a Kaehler metric is known as an almost Kaehler manifold. A complex manifold associated with a Kaehler metric is called a Kaehler manifold [47].

1.13 Contact Manifolds

"A $(2n+1)$-dimensional differentiable manifold $\overline{M}$ is said to have a contact structure, if there exist a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0,$$

everywhere on the manifold $\overline{M}$, where the exponent denotes the nth exterior power. The manifold $\overline{M}$ associated with the contact structure is known as a contact manifold. We call η a contact form of $\overline{M}$" [47].

1.14 Almost Contact Metric Manifolds

"Let $\overline{M}$ be a $(2n+1)$-dimensional differentiable manifold and ϕ, ξ and η be a tensor field of type $(1,1)$, a vector field, and a 1-form on $\overline{M}$, respectively. If ϕ, ξ and η satisfy the conditions

$$\begin{aligned} \eta \otimes \xi - I &= \phi^2 \\ \eta(\xi) &= 1 \\ \phi\xi &= 0 \\ (\eta \circ \phi) &= 0 \\ rank\ \phi &= 2n \end{aligned} \tag{1.14.1}$$

where I denote the identity transformation. The structure (ϕ, ξ, η) is known as an almost contact structure, and the structure $(\overline{M}, \phi, \xi, \eta)$ is said to be an almost contact manifold.

Let g be a Riemannian metric on almost contact manifold $\overline{M}$ which satisfies

$$g(\phi\mathcal{U}, \phi\mathcal{V}) = g(\mathcal{U}, \mathcal{V}) - \eta(\mathcal{U})\eta(\mathcal{V}),$$

$$\eta(\mathcal{U}) = g(\mathcal{U}, \xi),$$

$$g(\phi\mathcal{U}, \mathcal{V}) = -g(\mathcal{U}, \phi\mathcal{V}),$$

for all $\mathcal{U}, \mathcal{V} \in \Gamma(T\overline{M})$. The structure (ϕ, ξ, η, g) is called an almost contact metric structure, and the structure $(\overline{M}, \phi, \xi, \eta, g)$ is said to be an almost contact metric manifold"[47].

Let Ω denote the fundamental 2-form of $(\overline{M}, \phi, \xi, \eta, g)$, and is given by

$$\Omega(\mathcal{U}, \mathcal{V}) = g(\mathcal{U}, \phi\mathcal{V}), \ \ \mathcal{U}, \mathcal{V} \in \Gamma(T\overline{M}).$$

If $d\eta = \Omega$, then $(\overline{M}, \phi, \xi, \eta, g)$ is said to be a contact metric manifold.

Let $(\overline{M}, \phi, \xi, \eta, g)$ denote an almost contact metric manifold, then it is said to be a *Cosymplectic manifold* if 1-form η and fundamental 2-form Ω are closed, *i.e.*,

$$d\eta = 0, \quad \text{and} \quad d\Omega = 0.$$

1.15 Kenmotsu Manifolds

"Let $(\overline{M}, \phi, \xi, \eta, g)$ represents the almost contact metric manifold and $\overline{\nabla}$ be the Levi-Civita Connection on $\overline{M}$, if

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V} = -g(\mathcal{U}, \phi\mathcal{V})\xi - \eta\left(\mathcal{V}\right)\phi\mathcal{U} \tag{1.15.1}$$

and

$$\overline{\nabla}_{\mathcal{U}}\xi = \mathcal{U} - \eta(\mathcal{U})\xi, \tag{1.15.2}$$

then, the structure $(\overline{M}, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold" [44].

1.16 Sasakian Manifolds

"Let $(\overline{M}, \phi, \xi, \eta, g)$ represents the almost contact metric manifold and $\overline{\nabla}$ denotes the Levi-Civita Connection on $\overline{M}$, if

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V} = g(\mathcal{U}, \mathcal{V})\xi - \eta(\mathcal{V})\mathcal{U} \tag{1.16.1}$$

and

$$\overline{\nabla}_{\mathcal{U}}\xi = \phi\mathcal{U}, \tag{1.16.2}$$

then, the structure $(\overline{M}, \phi, \xi, \eta, g)$ is said to be a Sasakian manifold" [47].

1.17 List of Problems Considered

Differential geometry is a deep, and interesting branch of Mathematics. Theory of manifolds associated with some special structures, namely Kenmotsu, Sasakian, and quasi-Sasakian are fascinating topics in differential geometry. The thesis deals with the curves, Kenmotsu manifolds, adapted connections, and metallic manifolds. The brief outline of the problems taken in this thesis presented as

- Investigation of magnetic and slant curves in Kenmotsu manifolds.
- Contact CR-submanifold of a Kenmotsu manifold with Killing tensor field.
- Bronze and Copper differential geometry.
- Adapted connections on Kaehler-Norden Silver manifolds and harmonicity.
- Differential equations for the Tangent, Binormal, and Principal Normal indicatrices.

Chapter 2

Magnetic and Slant Curves in Kenmotsu Manifolds

2.1 Introduction

Suppose $(\overline{M}, g)$ denote the Riemannian manifold. A closed 2-form on $\overline{M}$ is known as the magnetic field. The concept of magnetic curves in $(\overline{M}, g)$ were investigated by many authors [40, 56, 57, 61, 109].

Călin *et al.* [20] investigated the slant curves in 3-dimensional f-Kenmotsu manifolds. They also gave a Classification of slant curves in the hyperbolic space $\mathbb{H}^3$. By using dynamical systems, Kalinin [28] studied the trajectories of the charge particles of magnetic fields on Kaehler manifolds of constant holomorphic sectional curvature. Moreover, Inoguchi *et al.* [38] studied the slant curves and obtained the torsion of slant curve in a 3-dimensional almost f-Kenmotsu manifold. They established that an almost f-Kenmotsu manifold is f-Kenmotsu manifold iff it is normal.

In two and three dimensional unit spheres, the Cabrerizo [40] have studied the Landau-Hall problem. Cabrerizo revealed that the magnetic flowlines are helices corresponding to the Killing vector fields. In 3-dimensional Riemannian manifolds, the study of contact magnetic field have been carried out by Cabrerizo *et al.* [41].

With an application to magnetic fields, they revealed that the metric g is adapted to the almost contact structure. In Sasakian 3-manifolds, the Lancret type problems for slant curves have been studied by Cho *et al.* [42]. They proved that a curve is of constant slope iff its ratio of curvature (κ) and torsion (τ) is constant in Euclidean 3-space.

In 3-dimensional space, Barros *et al.* [56] investigated the magnetic flow equipped with a Killing magnetic field. Barros *et al.* [57] also studied and solved some problems of magnetic fields in 3D. Under the action of Killing magnetic fields in $\mathbb{S}^2 \times \mathbb{R}$, Munteanu *et al.* [61] studied the trajectories of charged particles moving is space.

Inoguchi *et al.* [39] investigated the contact magnetic fields in quasi-Sasakian manifolds of dimension three. Additionally, they proposed a family of linear connections corresponding to the Okumura type connections. In [102, 103], Druţă-Romaniuc *et al.* presented the magnetic curves associated with the contact magnetic field on Sasakian manifolds as well as Cosymplectic manifolds.

In [77], Ikawa proposed a Sasaki-Kaehler submersion from a Sasakian manifold. On the other hand, Ikawa [78, 79] studied the motion of charged particles from the geometric view point as well as in two-step nilpotent Lie groups. Furthermore, Guvenc [97] studied the slant magnetic curves in S-manifolds. Guvenc also constructed the slant normal magnetic curves in $\mathbb{R}^{2n+s}(-3s)$.

Druţă-Romaniuc *et al.* [104] investigated the magnetic curves associated with the Killing magnetic fields in Euclidean space of dimension 3. The trajectories of charged particles under the action of a Kaehler magnetic field have been investigated by Adachi [109]. On a complex projective space, Adachi also study the magnetic field with respect to the Kaehler form. In [110], Adachi investigated the similarities between geodesics and trajectories on Kaehler manifolds of negative curvature.

Ozdemir *et al.* [122] studied magnetic curve in 3-dimensional semi-Riemannian manifolds and proposed some new kind of magnetic curves, which are known as $\mathcal{T}$-magnetic curves, $\mathcal{N}$-magnetic curves and $\mathcal{B}$-magnetic curves. Furthermore, they have deduced few examples of these curves.

2.2 Preliminaries

In this section, we discussed few definitions and results of Kenmotsu manifolds. Also, we recall some basic properties of the magnetic curves and Frenet curves.

2.2.1 Kenmotsu Manifolds

"Let $\overline{M}$ be a $(2n+1)$-dimensional differentiable manifold and ϕ, ξ and η be a tensor field of type $(1,1)$, a vector field, and a 1-form on $\overline{M}$, respectively. If ϕ, ξ and η satisfy

$$\begin{cases} \phi\xi = 0, \ \ \phi^2\mathcal{U} = -\mathcal{U} + \eta\left(\mathcal{U}\right)\xi, \\ \eta\left(\xi\right) = 1, \quad \eta\left(\phi\mathcal{U}\right) = 0, \end{cases} \tag{2.2.1}$$

then the structure (ϕ, ξ, η) is called an *almost contact structure*. A $(2n+1)$-dimensional manifold $\overline{M}$ together with (ϕ, ξ, η) is said to be an almost contact manifold $(\overline{M}, \phi, \xi, \eta)$

If Riemannian metric g on almost contact manifold satisfies

$$\begin{cases} \eta(\mathcal{U}) = g\left(\mathcal{U}, \xi\right), \\ g\left(\phi\mathcal{U}, \phi\mathcal{V}\right) = g\left(\mathcal{U}, \mathcal{V}\right) - \eta(\mathcal{U})\eta(\mathcal{V}), \end{cases} \tag{2.2.2}$$

for all the vector fields $\mathcal{U}$ and $\mathcal{V}$ on $\overline{M}$, then $(\overline{M}, \phi, \xi, \eta, g)$ is said to be an *almost contact metric manifold*" [47].

Let $\overline{\nabla}$ represents the Riemannian Connection on $\overline{M}$ and if it satisfies the following equalities:

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V} = g\left(\phi\mathcal{U}, \mathcal{V}\right)\xi - \eta\left(\mathcal{V}\right)\phi\mathcal{U} \tag{2.2.3}$$

and

$$\overline{\nabla}_{\mathcal{U}}\xi = \mathcal{U} - \eta\left(\mathcal{U}\right)\xi, \tag{2.2.4}$$

then the structure $(\overline{M}, \phi, \xi, \eta, g)$ is known as a *Kenmotsu manifold* [31, 44].

Let Ω denote the fundamental 2-form of $(\overline{M}, \phi, \xi, \eta, g)$ and is given as

$$g(\phi\mathcal{U}, \mathcal{V}) = \Omega(\mathcal{U}, \mathcal{V}), \tag{2.2.5}$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$.

If $d\eta = \Omega$, then $(\overline{M}, \phi, \xi, \eta, g)$ is called a contact metric manifold, where $d\eta$ denotes the exterior derivative. Here $d\eta$ is given by

$$d\eta(\mathcal{U}, \mathcal{V}) = \frac{1}{2}\left(\mathcal{U}\eta(\mathcal{V}) - \mathcal{V}\eta(\mathcal{U}) - \eta([\mathcal{U}, \mathcal{V}])\right), \tag{2.2.6}$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$ [102].

On a contact metric manifold, a 1-form η is called a contact form, if the following condition holds:

$$\eta \wedge (d\eta)^n \neq 0,$$

everywhere on the manifold $\overline{M}$.

2.2.2 Frenet Curves

Suppose that $(\overline{M}^3, g)$ denotes a Riemannian manifold of dimension three and $\overline{\nabla}$ denote the Riemannian connection defined on it. Let $\beta : I \to \overline{M}^3$ represents the Frenet curve and $\mathcal{T}'$, $\mathcal{N}'$, $\mathcal{B}'$ be the three orthonormal vectors of the Frenet frame $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$, and is given by

$$\mathcal{T}' = \frac{d\beta}{ds}, \qquad \mathcal{N}' = \frac{\mathcal{T}'}{\kappa}, \qquad \mathcal{B}' = \mathcal{T} \times \mathcal{N},$$

where $\mathcal{T}$, $\mathcal{N}$, $\mathcal{B}$, respectively, represent the tangent vector field, principal normal vector field and binormal vector field. The vector fields $\mathcal{T}$, $\mathcal{N}$, $\mathcal{B}$ are mutually orthogonal at every point on the curve β.

The Serret-Frenet formulas are given by

$$\begin{cases} \overline{\nabla}_{\mathcal{T}}\mathcal{T} = \kappa\mathcal{N} \\ \overline{\nabla}_{\mathcal{T}}\mathcal{N} = -\kappa\mathcal{T} + \tau\mathcal{B} \\ \overline{\nabla}_{\mathcal{T}}\mathcal{B} = -\tau\mathcal{N} \end{cases} \tag{2.2.7}$$

where κ and τ represent, respectively the *curvature* and the *torsion* of the curve β.

"Let β be a unit speed curve and is called a Frenet curve of osculating order r ($where\ r \geq 1$), if there exists an orthonormal set of vector fields $\{\beta' = \mathcal{T}, E_1, E_2, ..., E_{r-1}\}$ along the curve β such that

$$\begin{cases} \overline{\nabla}_{\mathcal{T}}\mathcal{T} = \kappa_1 E_1 \\ \overline{\nabla}_{\mathcal{T}}E_1 = -\kappa_1\mathcal{T} + \kappa_2 E_2 \\ \overline{\nabla}_{\mathcal{T}}E_j = -\kappa_j E_{j-1} + \kappa_{j+1}E_{j+1} \text{ for } j = 2, 3, \ldots, r-2 \\ \overline{\nabla}_{\mathcal{T}}E_{r-1} = -\kappa_{r-1}E_{r-2} \end{cases} \tag{2.2.8}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are positive C^∞ functions of the arc length parameter s and κ_j is known as the j-th curvature of β" [31].

If osculating order of a Frenet curve is one in $(\overline{M}, g)$, then a Frenet curve is called a geodesic. A Frenet curve is said to be a circle if its osculating order is two and the curvature κ_1 is constant. A helix of order r if all the curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are constants.

"Suppose $(\overline{M}, \phi, \xi, \eta, g)$ denotes the almost contact metric manifold and β be a Frenet curve of osculating order r on $\overline{M}$. A Frenet curve of osculating order two is said to be ϕ-curve if $\{\mathcal{T}, E_1, \xi\}$ spans a ϕ-invariant space. A curve of osculating order $r \geq 3$ is said to be ϕ-curve if $\{\mathcal{T}, E_1, E_2, \ldots, E_{r-1}\}$ is ϕ-invariant. Moreover, a ϕ-helix of order r is said to be a ϕ-curve of osculating order r if $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are constants" [102, 103].

The angle between the reeb vector field ξ and tangent to the curve β is said to be a *contact angle* θ of the curve β, that is,

$$g(\beta'(s), \xi) = \cos\theta(s).$$

A curve β is called a *slant curve* if the contact angle θ is constant. When the contact angle is $\frac{\pi}{2}$, then the curves are called *Legendre curve*, and a curve of contact angle 0 is known as a *Reeb flow*.

2.2.3 Magnetic Curves

Suppose $(\overline{M}, g)$ denote the Riemannian manifold. A closed 2-form on $(\overline{M}, g)$ is known as the *magnetic field*. The trajectories of charged particles moving on $(\overline{M}, g)$ with respect to a magnetic field F represents the *magnetic curves*. A tensor field Φ of type $(1,1)$ denotes the Lorentz force of a magnetic field F on $(\overline{M}, g)$. The Lorentz force equipped with magnetic field F is an endomorphism field Φ is given by

$$F(\mathcal{U}, \mathcal{V}) = g\left(\Phi\mathcal{U}, \mathcal{V}\right), \tag{2.2.9}$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on $\overline{M}$.

Let $\overline{\nabla}$ represent the Riemannian connection endowed with metric g and β be a regular curve. If a curve β satisfies the Lorentz equation (*or Newton's equation*)

$$\overline{\nabla}_{\beta'}\beta' = \Phi(\beta'), \tag{2.2.10}$$

then it is said to be a *magnetic curve*. The magnetic curve is called a *normal magnetic curve*, if speed of magnetic curve β is unity or a magnetic curve β is parametrized by the arc length s. A curve is said to be a *magnetic trajectory* if it satisfies the Lorentz equation. The magnetic trajectories are of constant speed. When Lorentz force vanishes, we obtain

$$\overline{\nabla}_{\beta'}\beta' = 0.$$

If $\overline{\nabla}F = 0$, then a magnetic field F is called as *uniform magnetic field* [102].

2.3 Magnetic Curves in Kenmotsu Manifolds

Suppose $(\overline{M}, \phi, \xi, \eta, g)$ denotes a contact metric manifold and Ω represents the fundamental 2-form given by the equation (2.2.5). Since $\Omega = d\eta$, the magnetic field F on $\overline{M}$ is given by

$$q\Omega\left(\mathcal{U}, \mathcal{V}\right) = F_q(\mathcal{U}, \mathcal{V}), \tag{2.3.1}$$

where $\mathcal{U}$, $\mathcal{V}$ are vector fields on $\overline{M}$, and q denotes a real constant. Here F_q is called the contact magnetic field with the strength q. The magnetic curves are the geodesics on $\overline{M}$ and contact magnetic field vanishes if a real constant $q = 0$. Now, we assume $q \neq 0$.

The equation of Lorentz force Φ_q can be obtained by clubbing equations (2.2.5) and (2.2.9), *i.e.*,

$$\Phi_q = q\phi. \tag{2.3.2}$$

Thus, the Lorentz equation (2.2.10) is given by

$$\overline{\nabla}_{\beta'}\beta' = q\phi\beta', \tag{2.3.3}$$

where β denotes a Frenet curve parametrized by arc length s. The solutions of the equation (2.3.3) are said to be normal magnetic curves.

A classification result in the form of theorem for the normal magnetic curves with F_q on Kenmotsu manifold is given as follows:

Theorem 2.3.1. *Suppose $(\overline{M}, \phi, \xi, \eta, g)$ denotes a Kenmotsu manifold and F_q represents the contact magnetic field for $q \neq 0$. Then the Frenet curve β be a normal magnetic curve corresponding to F_q, if β belongs to the following list:*

1. *The geodesics are deduced as integral curve of ξ.*
2. *For a non-geodesic ϕ-circle of curvature $\kappa_1 = \sqrt{q^2 - \sin^2\theta}$ for $|q| > 1$, and having the constant angle $\theta = arc\ cot\frac{1}{|q|}$.*
3. *Legendre ϕ-curves with curvatures $\kappa_1 = |q|$ and $\kappa_2 = 1$, for $\theta = \frac{\pi}{2}$.*
4. *ϕ-helices of order three with axis ξ having curvatures $\kappa_1 = |q|$ and $\kappa_2 = \left|q\ \cos\theta + sgn(q)\ \sqrt{1 - \cos^2\theta}\right|$, for $\theta \neq \frac{\pi}{2}$.*

Proof. *When the magnetic curve β is a geodesic, by the Lorentz equation* (2.3.3) *we get $\phi\beta' = 0$, thus β' is parallel to ξ. As we know that, β' and ξ are both unit vector fields, then $\beta' = \pm\xi$. Therefore, the Frenet curve β is an integral curve of ξ.*

Suppose β *denotes a non-geodesic magnetic curve of osculating order* $r > 1$, *we get*

$$\begin{aligned} 0 &= g(q\phi\mathcal{T},\xi) = g(\overline{\nabla}_{\mathcal{T}}\mathcal{T},\xi) \\ &= \frac{d}{ds}g\left(\mathcal{T},\xi\right) - g(\mathcal{T},\overline{\nabla}_{\mathcal{T}}\xi). \end{aligned}$$

Making use of equation (2.2.4), *we find,* $\frac{d}{ds}g\left(\mathcal{T},\xi\right)$ *vanishes.*

Therefore, θ *is a constant angle between* ξ *and* $\mathcal{T}$. *As a result, we have*

$$\cos\theta = \eta\left(\mathcal{T}\right). \tag{2.3.4}$$

From the Lorentz equation and first Frenet formula, we obtain

$$\kappa_1 E_1 = q\phi\mathcal{T}, \tag{2.3.5}$$

so that, the first curvature is given as follows

$$\kappa_1 = |q|\sqrt{1-\cos^2\theta}. \tag{2.3.6}$$

From the equations (2.3.5) *and* (2.3.6), *we get*

$$\phi\mathcal{T} = sgn(q)\sqrt{1-\cos^2\theta}\ E_1, \tag{2.3.7}$$

where $sgn(q)$ *represents the real number.*

If $\kappa_2 = 0$, *that is, second curvature vanishes, then* β *is a Frenet curve of osculating order* 2. *Since* κ_1 *is a constant, then* β *will be a circle.*

By applying η *to equation* (2.3.7), *we have*

$$\begin{aligned} \eta\left(\phi\mathcal{T}\right) &= 0, \\ sgn(q)\ \eta(E_1)\ \sqrt{1-\cos^2\theta} &= 0. \end{aligned}$$

that is,

$$\eta\left(E_1\right) = 0.$$

Now, taking covariant derivative w.r.t. $\mathcal{T}$, above equation yields

$$\overline{\nabla}_{\mathcal{T}}(\eta\,(E_1)) = 0$$

$$g(\overline{\nabla}_{\mathcal{T}}E_1, \xi) + g(E_1, \overline{\nabla}_{\mathcal{T}}\xi) = 0$$

$$\sqrt{1 - cos^2\theta}\left(\sqrt{1 - cos^2\theta} - |q|\cos\theta\right) = 0$$

As β is non-geodesic, the following result is obtained.

$$\cot\theta = \frac{1}{|q|}\ .$$

For $|q| > 1$, the equation (2.3.6) gives

$$\kappa_1 = \sqrt{q^2 - 1 + \cos^2\theta}.$$

For $\kappa_2 \neq 0$, from equations (2.2.1) and (2.3.4), we obtain

$$\phi^2\mathcal{T} = -\mathcal{T} + \cos\theta\ \xi. \tag{2.3.8}$$

Now,

$$\begin{aligned}\overline{\nabla}_{\mathcal{T}}\phi\mathcal{T} &= (\overline{\nabla}_{\mathcal{T}}\phi)\mathcal{T} + \phi(\overline{\nabla}_{\mathcal{T}}\mathcal{T})\\ &= g(\phi\mathcal{T}, \mathcal{T})\xi + \eta(\mathcal{T})\phi\mathcal{T} + \phi(q\phi\mathcal{T})\\ &= q\ \xi\ \cos\theta - q\ \mathcal{T} + sgn(q)\ \xi\ \sqrt{1 - cos^2\theta} - sgn(q)\ \mathcal{T}\ \cos\theta\ \sqrt{1 - cos^2\theta}.\end{aligned} \tag{2.3.9}$$

By taking the covariant derivative of the equation (2.3.7) w.r.t. $\mathcal{T}$ and using second Frenet formula, we obtain

$$\begin{aligned}\overline{\nabla}_{\mathcal{T}}\phi\mathcal{T} &= sgn(q)\sqrt{1 - cos^2\theta}\ (\overline{\nabla}_{\mathcal{T}}E_1)\\ &= sgn\,(q)\ \sqrt{1 - cos^2\theta}\ (-\,|q|\ \mathcal{T}\ \sin\theta + \kappa_2 E_2).\end{aligned} \tag{2.3.10}$$

From equations (2.3.9) and (2.3.10), we get

$$\begin{aligned}\xi\ sgn(q)\ \sqrt{1 - cos^2\theta} - q\ \mathcal{T} + q\ \xi\ \cos\theta - \mathcal{T}\ sgn(q)\ \cos\theta\ \sqrt{1 - cos^2\theta}\\ = (sgn(q)\ \kappa_2 E_2 - q\ \mathcal{T}\ \sin\theta)\sqrt{1 - cos^2\theta}.\end{aligned}$$

A direct calculation yields,

$$(\xi - \mathcal{T}\,\cos\theta)\,(sgn(q)\,\sin\theta + q\,\cos\theta) = sgn(q)\,\sqrt{1 - cos^2\theta}\,\kappa_2 E_2. \tag{2.3.11}$$

Thus, we have

$$\kappa_2 = \left| q\,\cos\theta + sgn(q)\,\sqrt{1 - cos^2\theta} \right|.$$

From above equation and (2.3.11)*, the* ξ *can be expressed in terms of Frenet frame of* β *as*

$$\xi = (\mathcal{T}\cos\theta + \varepsilon\,sgn(q)\sqrt{1 - cos^2\theta}\,E_2), \tag{2.3.12}$$

where $\varepsilon = sgn(q\,\cos\theta + sgn(q)\,\sqrt{1 - cos^2\theta})$

Now, operating ϕ *on equation* (2.3.12)*, we have*

$$\phi\xi = \phi\mathcal{T}\,\cos\,\theta + \varepsilon\,sgn(q)\,\sin\,\theta\,\phi E_2,$$

or

$$\phi E_2 = -\varepsilon E_1\sqrt{1 - \sin^2\theta}.$$

Applying ϕ *on* (2.3.7) *and making use of* (2.3.4), (2.3.12)*, we obtain*

$$\begin{aligned}\phi E_1 &= \frac{-\mathcal{T}}{\sin\theta\,sgn(q)} + \frac{\xi\,\cos\,\theta}{\sin\theta\,sgn(q)}\\ &= \varepsilon\,E_2\,\cos\theta - sgn(q)\,\mathcal{T}\,\sqrt{1 - cos^2\theta}.\end{aligned} \tag{2.3.13}$$

After applying η *to equation* (2.3.13)*, we have*

$$\eta E_2 = \varepsilon sgn(q)\sqrt{1 - \cos^2\theta}.$$

For $\theta = \frac{\pi}{2}$*, the equation* (2.3.12) *gives*

$$E_2 = -\xi\,sgn(q),$$

and the curvatures are $\kappa_1 = |q|$, $\kappa_2 = 1$ *and* $\kappa_3 = 0$.

For $\theta \neq \frac{\pi}{2}$*, taking covariant derivative of equation* (2.3.13) *w.r.t.* $\mathcal{T}$*, we obtain*

$$(\overline{\nabla}_{\mathcal{T}}\phi)E_1 + \phi(\overline{\nabla}_{\mathcal{T}}E_1) = -sgn(q)\,\sin\,\theta\,(\overline{\nabla}_{\mathcal{T}}\mathcal{T}) + \varepsilon\,\cos\,\theta\,(\overline{\nabla}_{\mathcal{T}}E_2),$$

or

$$\overline{\nabla}_{\mathcal{T}} E_2 = (q \cos\theta - \sqrt{1-\cos^2\theta}\,\varepsilon\ sgn(q)\ \xi\ \cos\theta - \sqrt{1-\cos^2\theta}\ sgn(q))\ E_1.$$

and hence $\kappa_3 = 0$.

Therefore, the non-geodesic magnetic curves are Frenet curves of osculating order three on the Kenmotsu manifolds with constant curvatures κ_1 *and* κ_2.

Remark 2.3.1. *Since* $\xi \in span\{\mathcal{T}, E_2\}$, *thus* ξ *can be written as*

$$\xi = \mathcal{T}\sqrt{1-\sin^2\theta} + \rho\ E_2. \tag{2.3.14}$$

Taking norm on both sides, we get

$$\rho^2 = 1 - \cos^2\theta.$$

When $\theta = \frac{\pi}{2}$, *we obtain*

$$\rho^2 = 1 \qquad \text{and} \qquad \xi = \rho\ E_2.$$

Proposition 2.1. *In Kenmotsu manifold, if* β *denotes a non-geodesic Legendre* ϕ*-curve of order* 3*, then* $\kappa_2 = 1$ *and* $E_2 = \pm\xi$.

2.4 Slant Curves in Kenmotsu Manifolds

Suppose a Frenet curve β is parametrized by the arc length s in an almost contact metric 3-manifold. The contact angle of the unit speed curve β is defined by

$$g\left(\beta'(s), \xi\right) = \cos\theta(s),$$

where $\theta(s) = (0, \pi)$.

Now, taking covariant derivative of above formula along the curve β, we obtain

$$\begin{aligned} -\theta' \sin\theta &= g\left(\mathcal{T}, \overline{\nabla}_{\mathcal{T}}\xi\right) + g\left(\kappa\mathcal{N}, \xi\right) \\ &= 1 - \cos^2\theta + \kappa\ \eta(\mathcal{N}). \end{aligned}$$

From the above relation, we obtain

Proposition 2.2. *Let β denotes a unit speed curve. If β is a slant curve in a Kenmotsu manifold, then we obtain;*

$$\eta(\mathcal{N}) = \left(\frac{\cos^2\theta - 1}{\kappa}\right). \tag{2.4.1}$$

Making use of the Frenet Frame field $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$, then ξ can be expressed as

$$\xi = (\sqrt{1-\sin^2\theta})\ \mathcal{T} + \left(\frac{\cos^2\theta - 1}{\kappa}\right)\mathcal{N} + \eta(\mathcal{B})\,\mathcal{B}.$$

Since ξ represents the unitary vector field, the above expression yields

$$\eta(\mathcal{B}) = \frac{\sqrt{1-\cos^2\theta}}{\kappa}\sqrt{\kappa^2 + \cos^2\theta - 1}. \tag{2.4.2}$$

Remark 2.4.1. *For a slant curve β, the expression for ξ in the Frenet Frame field is given by*

$$\xi = (\sqrt{1-\sin^2\theta})\ \mathcal{T} + \left(\frac{\cos^2\theta - 1}{\kappa}\right)\mathcal{N} + \left(\frac{\sqrt{1-\cos^2\theta}}{\kappa}\sqrt{\kappa^2 + \cos^2\theta - 1}\right)\mathcal{B}. \tag{2.4.3}$$

2.5 Curvature and Torsion

Suppose β denotes a non geodesic curve, thus β cannot be an integral curve of characteristic vector field ξ. Now, choose the orthonormal frame field in almost contact metric 3-manifold $\overline{M}$, along the non geodesic curve β as

$$e_1 = (\beta') = \mathcal{T}, \qquad e_2 = \left(\frac{\phi\beta'}{\sqrt{1-\cos^2\theta}}\right), \qquad e_3 = \left(\frac{\xi - \beta'\cos\theta}{\sqrt{1-\cos^2\theta}}\right). \tag{2.5.1}$$

The reeb vector field ξ can be expressed as

$$\xi = (\sqrt{1-\sin^2\theta})\, e_1 + (\sqrt{1-\cos^2\theta})\, e_3. \tag{2.5.2}$$

Therefore, for a slant curve β, we obtain

$$\begin{cases} \overline{\nabla}_{\beta'} e_1 = \delta\sqrt{1-\cos^2\theta}\, e_2 \\ \overline{\nabla}_{\beta'} e_2 = \delta\sqrt{1-\cos^2\theta}\ e_1 - \cos\theta\ e_2 - \delta\cos\theta\ e_3 \\ \overline{\nabla}_{\beta'} e_3 = \sqrt{1-\cos^2\theta}\ e_1 - \delta\cos\theta\ e_2 - \cos\theta\ \ e_3 \end{cases} \tag{2.5.3}$$

where $\delta = \frac{g(\overline{\nabla}_{\beta'}\beta', \phi\beta')}{1-\cos^2\theta}$.

From the equation (2.2.4), we obtain

$$\begin{aligned}\overline{\nabla}_{\beta'}\xi &= \beta' - \eta(\beta')\xi \\ &= (1-\cos^2\theta)\ e_1 - \left(\sqrt{1-\cos^2\theta}\sqrt{1-\sin^2\theta}\right)\ e_3.\end{aligned}$$

The cross product of two vector fields in arbitrary oriented $(\overline{M}^3, g)$ is given by

$$dv_g(\mathcal{U}, \mathcal{V}, \mathcal{W}) = g\left(\mathcal{U} \times \mathcal{V}, \mathcal{W}\right),$$

for any vector fields $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ on $\overline{M}^3$, where dv_g denotes the volume given by the metric g. The cross product of two vector fields on the almost contact metric 3-manifold is defined by

$$\mathcal{U} \times \mathcal{V} = g\left(\phi\mathcal{U}, \mathcal{V}\right)\xi - \eta\left(\mathcal{V}\right)\phi\mathcal{U} + \eta(\mathcal{U})\phi(\mathcal{V}).$$

Since $\mathcal{U}$ is orthogonal to characteristic vector field ξ. Suppose $\mathcal{U}$, $\phi\mathcal{U}$ and ξ are basis vectors, taken in the anti-clockwise sense, so that

$$\phi\beta' = \xi \times \beta'.$$

As we know that $\beta' = \mathcal{T}$, the magnetic equation is

$$\overline{\nabla}_{\beta'}\beta' = q\left(\xi \times \beta'\right) = \kappa\mathcal{N}. \tag{2.5.4}$$

Consequently, we have

$$\begin{aligned}\kappa^2 &= q^2\ g\left(\xi \times \beta', \xi \times \beta'\right) \\ &= q^2\ (1-\cos^2\theta).\end{aligned}$$

Hence, the curve β has constant curvature *i.e.* $\kappa = |q|\sqrt{1-\cos^2\theta}$. From the equation (2.5.4), we obtain

$$\mathcal{N} = \frac{q}{\kappa}\phi\beta'.$$

The Binormal vector field $\mathcal{B}$ is defined as

$$\begin{aligned}\mathcal{B} &= \beta' \times \mathcal{N} \\ &= \beta' \times \left[\frac{q}{\kappa}(\xi \times \beta')\right] \\ &= \frac{q}{\kappa}\left[\xi - \left(\sqrt{1-\sin^2\theta}\right)\beta'\right].\end{aligned}$$

Differentiating above equation along β' by operating Levi-Civita connection $\overline{\nabla}$ and making use of the equations (2.2.4) and (2.5.2), we get

$$\begin{aligned}\overline{\nabla}_{\beta'}\mathcal{B} &= \frac{q}{\kappa}\left[\overline{\nabla}_{\beta'}\xi - \left(\sqrt{1-\sin^2\theta}\right)\ \overline{\nabla}_{\beta'}\beta'\right] \\ &= \frac{q}{\kappa}\left[\beta' - (1-\sin^2\theta)\ e_1 - \left(q\sin\theta\sqrt{1-\sin^2\theta}\right)\ e_2 - \left(\sin\theta\ \sqrt{1-\sin^2\theta}\right)\ e_3\right].\end{aligned}$$

Now, using $\overline{\nabla}_{\beta'}\mathcal{B} = -\tau\mathcal{N}$ and equation (2.5.1), we get

$$\tau = q\sqrt{1-\sin^2\theta}.$$

2.6 Conclusion

In this chapter, the magnetic curves and slant curves have been investigated in Kenmotsu manifolds. The magnetic trajectories corresponding to contact magnetic fields have been investigated and a classification theorem is obtained for the normal magnetic curves. Additionally, we have obtained a characterization result for the Frenet curve to be a slant curve. Furthermore, we have given some results on curvature and torsion. This work may be beneficial in the study of some other manifolds like f-Kenmotsu manifolds and almost Kenmotsu manifolds.

Chapter 3

Contact CR-submanifold of a Kenmotsu Manifold with Killing Tensor Field

3.1 Introduction

For the last forty years, CR-submanifolds are an active field of research in differential geometry. In 1978, The concept of CR-submanifolds of a Kaehler manifold was established by A. Bejancu [2], which generalize the complex as well as totally real submanifolds. In [1], A. Bejancu investigated some problems of geometry of CR-submanifolds. Later, Atceken *et al.* [54, 55] studied some interesting properties of the Contact CR-submanifolds in Kenmotsu manifolds.

In 1971, Professor D. E. Blair [29] introduced the concept of a Killing tensor field. K. Kenmotsu [44] investigated a certain class of almost contact manifold in 1972, which is known as a Kenmotsu manifold and then many other authors investigated this manifold [4, 111, 112].

N. Papaghuic [76] and M. Kobayashi [63] studied the geometry of semi-invariant submanifolds of a Kenmotsu manifold. Gupta *et al.* [90] studied the slant submanifold in a Kenmotsu manifold and obtained some examples. In [81], Pandey *et al.* characterized a slant submanifold in Kenmotsu manifold by using Killing tensor fields. Moreover, Pandey *et al.* studied B. Y. Chen's inequalities for bi-slant submanifolds, and obtained Existence and uniqueness theorem for slant immersions in Kenmotsu space forms [82, 83]. Avik De [4] obtained an example of a 3-dimensional Kenmotsu manifold with η-parallel Ricci tensor and also obtained Killing vector field condition in Kenmotsu manifold.

3.2 Preliminaries

"Suppose $\overline{M}$ be a $(2n+1)$-dimensional differentiable manifold. Let ϕ, ξ, η and g represent a tensor field of type $(1,1)$, a vector field, a 1-form and a Riemannian metric on $\overline{M}$, respectively. If ϕ, ξ, η and g satisfy the following conditions:

$$\begin{cases} \phi\xi = 0, \ \phi^2\mathcal{U} = -\mathcal{U} + \eta(\mathcal{U})\xi, \\ \eta(\xi) = 1, \quad \eta(\phi\mathcal{U}) = 0, \end{cases} \tag{3.2.1}$$

and

$$\begin{cases} g(\phi\mathcal{U}, \phi\mathcal{V}) = g(\mathcal{U}, \mathcal{V}) - \eta(\mathcal{U})\eta(\mathcal{V}), \\ \qquad \eta(\mathcal{U}) = g(\mathcal{U}, \xi), \end{cases} \tag{3.2.2}$$

for all vector fields $\mathcal{U}$ and $\mathcal{V}$ on $\overline{M}$, then the structure (ϕ, ξ, η, g) is called an *almost contact metric structure*. A $(2n+1)$-dimensional manifold $\overline{M}$ together with an almost contact metric structure is said to be an *almost contact metric manifold* [30].

Let $\overline{\nabla}$ denotes the Levi-Civita Connection on $\overline{M}$ and if the following conditions are satisfied

$$\begin{cases} (\overline{\nabla}_{\mathcal{U}}\phi)\mathcal{V} = g(\phi\mathcal{U}, \mathcal{V})\xi - \eta(\mathcal{V})\phi\mathcal{U}, \\ \qquad \overline{\nabla}_{\mathcal{U}}\xi = \mathcal{U} - \eta(\mathcal{U})\xi, \end{cases} \tag{3.2.3}$$

then, the structure $(\overline{M}, \phi, \xi, \eta, g)$ is called a *Kenmotsu manifold*" [44, 47].

Suppose M denote an isometrically immersed submanifold in $\overline{M}$. Suppose ∇ and $\overline{\nabla}$, respectively, denote the Riemannian connections on M and $\overline{M}$. Therefore, the *Gauss* and *Weingarten* formulas are as follows:

$$\overline{\nabla}_{\mathcal{U}}\mathcal{V} = h(\mathcal{U},\mathcal{V}) + \nabla_{\mathcal{U}}\mathcal{V}, \tag{3.2.4}$$

and

$$\overline{\nabla}_{\mathcal{U}}\mathcal{X} = \nabla^{\perp}_{\mathcal{U}}\mathcal{X} - A_{\mathcal{X}}\mathcal{U}, \tag{3.2.5}$$

for any vector fields $\mathcal{U}$, $\mathcal{V}$ in $\Gamma(TM)$ and $\mathcal{X} \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ represents the normal connection on $T^{\perp}M$, A be the shape operator and h denote the second fundamental form of M in Kenmotsu manifold $\overline{M}$.

The shape operator A and second fundamental form h are connected as

$$g(h(\mathcal{U},\mathcal{V}),\mathcal{X}) = g(A_{\mathcal{X}}\mathcal{U},\mathcal{V}). \tag{3.2.6}$$

In Kenmotsu manifold, M denotes an isometrically immersed submanifold. Let $\mathcal{U}$ be any vector field tangent to M, then we have

$$\phi\mathcal{U} = t\mathcal{U} + \omega\mathcal{U}, \tag{3.2.7}$$

where $t\mathcal{U}$ and $\omega\mathcal{U}$ respectively, denote the tangential and the normal component of $\phi\mathcal{U}$.

The covariant derivative of t and ω are expressed as

$$(\nabla_{\mathcal{U}}t)\mathcal{V} = \nabla_{\mathcal{U}}t\mathcal{V} - t\nabla_{\mathcal{U}}\mathcal{V},$$

and

$$(\nabla_{\mathcal{U}}\omega)\mathcal{V} = \nabla^{\perp}_{\mathcal{U}}\omega\mathcal{V} - \omega\nabla_{\mathcal{U}}\mathcal{V}.$$

Analogously, for any vector field $\mathcal{X}$ normal to M, we put

$$\phi\mathcal{X} = B\mathcal{X} + C\mathcal{X}, \tag{3.2.8}$$

where $B\mathcal{X}$ and $C\mathcal{X}$ respectively, denote the tangential and the normal component of $\phi\mathcal{X}$.

The covariant derivative of B and C are expressed as

$$(\nabla_{\mathcal{U}}B)\mathcal{X} = \nabla_{\mathcal{U}}B\mathcal{X} - B\nabla^{\perp}_{\mathcal{U}}\mathcal{X},$$

and

$$(\nabla_{\mathcal{U}}C)\mathcal{X} = \nabla^{\perp}_{\mathcal{U}}C\mathcal{X} - C\nabla^{\perp}_{\mathcal{U}}\mathcal{X}.$$

If endomorphism t is defined by equation (3.2.7), then we have

$$g\left(t\mathcal{U},\mathcal{V}\right)+g\left(\mathcal{U},t\mathcal{V}\right)=0. \tag{3.2.9}$$

Definition 3.2.1. *[55] "Let M be a submanifold of a Kenmotsu manifold $\overline{M}$. Then M is said to be a contact CR-submanifold of $\overline{M}$ if there exist a differentiable distribution $D : p \to D_p \subseteq T_p(M)$ on M satisfying the following conditions:*

(i) $TM = D \oplus D^{\perp}, \quad \xi \in D,$

(ii) D is a invariant with respect to ϕ, i.e., $\phi D_p = D_p$,

(iii) the orthogonal complementary distribution $D^{\perp} : p \to D^{\perp}_p \subseteq T_p(M)$ is anti-invariant, i.e., $\phi D^{\perp}_p \subseteq T^{\perp}_p(M)$, *for each $p \in M$."*

If $dim\ D_p = 0$, then M is said to be a $totally\ real\ submanifold$ and if $dim\ D^{\perp}_p = 0$, then M is called a $complex\ submanifold$. A contact CR-submanifold is known as proper, if it is neither complex nor totally real.

Suppose M denotes a contact CR-submanifold in $\overline{M}$ and let $\mathcal{U}$ and $\mathcal{V}$ are vector fields in $\Gamma(TM)$. Then, from equations (3.2.3), (3.2.7), (3.2.8) together with the Gauss and Weingarten formulas [55], we obtain the following equation:

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V} = \overline{\nabla}_{\mathcal{U}}\phi\mathcal{V} - \phi\overline{\nabla}_{\mathcal{U}}\mathcal{V}, \tag{3.2.10}$$

or,

$$g\left(\phi\mathcal{U},\mathcal{V}\right) - \eta\left(\mathcal{V}\right)\ \phi\mathcal{U} = \overline{\nabla}_{\mathcal{U}}t\mathcal{V} + \overline{\nabla}_{\mathcal{U}}\omega\mathcal{V} - \phi\nabla_{\mathcal{U}}\mathcal{V} - \phi h(\mathcal{U},\mathcal{V}).$$

On comparing the tangential and normal component of the both sides of above equation, the following equations are obtained:

$$(\nabla_{\mathcal{U}}t)\mathcal{V} = A_{\omega\mathcal{V}}\mathcal{U} + Bh\left(\mathcal{U},\mathcal{V}\right) + g\left(t\mathcal{U},\mathcal{V}\right)\xi - \eta\left(\mathcal{V}\right)t\mathcal{U}, \tag{3.2.11}$$

and

$$\left(\nabla_{\mathcal{U}}\omega\right)\mathcal{V} = Ch\left(\mathcal{U},\mathcal{V}\right) - h\left(\mathcal{U},t\mathcal{V}\right) - \eta(\mathcal{V})\omega\mathcal{U}. \tag{3.2.12}$$

Since the structure vector field ξ is tangent to M. Then, from equations (3.2.3) and (3.2.6), we get

$$A_{\mathcal{X}}\xi = h\left(\mathcal{U},\xi\right) = 0, \tag{3.2.13}$$

for any vector fields $\mathcal{U}$ in $\Gamma(TM)$ and $\mathcal{X}$ in $\Gamma(T^{\perp}M)$. Thus, the equation (3.2.11) becomes

$$(\nabla_{\mathcal{U}}t)\mathcal{V} = g\left(t\mathcal{U},\mathcal{V}\right)\xi - \eta\left(\mathcal{V}\right)t\mathcal{U}, \tag{3.2.14}$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields in $\Gamma(D)$. Therefore, the induced structure t is called a Kenmotsu structure on M [55].

Suppose M represent the contact CR-submanifold in $\overline{M}$. Thus, the equation (3.2.11) reduces to

$$(\nabla_{\mathcal{U}}t)\mathcal{V} = Bh\left(\mathcal{U},\mathcal{V}\right) + g\left(t\mathcal{U},\mathcal{V}\right)\xi - \eta\left(\mathcal{V}\right)t\mathcal{U}, \tag{3.2.15}$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields in $\Gamma(D)$ [54].

If $h = 0$, *i.e.*, the second fundamental form vanishes, then M is said to be a totally geodesic submanifold. A submanifold M is known as totally umbilical if Riemannian metric g and second fundamental form h satisfies the following equality

$$g\left(\mathcal{U},\mathcal{V}\right)H = h\left(\mathcal{U},\mathcal{V}\right),$$

where H denotes the mean curvature vector. Moreover, if $H = 0$, then the submanifold M is called minimal.

Let ϕ denotes the tensor field of type $(1,1)$ and is said to be Killing tensor field [29], if following condition is satisfied:

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V} + \left(\overline{\nabla}_{\mathcal{V}}\phi\right)\mathcal{U} = 0. \tag{3.2.16}$$

3.3 Contact CR-submanifold of a Kenmotsu Manifold $\overline{M}$ with Killing Tensor Field

In this section, some interesting results have been examined for contact CR-submanifold with a Killing tensor field in Kenmotsu manifold.

Theorem 3.3.1. *Suppose $\overline{M}$ is a Kenmotsu manifold and M denotes a contact CR-submanifold of $\overline{M}$ with Killing tensor field ϕ, then*

$$(\overline{\nabla}_{\mathcal{U}}t\mathcal{V}+\overline{\nabla}_{\mathcal{V}}t\mathcal{U})-t(\overline{\nabla}_{\mathcal{U}}\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\mathcal{U})=\omega(\overline{\nabla}_{\mathcal{U}}\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\mathcal{U})-\left(\overline{\nabla}_{\mathcal{U}}\omega\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\omega\mathcal{U}\right). \quad (3.3.1)$$

Proof. *From the equation* (3.2.10)*, we have*

$$(\overline{\nabla}_{\mathcal{U}}\phi)\mathcal{V}=\overline{\nabla}_{\mathcal{U}}\phi\mathcal{V}-\phi\overline{\nabla}_{\mathcal{U}}\mathcal{V}.$$

Interchanging $\mathcal{U}$ and $\mathcal{V}$ in the above equation, we have

$$\left(\overline{\nabla}_{\mathcal{V}}\phi\right)\mathcal{U}=\overline{\nabla}_{\mathcal{V}}\phi\mathcal{U}-\phi\overline{\nabla}_{\mathcal{V}}\mathcal{U}.$$

By adding above two equations, we obtain

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V}+\left(\overline{\nabla}_{\mathcal{V}}\phi\right)\mathcal{U}=\overline{\nabla}_{\mathcal{U}}\phi\mathcal{V}-\phi\overline{\nabla}_{\mathcal{U}}\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\phi\mathcal{U}-\phi\overline{\nabla}_{\mathcal{V}}\mathcal{U}.$$

Now, making use of equation (3.2.16)*, we get*

$$0=\overline{\nabla}_{\mathcal{U}}\phi\mathcal{V}-\phi\overline{\nabla}_{\mathcal{U}}\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\phi\mathcal{U}-\phi\overline{\nabla}_{\mathcal{V}}\mathcal{U}. \quad (3.3.2)$$

By the use of (3.2.7)*, above equation becomes*

$$(\overline{\nabla}_{\mathcal{U}}t\mathcal{V}+\overline{\nabla}_{\mathcal{V}}t\mathcal{U})-t(\overline{\nabla}_{\mathcal{U}}\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\mathcal{U})=\omega(\overline{\nabla}_{\mathcal{U}}\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\mathcal{U})-\left(\overline{\nabla}_{\mathcal{U}}\omega\mathcal{V}+\overline{\nabla}_{\mathcal{V}}\omega\mathcal{U}\right).$$

Theorem 3.3.2. *Suppose M denotes a contact CR-submanifold with Killing tensor field ϕ of a Kenmotsu manifold $\overline{M}$, then*

$$\eta(\mathcal{V})\,t\mathcal{U}+\eta(\mathcal{U})\,t\mathcal{V}=0 \quad (3.3.3)$$

and

$$\eta(\mathcal{V})\,\omega\mathcal{U}+\eta(\mathcal{U})\,\omega\mathcal{V}=0. \quad (3.3.4)$$

Proof. *From the equation* (3.2.3), *we have*

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V} = g\left(\phi\mathcal{U},\mathcal{V}\right)\xi - \eta\left(\mathcal{V}\right)\phi\mathcal{U}.$$

By swapping $\mathcal{U}$ *and* $\mathcal{V}$*, above equation yields*

$$\left(\overline{\nabla}_{\mathcal{V}}\phi\right)\mathcal{U} = -g\left(\phi\mathcal{U},\mathcal{V}\right)\xi - \eta\left(\mathcal{U}\right)\phi\mathcal{V}.$$

Now, combining above two equations, we obtain

$$\left(\overline{\nabla}_{\mathcal{U}}\phi\right)\mathcal{V} + \left(\overline{\nabla}_{\mathcal{V}}\phi\right)\mathcal{U} = -\eta\left(\mathcal{V}\right)\phi\mathcal{U} - \eta\left(\mathcal{U}\right)\phi\mathcal{V}.$$

By using the equation (3.2.16), *we get*

$$-\eta\left(\mathcal{V}\right)\phi\mathcal{U} - \eta\left(\mathcal{U}\right)\phi\mathcal{V} = 0. \tag{3.3.5}$$

Making use of (3.2.7) *in equation* (3.3.5), *then comparing the tangential and normal components. The desired result have been acquired.*

Theorem 3.3.3. *Suppose* $\overline{M}$ *is a Kenmotsu manifold and* M *denotes a contact CR-submanifold of* $\overline{M}$ *with Killing tensor field* ϕ*, then the induced structure* t *satisfies*

$$(\nabla_{\mathcal{U}}t)\mathcal{V} + (\nabla_{\mathcal{V}}t)\mathcal{U} = 0. \tag{3.3.6}$$

Proof. *From the equation* (3.2.14), *we have*

$$(\nabla_{\mathcal{U}}t)\mathcal{V} = g\left(t\mathcal{U},\mathcal{V}\right)\xi - \eta\left(\mathcal{V}\right)t\mathcal{U}.$$

Now swapping $\mathcal{U}$ *and* $\mathcal{V}$ *in above equation, we obtain*

$$(\nabla_{\mathcal{V}}t)\mathcal{U} = g\left(\mathcal{U},t\mathcal{V}\right)\xi - \eta\left(\mathcal{U}\right)t\mathcal{V}.$$

On clubbing above two equations, we have

$$(\nabla_{\mathcal{U}}t)\mathcal{V} + (\nabla_{\mathcal{V}}t)\mathcal{U} = -\eta\left(\mathcal{V}\right)t\mathcal{U} - \eta\left(\mathcal{U}\right)t\mathcal{V}.$$

Making use of (3.3.3) *in above equation, the required result have been obtained.*

Theorem 3.3.4. *Suppose* $\overline{M}$ *is a Kenmotsu manifold and* M *denotes a contact CR-submanifold of* $\overline{M}$ *with Killing tensor field* ϕ. *If second fundamental form h is parallel then* M *is a totally geodesic manifold.*

Proof. *Interchanging* $\mathcal{U}$ *and* $\mathcal{V}$ *in equation* (3.2.15), *we get*

$$(\nabla_{\mathcal{V}}t)\mathcal{U} = Bh(\mathcal{U},\mathcal{V}) - g\left(\mathcal{V},t\mathcal{U}\right)\xi - \eta\left(\mathcal{U}\right)t\mathcal{V}. \tag{3.3.7}$$

By combining equations(3.2.15) *and* (3.3.7), *we obtain*

$$(\nabla_{\mathcal{U}}t)\mathcal{V} + (\nabla_{\mathcal{V}}t)\mathcal{U} = 2Bh\left(\mathcal{U},\mathcal{V}\right) - \eta\left(\mathcal{V}\right)t\mathcal{U} - \eta\left(\mathcal{U}\right)t\mathcal{V}.$$

Now, making use of equations (3.3.3) *and* (3.3.6), *we get*

$$h\left(\mathcal{U},\mathcal{V}\right) \ \textit{vanishes}.$$

where $\mathcal{U}$ *and* $\mathcal{V}$ *are any vector fields in* $\Gamma(TM)$.

Lemma 3.3.1. *Suppose* M *denotes a contact CR-submanifold of a Kenmotsu manifold* $\overline{M}$ *with Killing tensor field* ϕ, *then*

$$A_{\omega\mathcal{V}}\mathcal{U} + A_{\omega\mathcal{U}}\mathcal{V} + 2Bh\left(\mathcal{U},\mathcal{V}\right) = 0. \tag{3.3.8}$$

Proof. *By interchanging* $\mathcal{U}$ *and* $\mathcal{V}$ *in equation* (3.2.11), *we have*

$$(\nabla_{\mathcal{V}}t)\mathcal{U} = A_{\omega\mathcal{U}}\mathcal{V} + Bh\ \left(\mathcal{U},\mathcal{V}\right) + g\left(t\mathcal{V},\mathcal{U}\right)\xi - \eta\left(\mathcal{U}\right)t\mathcal{V}. \tag{3.3.9}$$

On Clubbing equations (3.2.11) *and* (3.3.9), *we get the following equation;*

$$\begin{aligned}(\nabla_{\mathcal{U}}t)\mathcal{V} + (\nabla_{\mathcal{V}}t)\mathcal{U} = \ & A_{\omega\mathcal{V}}\mathcal{U} + A_{\omega\mathcal{U}}\mathcal{V} + 2Bh\left(\mathcal{U},\mathcal{V}\right) + g\left(t\mathcal{U},\mathcal{V}\right)\xi \\ & + g\left(t\mathcal{V},\mathcal{U}\right)\xi - \eta\left(\mathcal{U}\right)t\mathcal{V} - \eta\left(\mathcal{V}\right)t\mathcal{U}.\end{aligned}$$

Making use of equation (3.2.9), *then we get*

$$(\nabla_{\mathcal{U}}t)\mathcal{V} + (\nabla_{\mathcal{V}}t)\mathcal{U} = A_{\omega\mathcal{V}}\mathcal{U} + A_{\omega\mathcal{U}}\mathcal{V} + 2Bh\left(\mathcal{U},\mathcal{V}\right) - \eta\left(\mathcal{U}\right)t\mathcal{V} - \eta\left(\mathcal{V}\right)t\mathcal{U}.$$

Since t *satisfies* (3.3.3) *and* (3.3.6). *Hence, the desired result have been acquired.*

Proposition 3.1. *Let M represents a contact CR-submanifold of $\overline{M}$ with Killing tensor field ϕ. If the endomorphism t is parallel, thus M is an anti-invariant submanifold in $\overline{M}$.*

Proof. *By swapping $\mathcal{U}$ and $\mathcal{V}$ in* (3.2.15)*, we obtain*

$$(\nabla_{\mathcal{V}}t)\,\mathcal{U} = Bh\,(\mathcal{U},\mathcal{V}) + g\,(t\mathcal{V},\mathcal{U})\,\xi - \eta\,(\mathcal{U})\,t\mathcal{V},$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields in $\Gamma(D)$.

Now, combining above equation with (3.2.15)*, we have*

$$(\nabla_{\mathcal{U}}t)\,\mathcal{V} + (\nabla_{\mathcal{V}}t)\,\mathcal{U} = 2Bh\,(\mathcal{U},\mathcal{V}) + g\,(t\mathcal{U},\mathcal{V})\,\xi + g\,(t\mathcal{V},\mathcal{U})\,\xi - \eta\,(\mathcal{V})\,t\mathcal{U} - \eta\,(\mathcal{U})\,t\mathcal{V}.$$

Making use of equations (3.2.9) *and* (3.3.6) *in above equation, we get*

$$2Bh\,(\mathcal{U},\mathcal{V}) - \eta\,(\mathcal{V})\,t\mathcal{U} - \eta\,(\mathcal{U})\,t\mathcal{V} = 0.$$

Choosing $\xi = \mathcal{V}$ and taking into consideration of (3.2.1) *and* (3.2.13)*, we conclude that*

$$t\mathcal{U} = 0.$$

Hence, M is an anti-invariant submanifold.

Proposition 3.2. *Let M denotes a contact CR-submanifold of $\overline{M}$ with Killing tensor field ϕ. Then M is called invariant submanifold in $\overline{M}$ if the endomorphism ω is parallel.*

Proof. *Interchanging $\mathcal{U}$ and $\mathcal{V}$ in equation* (3.2.12)*, we obtain the following equation.*

$$(\nabla_{\mathcal{V}}\omega)\,\mathcal{U} = Ch\,(\mathcal{U},\mathcal{V}) - h\,(\mathcal{V},t\mathcal{U}) - \eta\,(\mathcal{U})\,\omega\mathcal{V}, \tag{3.3.10}$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields in $\Gamma(TM)$.

Adding equations (3.2.12) *and* (3.3.10)*, we obtain*

$$(\nabla_{\mathcal{U}}\omega)\,\mathcal{V} + (\nabla_{\mathcal{V}}\omega)\,\mathcal{U} = 2C\;h\,(\mathcal{U},\mathcal{V}) - h\,(\mathcal{U},t\mathcal{V}) - h\,(\mathcal{V},t\mathcal{U}) - \eta\,(\mathcal{V})\,\omega\mathcal{U} - \eta\,(\mathcal{U})\,\omega\mathcal{V}.$$

If endomorphism ω is parallel, above equation yields

$$2C\ h(\mathcal{U},\mathcal{V}) - h(\mathcal{U},t\mathcal{V}) - h(\mathcal{V},t\mathcal{U}) - \eta(\mathcal{V})\,\omega\mathcal{U} - \eta(\mathcal{U})\,\omega\mathcal{V} = 0.$$

Now, choosing $\xi = \mathcal{V}$ and taking into consideration of (3.2.1) *and* (3.2.13)*, we conclude that*

$$\omega\mathcal{U} = 0.$$

Therefore, M is an invariant submanifold.

Lemma 3.3.2. *Suppose $\overline{M}$ represents a Kenmotsu manifold and M denotes a contact CR-submanifold of $\overline{M}$ with Killing tensor field ϕ, then*

$$(\nabla_{\mathcal{U}}\omega)\,\mathcal{V} + (\nabla_{\mathcal{V}}\omega)\,\mathcal{U} = 0. \tag{3.3.11}$$

iff

$$2Ch(\mathcal{U},\mathcal{V}) = h(\mathcal{U},t\mathcal{V}) + h(\mathcal{V},t\mathcal{U}). \tag{3.3.12}$$

Proof. *From the equation* (3.2.12)*, we get*

$$(\nabla_{\mathcal{U}}\omega)\,\mathcal{V} = Ch(\mathcal{U},\mathcal{V}) - h(\mathcal{U},t\mathcal{V}) - \eta(\mathcal{V})\omega\mathcal{U}.$$

Now combining above equation and (3.3.10)*, we get*

$$(\nabla_{\mathcal{U}}\omega)\,\mathcal{V} + (\nabla_{\mathcal{V}}\omega)\,\mathcal{U} = 2Ch(\mathcal{U},\mathcal{V}) - h(\mathcal{U},t\mathcal{V}) - h(\mathcal{V},t\mathcal{U}) - \eta(\mathcal{V})\,\omega\mathcal{U} - \eta(\mathcal{U})\,\omega\mathcal{V}.$$

Making use of equation (3.3.4)*, we obtain*

$$(\nabla_{\mathcal{U}}\omega)\,\mathcal{V} + (\nabla_{\mathcal{V}}\omega)\,\mathcal{U} = 2Ch(\mathcal{U},\mathcal{V}) - h(\mathcal{U},t\mathcal{V}) - h(\mathcal{V},t\mathcal{U}).$$

Thus, the result follows.

3.4 Examples

In this part, few examples have been deduced for Kenmotsu manifolds which satisfies the Killing tensor field ϕ.

Example 3.4.1. *Suppose* $\overline{M} = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$ *be the* 3*-dimensional manifold and* (x, y, z) *denote the standard coordinates in* $\mathbb{R}^3$*. Let* g *represents the metric on the manifold* $\overline{M}$ *given by*

$$g = e^{2z}(dx \otimes dx + dy \otimes dy) + \eta \otimes \eta.$$

Now, we choose

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} = \xi.$$

The above vector fields are linearly independent at the every point of $\overline{M}$ *such that* $g(e_i, e_j) = 0$ *for* $i \neq j$ *and* $g(e_i, e_j) = 1$ *for* $i = j$*, for* $1 \leq i, j \leq 3$*. The* 1*-form* η *is given by* $\eta(\mathcal{U}) = g(\mathcal{U}, e_3)$ *for chosen vector field* $\mathcal{U}$ *on the manifold* $\overline{M}$*. Suppose* ϕ *denotes the* $(1,1)$*-tensor field and is defined by* $\phi(e_1) = 0, \quad \phi(e_2) = 0, \quad \phi(e_3) = 0.$ *Now, using the linearity property of* ϕ *and* g*, we get*

$$\phi^2\mathcal{U} = -\mathcal{U} + \eta(\mathcal{U})\xi, \quad \eta(e_3) = 1, \quad g(\phi\mathcal{U}, \phi\mathcal{V}) = g(\mathcal{U}, \mathcal{V}) - \eta(\mathcal{U})\eta(\mathcal{V}),$$

where $\mathcal{U}$ *and* $\mathcal{V}$ *are chosen vector fields on the manifold* $\overline{M}$*.*

A direct calculation yields,

$$\begin{aligned}
\overline{\nabla}_{e_1} e_1 = -e_3, \quad \overline{\nabla}_{e_1} e_2 = 0, \quad \overline{\nabla}_{e_1} e_3 = e_1,\\
\overline{\nabla}_{e_2} e_1 = 0, \quad \overline{\nabla}_{e_2} e_2 = -e_3, \quad \overline{\nabla}_{e_2} e_3 = e_2,\\
\overline{\nabla}_{e_3} e_1 = e_1, \quad \overline{\nabla}_{e_3} e_2 = e_2, \quad \overline{\nabla}_{e_3} e_3 = 0.
\end{aligned}$$

By making use of the above relations, we see that the manifold satisfies the equation $\overline{\nabla}_{\mathcal{U}}\xi = \mathcal{U} - \eta(\mathcal{U})\xi$ *for* $e_3 = \xi$*. Therefore, the manifold is a Kenmotsu manifold. From the above relations, we get the following equations*

$$\begin{cases}
(\overline{\nabla}_{e_1}\phi)e_1 + (\overline{\nabla}_{e_1}\phi)e_1 = 0, \quad (\overline{\nabla}_{e_1}\phi)e_2 + (\overline{\nabla}_{e_2}\phi)e_1 = 0,\\
(\overline{\nabla}_{e_1}\phi)e_3 + (\overline{\nabla}_{e_3}\phi)e_1 = 0, \quad (\overline{\nabla}_{e_2}\phi)e_1 + (\overline{\nabla}_{e_1}\phi)e_2 = 0,\\
(\overline{\nabla}_{e_2}\phi)e_2 + (\overline{\nabla}_{e_2}\phi)e_2 = 0, \quad (\overline{\nabla}_{e_2}\phi)e_3 + (\overline{\nabla}_{e_3}\phi)e_2 = 0,\\
(\overline{\nabla}_{e_3}\phi)e_1 + (\overline{\nabla}_{e_1}\phi)e_3 = 0, \quad (\overline{\nabla}_{e_3}\phi)e_2 + (\overline{\nabla}_{e_2}\phi)e_3 = 0,\\
(\overline{\nabla}_{e_3}\phi)e_3 + (\overline{\nabla}_{e_3}\phi)e_3 = 0.
\end{cases} \tag{3.4.1}$$

From the equation (3.4.1), *it follows that* ϕ *is the Killing tensor field. Consequently, the manifold* $\overline{M}$ *is a Kenmotsu manifold with the Killing tensor field* ϕ*. Furthermore, we have*

$$\begin{cases} \overline{\nabla}_{e_1}\phi e_1 - \phi\overline{\nabla}_{e_1}e_1 + \overline{\nabla}_{e_1}\phi e_1 - \phi\overline{\nabla}_{e_1}e_1 = 0, \\ \overline{\nabla}_{e_1}\phi e_2 - \phi\overline{\nabla}_{e_1}e_2 + \overline{\nabla}_{e_2}\phi e_1 - \phi\overline{\nabla}_{e_2}e_1 = 0, \\ \overline{\nabla}_{e_1}\phi e_3 - \phi\overline{\nabla}_{e_1}e_3 + \overline{\nabla}_{e_3}\phi e_1 - \phi\overline{\nabla}_{e_3}e_1 = 0, \\ \overline{\nabla}_{e_2}\phi e_1 - \phi\overline{\nabla}_{e_2}e_1 + \overline{\nabla}_{e_1}\phi e_2 - \phi\overline{\nabla}_{e_1}e_2 = 0, \\ \overline{\nabla}_{e_2}\phi e_2 - \phi\overline{\nabla}_{e_2}e_2 + \overline{\nabla}_{e_2}\phi e_2 - \phi\overline{\nabla}_{e_2}e_2 = 0, \\ \overline{\nabla}_{e_2}\phi e_3 - \phi\overline{\nabla}_{e_2}e_3 + \overline{\nabla}_{e_3}\phi e_2 - \phi\overline{\nabla}_{e_3}e_2 = 0, \\ \overline{\nabla}_{e_3}\phi e_1 - \phi\overline{\nabla}_{e_3}e_1 + \overline{\nabla}_{e_1}\phi e_3 - \phi\overline{\nabla}_{e_1}e_3 = 0, \\ \overline{\nabla}_{e_3}\phi e_2 - \phi\overline{\nabla}_{e_3}e_2 + \overline{\nabla}_{e_2}\phi e_3 - \phi\overline{\nabla}_{e_2}e_3 = 0, \\ \overline{\nabla}_{e_3}\phi e_3 - \phi\overline{\nabla}_{e_3}e_3 + \overline{\nabla}_{e_3}\phi e_3 - \phi\overline{\nabla}_{e_3}e_3 = 0. \end{cases} \tag{3.4.2}$$

and

$$\begin{cases} \eta(e_1)\phi(e_1) + \eta(e_1)\phi(e_1) = 0, & \eta(e_2)\phi(e_1) + \eta(e_1)\phi(e_2) = 0, \\ \eta(e_3)\phi(e_1) + \eta(e_1)\phi(e_3) = 0, & \eta(e_1)\phi(e_2) + \eta(e_2)\phi(e_1) = 0, \\ \eta(e_2)\phi(e_2) + \eta(e_2)\phi(e_2) = 0, & \eta(e_3)\phi(e_2) + \eta(e_2)\phi(e_3) = 0, \\ \eta(e_1)\phi(e_3) + \eta(e_3)\phi(e_1) = 0, & \eta(e_2)\phi(e_3) + \eta(e_3)\phi(e_2) = 0, \\ \eta(e_3)\phi(e_3) + \eta(e_3)\phi(e_3) = 0. \end{cases} \tag{3.4.3}$$

The equations (3.4.1) *and* (3.4.2) *satisfy the equation* (3.3.2) *and the equations* (3.4.1) *and* (3.4.3) *satisfy the equation* (3.3.5).

Analogous to [111], we have the following example of 5-dimensional Kenmotsu manifold with the Killing tensor field.

Example 3.4.2. *Suppose* $\overline{M} = \{(x_1, x_2, x_3, x_4, v) \in \mathbb{R}^5, v \neq 0\}$ *be the* 5*-dimensional manifold and* (x_1, x_2, x_3, x_4, v) *denote the standard coordinates in* $\mathbb{R}^5$*. Let metric* g *on* $\overline{M}$ *is given by*

$$g = \eta \otimes \eta + e^{2v} \sum_{i=1}^{4} dx_i \otimes dx_i.$$

Now, we choose

$$e_1 = e^{-v}\frac{\partial}{\partial x_1}, \quad e_2 = e^{-v}\frac{\partial}{\partial x_2}, \quad e_3 = e^{-v}\frac{\partial}{\partial x_3}, \quad e_4 = e^{-v}\frac{\partial}{\partial x_4}, \quad e_5 = \frac{\partial}{\partial v} = \xi.$$

The above vector fields are linearly independent at the every point of $\overline{M}$ *such that* $g(e_i, e_j) = 0$ *for* $i \neq j$ *and* $g(e_i, e_j) = 1$ *for* $i = j$, *where* $i, j = 1, 2, 3, 4, 5$. *The* 1*-form* η *is given by* $\eta(\mathcal{U}) = g(\mathcal{U}, e_5)$ *for the chosen vector field* $\mathcal{U}$ *on the manifold* $\overline{M}$. *Suppose* ϕ *represents the tensor field of type* $(1, 1)$ *and is defined by*

$$\phi(e_1) = 0, \quad \phi(e_2) = 0, \quad \phi(e_3) = 0, \quad \phi(e_4) = 0, \quad \phi(e_5) = 0.$$

Now, using the linearity property of g *and* ϕ, *we have*

$$\phi^2\mathcal{U} = -\mathcal{U} + \eta(\mathcal{U})\xi, \quad \eta(e_5) = 1, \quad g(\phi\mathcal{U}, \phi\mathcal{V}) = g(\mathcal{U}, \mathcal{V}) - \eta(\mathcal{U})\eta(\mathcal{V}),$$

where $\mathcal{U}$ *and* $\mathcal{V}$ *are the chosen vector fields on the manifold* $\overline{M}$.

A simple computation yields,

$$\begin{aligned}
&\overline{\nabla}_{e_1}e_1 = -e_5, \quad \overline{\nabla}_{e_1}e_2 = 0, \quad \overline{\nabla}_{e_1}e_3 = 0, \quad \overline{\nabla}_{e_1}e_4 = 0, \quad \overline{\nabla}_{e_1}e_5 = e_1,\\
&\overline{\nabla}_{e_2}e_1 = 0, \quad \overline{\nabla}_{e_2}e_2 = -e_5, \quad \overline{\nabla}_{e_2}e_3 = 0, \quad \overline{\nabla}_{e_2}e_4 = 0, \quad \overline{\nabla}_{e_2}e_5 = e_2,\\
&\overline{\nabla}_{e_3}e_1 = 0, \quad \overline{\nabla}_{e_3}e_2 = 0, \quad \overline{\nabla}_{e_3}e_3 = -e_5, \quad \overline{\nabla}_{e_3}e_4 = 0, \quad \overline{\nabla}_{e_3}e_5 = e_3,\\
&\overline{\nabla}_{e_4}e_1 = 0, \quad \overline{\nabla}_{e_4}e_2 = 0, \quad \overline{\nabla}_{e_4}e_3 = 0, \quad \overline{\nabla}_{e_4}e_4 = -e_5, \quad \overline{\nabla}_{e_4}e_5 = e_4,\\
&\overline{\nabla}_{e_5}e_1 = e_1, \quad \overline{\nabla}_{e_5}e_2 = e_2, \quad \overline{\nabla}_{e_5}e_3 = e_3, \quad \overline{\nabla}_{e_5}e_4 = e_4, \quad \overline{\nabla}_{e_5}e_5 = 0.
\end{aligned}$$

By using the above relations, it follows that the manifold satisfies the equation $\overline{\nabla}_{\mathcal{U}}\xi = \mathcal{U} - \eta(\mathcal{U})\xi$ *for* $\xi = e_5$. *Furthermore, on the similar pattern of Example* 3.4.1, *it follows that* ϕ *is a Killing tensor field. Therefore,* $\overline{M}$ *be a* 5*-dimensional Kenmotsu manifold with the Killing tensor field. Also, analogous to Example* 3.4.1, *it can be seen that the equations* (3.3.2) *and* (3.3.5) *are satisfied.*

3.5 Conclusion

In this chapter, we have attempted to examine the properties of the contact CR-submanifold with Killing tensor field in Kenmotsu manifold. Further, it has been accomplished that if the second fundamental form h is parallel then contact CR-submanifold is totally geodesic in Kenmotsu manifold. Finally, some examples have been provided in the Kenmotsu manifold which satisfies the condition of Killing tensor field.

Chapter 4

Bronze and Copper Differential Geometry

4.1 Bronze Differential Geometry

4.1.1 Introduction

Yano [46], generalized the notion of almost contact structure and almost complex structure in 1963. Yano proposed the concept of an f-structure which satisfies the condition $f^3 + f = 0$, where f is a $(1,1)$-tensor field of constant rank on a manifold $\overline{M}$. In 1970, the notion of the polynomial structure on a manifold was introduced by Goldberg and Yano [99].

The role of celebrity numbers π, e and the Golden proportion $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ which is the positive real solution of the equation $x^2 - x - 1 = 0$, has always fascinated the mathematicians. Nevertheless, the Silver ratio and the Bronze ratio are also well known and much-sought-after numbers by mathematicians due to their own elegance and use in design, architecture and physics. Basically, the positive root $x = \frac{n+\sqrt{n^2+4}}{2}$ of the algebraic equation $x^2 - nx - 1 = 0$ for $n = 1, 2, 3, 4, \ldots$ forms a series called the metallic proportions e.g. $(n = 1)$ Golden proportion, $(n = 2)$ Silver proportion, $(n = 3)$ Bronze proportion, $(n = 4)$ Copper proportion, $(n = 5)$ Nickel proportion and so on [87, 114].

Inspired by several fascinating properties of the Golden proportion, Hreţcanu [23] introduced the notion of Golden structure on manifolds, by using a corresponding almost product structure. Crăşmăreanu *et al.* investigated the Golden differential geometry in [60]. They obtained some inquisitive properties and some results of Golden structure. Moreover, the Golden structure has been studied by many other authors [5, 6, 15, 65, 66, 69, 71].

In 2007, Primo *et al.* [9] obtained some algebraic and geometric characterizations of the Silver proportion. More recently, the geometry of Silver structure was introduced and studied by Ozkan *et al.* [67]. The concept of the Silver structure was encouraged by the Silver proportion $\theta = 1 + \sqrt{2} \approx 2.414\ldots$ which is the positive real root of the algebraic equation $x^2 - 2x - 1 = 0$.

Motivated by the recent study of Golden structure and Silver structure on manifolds, we have investigated the Bronze structure, by making use of the Bronze proportion $\psi = \frac{3+\sqrt{13}}{2} \approx 3.302\ldots$, which is the positive real solution of the equation $x^2 - 3x - 1 = 0$. Since the convergence of the Golden proportion is most slow, *i.e.*, the Golden proportion is the most irrational among all irrational ratios [114], for this reason the Silver proportion and the Bronze proportion exhibit certain important properties which in turn makes the study of Silver structure and the Bronze structure more interesting.

4.1.2 Preliminaries of Bronze Structure

In this part, we give some important definitions and basics of the Bronze structure for our later use.

Definition 4.1.1. *[60] "Let $\overline{M}$ be a C^∞-differentiable manifold. If a $(1,1)$-tensor field Φ satisfies the equation*

$$\Phi^2 = \Phi + I, \tag{4.1.1}$$

then Φ is called a Golden structure on $\overline{M}$ and $(\overline{M}, \Phi)$ is a Golden manifold."

Definition 4.1.2. *[67] "On a C^∞-differentiable manifold $\overline{M}$, a $(1,1)$-tensor field Θ that satisfies the equation*

$$\Theta^2 = 2\Theta + I \tag{4.1.2}$$

is called a Silver structure on $\overline{M}$ and $(\overline{M}, \Theta)$ is a Silver manifold."

A Bronze structure Ψ on $\overline{M}$ can be defined by

Definition 4.1.3. *Suppose $\overline{M}$ denotes a differentiable manifold and Ψ represents a $(1,1)$-tensor field on $\overline{M}$. The tensor field Ψ is said to be a Bronze structure if it satisfies*

$$\Psi^2 = 3\Psi + I, \tag{4.1.3}$$

where I denote the identity tensor field of type $(1,1)$ on $\overline{M}$ and $(\overline{M}, \Psi)$ is said to be a Bronze manifold.

The Bronze structure is inspired by the Bronze proportion $\psi = \frac{3+\sqrt{13}}{2} \approx 3.302\ldots$, which is a positive zero of the algebraic equation $x^2 - 3x - 1 = 0$.

Proposition 4.1. *(i) The eigenvalues of the Ψ are $3 - \psi$ and the Bronze ratio ψ.*

(ii) For every $p \in \overline{M}$, the Ψ is an isomorphism on $T_p\overline{M}$.

(iii) Thus, the Bronze structure is invertible, and its inverse $\Psi^{-1} = \hat{\Psi}$ satisfying

$$\hat{\Psi}^2 = -3\hat{\Psi} + I. \tag{4.1.4}$$

4.1.3 Geometry of the Bronze Structure

In this part, the geometry of a Bronze structure Ψ has been discussed. From [60] "if T, P and J are an almost tangent structure, an almost product structure, and an almost complex structure, respectively, then $-T$, $-P$ and $-J$ are also an almost tangent structure, an almost product structure and an almost complex structure, respectively."

Proposition 4.2. *If Ψ denotes a Bronze structure then $\tilde{\Psi} = 3I - \Psi$ is also a Bronze structure.*

Now, we have the theorem which provides a relation between an almost product structure P and the Bronze structure Ψ on manifold $\overline{M}$.

Theorem 4.1.1. *Suppose P denotes an almost product structure, then P yields a Bronze structure Ψ on $\overline{M}$ as*

$$\Psi = \frac{1}{2}(3I + \sqrt{13}P). \tag{4.1.5}$$

On the other hand, if Ψ be a Bronze structure on $\overline{M}$, then Ψ induces an almost product structure on $\overline{M}$ in the following manner

$$P = \frac{1}{\sqrt{13}}\left(2\Psi - 3I\right). \tag{4.1.6}$$

On the pattern of structure (4.1.5), we have

Definition 4.1.4. *Let $\overline{M}$ be endowed with an almost tangent structure T and $(\overline{M}, T)$ denotes the almost tangent manifold. Let Ψ_t be a tensor field and is said to be a tangent Bronze structure on $(\overline{M}, T)$, if*

$$\Psi_t = \frac{1}{2}(3I + \sqrt{13}T). \tag{4.1.7}$$

The tangent Bronze structure Ψ_t satisfies the condition

$$\Psi_t^2 - 3\Psi_t + \frac{9}{4}I = 0. \tag{4.1.8}$$

Above equation over the reals $\mathbb{R}$, that is, $x^2 - 3x + \frac{9}{4} = 0$; we get the tangent real Bronze ratio $\psi_t = \frac{3}{2}$.

Definition 4.1.5. *Let $\overline{M}$ be endowed with an almost complex structure J. Let Ψ_c be a tensor field and is said to be a complex Bronze structure on $(\overline{M}, J)$, if*

$$\Psi_c = \frac{1}{2}(3I + \sqrt{13}J). \tag{4.1.9}$$

The complex Bronze structure Ψ_c satisfies the equation

$$\Psi_c^2 - 3\Psi_c + \frac{11}{2} I = 0. \qquad (4.1.10)$$

For $\overline{M} = \mathbb{R}$, we have

$$x^2 - 3x + \frac{11}{2} = 0, \qquad (4.1.11)$$

with solutions $\frac{3}{2} + i\frac{\sqrt{13}}{2}$ and $\frac{3}{2} - i\frac{\sqrt{13}}{2}$.

Definition 4.1.6. *The complex quantity $\psi_c = \frac{3}{2} + i\frac{\sqrt{13}}{2}$ will be called a complex Bronze ratio.*

As a consequence, we have a relationship between the complex Bronze ratio, tangent real Bronze ratio and Bronze ratio.

Bronze ratio: $\psi = \frac{3}{2} + \frac{\sqrt{13}}{2}$.

Tangent real Bronze ratio: $\psi_t = \frac{3}{2}$.

Complex Bronze ratio: $\psi_c = \frac{3}{2} + i\frac{\sqrt{13}}{2}$.

Thus,

$$\psi_c = \psi_t + i(\psi - \psi_t).$$

4.1.4 Examples

In this part, some illustrative examples have been carried out for the Bronze Structure.

Example 4.1.1. (Clifford Algebras).

Suppose $C'(n)$ represents the real Clifford algebra. Let $C'(n)$ be the positive definite form $\sum_{i=1}^{n}(x^i)^2$ over $\mathbb{R}^n$ and $\{e_1, e_{2,}, \ldots, e_n\}$ denotes the orthonormal basis of $\mathbb{R}^n$, then we have the following relations of $C'(n)$ [26]:

$$\begin{cases} e_i e_j = -e_j e_i, \quad i \neq j, \\ e_i^2 = 1. \end{cases} \qquad (4.1.12)$$

Now making use of $\Psi_i = \frac{1}{2}(3+\sqrt{13}e_i)$*, we obtain new presentation relations of* $C'(n)$*:*

$$\begin{cases} \Psi_i = Bronze\,Structure, \\ \Psi_i\Psi_j + \Psi_j\Psi_i = 3\left(\Psi_i + \Psi_j\right) - \frac{9}{2} \quad i \neq j. \end{cases} \tag{4.1.13}$$

In [26], $C'(2)$ *is given as*

$$1 = l_2, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, we have

$$\Psi_1 = \frac{1}{2}(3I_2 + \sqrt{13}e_1) = \begin{pmatrix} \frac{3}{2} + \frac{\sqrt{13}}{2} & 0 \\ 0 & \frac{3}{2} - \frac{\sqrt{13}}{2} \end{pmatrix} = \begin{pmatrix} \psi & 0 \\ 0 & 3-\psi \end{pmatrix}. \tag{4.1.14}$$

$$\Psi_2 = \frac{1}{2}(3I_2 + \sqrt{13}e_2) = \begin{pmatrix} \frac{3}{2} & \frac{\sqrt{13}}{2} \\ \frac{\sqrt{13}}{2} & \frac{3}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & \sqrt{13} \\ \sqrt{13} & 3 \end{pmatrix}. \tag{4.1.15}$$

Example 4.1.2. (2D Bronze Matrices)*.*

Let Ψ *be a matrix in* $\mathbb{R}^n_n$ *and is known as a Bronze matrix, if* Ψ *holds the following condition*

$$\Psi^2 = 3\Psi + I_n, \tag{4.1.16}$$

where I_n *represents the identity matrix on* $\mathbb{R}^n_n$*.*

For $n = 2$*, solving the equation* (4.1.16)*, the Bronze structures in* $\mathbb{R}^2_2$ *is given by*

(i) Let a *and* d *belongs to* $\mathbb{R}$ *and* b *belongs to* $\mathbb{R} - \{0\}$*, then we have*

$$\Psi_{a,b} = \begin{pmatrix} a & -\frac{1}{b}(a^2 - 3a - 1) \\ b & 3-a \end{pmatrix} \quad or \quad \Psi_{b,d} = \begin{pmatrix} 3-d & -\frac{1}{b}(d^2 - 3d - 1) \\ b & d \end{pmatrix}. \tag{4.1.17}$$

(ii) Let a *equals to* Ψ *and* b *in* $\mathbb{R}$*, then we get*

$$\Psi_{\psi,b} = \begin{pmatrix} \psi & b \\ 0 & 3-\psi \end{pmatrix} \quad or \quad \Psi_{3-\psi,b} = \begin{pmatrix} 3-\psi & b \\ 0 & \psi \end{pmatrix}.$$

(iii) Let a equals to ψ and b is 0*, then we get*

$$\Psi_{\psi,0} = \begin{pmatrix} \psi & 0 \\ 0 & 3-\psi \end{pmatrix} \quad or \quad \Psi_{3-\psi,0} = \begin{pmatrix} 3-\psi & 0 \\ 0 & \psi \end{pmatrix}.$$

Therefore from equations (4.1.14)*,* (4.1.15) *and* (4.1.17)*, we obtain*

$$\Psi_1 = \lim_{b\to 0} \Psi_{\psi,b} \quad and \quad \Psi_2 = \Psi_{\frac{3}{2},\frac{\sqrt{13}}{2}}.$$

Example 4.1.3. (Bronze Reflections).

Let $(E, <,>)$ denotes the Euclidean space and the reflection associated with a hyperplane H w.r.t. the normal $\nu \in E - \{0\}$ in $(E, <,>)$ satisfies the following condition

$$r_\nu(x) = x - \frac{2 < x, \nu >}{< \nu, \nu >}\nu,$$

and $r_\nu^2 = I_E$ denotes the identity on E [26].

Now, the Bronze reflection corresponding to ν can be defined as

$$\Psi_\nu = \frac{1}{2}(3I_E + \sqrt{13}r_\nu),$$

and then ν is an eigenvector of Bronze reflection Ψ_ν with respect to eigenvalue $3-\psi$. Also, from the lemma [[26], p.314]*, we have*

$$\mathcal{U}\Psi_\nu\mathcal{U}^{-1} = \Psi_{\mathcal{U}(\nu)},$$

where $\mathcal{U}$ in the orthogonal group of E, i.e., $\mathcal{U} \in O(E, <,>)$.

The explicit expression of this linear transformation can be given by

$$\Psi_\nu(x) = \psi x - \sqrt{13}\frac{< x, \nu >}{< \nu, \nu >}\nu.$$

Example 4.1.4. (Triple structure in terms of Bronze structures).

"Let F and P denote two tensor fields of type (1, 1) *on the manifold $\overline{M}$. Now, we have four structures with the triplet $(F, P, J = P \circ F)$:*

(i) $P^2 = I = F^2$ and $F \circ P + P \circ F = 0$; then $J^2 = -I$,

(ii) $P^2 = I = F^2$ *and* $-F \circ P + P \circ F = 0$; *then* $J^2 = I$,

(iii) $P^2 = -I = F^2$ *and* $F \circ P + P \circ F = 0$; *then* $J^2 = -I$,

(iv) $P^2 = -I = F^2$ *and* $-F \circ P + P \circ F = 0$; *then* $J^2 = I$,

are known as, almost biproduct complex (*abpc*), *almost hyperproduct* (*ahp*), *almost hypercomplex* (*ahc*) *and almost product bicomplex* (*apbc*), *respectively" [60, 113].*

From the equation (4.1.5), *for* $J = P \circ F$, *we have*

$$\Psi_F = \frac{3}{2}I + \frac{\sqrt{13}}{2}F, \quad \Psi_P = \frac{3}{2}I + \frac{\sqrt{13}}{2}P, \quad \Psi_J = \frac{3}{2}I + \frac{\sqrt{13}}{2}J.$$

Thus, we get the following relation

$$\frac{\sqrt{13}}{2}\Psi_J = \frac{2}{3}\Psi_P\Psi_F - \Psi_P - \Psi_F + \psi I.$$

As a result, the triplet (Ψ_F, Ψ_P, Ψ_J) *is:*

(i) An almost biproduct complex (*abpc*)*-structure iff:* Ψ_F, Ψ_P *are Bronze structures and* $4(\Psi_P\Psi_F + \Psi_F\Psi_P) = 12(\Psi_P + \Psi_F) - 18I$; *then* Ψ_J *is also a complex Bronze structure.*

(ii) An almost hyperproduct (*ahp*)*-structure iff:* Ψ_F, Ψ_P *are Bronze structures and* $\Psi_F\Psi_P = \Psi_P\Psi_F$; *then* Ψ_J *is also a Bronze structure.*

(iii) An almost hypercomplex (*ahc*)*-structure iff:* Ψ_F, Ψ_P *are complex Bronze structures and* $4(\Psi_P\Psi_F + \Psi_F\Psi_P) = 12(\Psi_P + \Psi_F) - 18I$; *then* Ψ_J *is also a complex Bronze structure.*

(iv) An almost product bicomplex (*apbc*)*-structure iff:* Ψ_F, Ψ_P *are complex Bronze structures and* $\Psi_F\Psi_P = \Psi_P\Psi_F$; *then* Ψ_J *is also a Bronze structure.*

Example 4.1.5. (Quaternion Algebras).

Let $\mathbb{H}$ *represents the quaternion algebra and* $\{1, e_1, e_2, e_3\}$ *be the base of it, satisfying* $e_1^2 = e_2^2 = e_3^2 = -1$ *and*

$$e_1 = -e_3e_2 = e_2e_3, \quad e_2 = -e_1e_3 = e_3e_1, \quad e_3 = -e_2e_1 = e_1e_2.$$

The quaternion q can be written by the following expression

$$q = S_q + \vec{V}_q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3,$$

where $S_q = a_0$ *denotes the scalar component and* $\vec{V}_q = a_1 e_1 + a_2 e_2 + a_3 e_3$ *denotes the vector component of q. Since* $\sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2} = ||q||$ *represents the norm of quaternion q. For any* $q \neq 0$, $q_0 = \frac{q}{||q||}$ *is the unit quaternion.*

Furthermore, $q_0 = \cos\alpha + \vec{\varepsilon}_0 \sin\alpha$ *be the another representation form of the every unit quaternion, where* $\vec{\varepsilon}_0$ *is the unit vector holds the condition* $\vec{\varepsilon}_0^{\,2} = -1$. *Therefore, analogous to [67] (see eg. 3.3),*

(i) the Bronze split quaternion structure can be define as

$$\Psi_q = \frac{3}{2} + \frac{\sqrt{13}}{2}\vec{\varepsilon}_0,$$

where $\vec{\varepsilon}_0^{\,2} = 1$.

(ii) the Bronze biquaternion structure can be define as

$$\Psi_q = \frac{3}{2} + i\frac{\sqrt{13}}{2}\vec{\varepsilon}_0,$$

where $i^2 = -1$ *and* $\vec{\varepsilon}_0^{\,2} = -1$.

4.1.5 Connections as Bronze Structure

4.1.5.1 Connections in principal fibre bundles

Suppose principal fibre bundle is denoted by $P(\overline{M}, \pi, G)$, where $\overline{M}$ be a base space, π be a projection, G be a structure group and P be a total space. Let K denote a vertical distribution ($K = ker\ \pi_*$) on P and H represents a horizontal distribution (complementary distribution, *i.e.*, $TP = K \oplus H$ and H is G-invariant).

Let h and ν are the corresponding projectors of horizontal H and vertical distribution K, respectively. The tensor field F of type $(1,1)$ is an almost product structure on P, if

$$F = \nu - h.$$

In [60], an almost product structure F denote a principal connection iff the following conditions are satisfied:

(i) $dR_a \circ F_u = F_{ua} \circ dR_a$ for each $u \in P$ and $a \in G$.

(ii) $\mathcal{U}$ be a vertical vector field $\Longleftrightarrow \mathcal{U} = F(\mathcal{U})$.

Considering the operation (4.1.5), for a Bronze structure Ψ, we have

Proposition 4.3. *Suppose Ψ represents a Bronze structure on P. The Bronze structure is associated to a principal connection iff the following conditions are satisfied:*

(i) $dR_a \circ \Psi_u = \Psi_{ua} \circ dR_a$ for each $u \in P$ and $a \in G$.

(ii) $\mathcal{U}$ is a vertical vector field iff $\mathcal{U}$ be the eigenvector of Bronze structure Ψ w.r.t. the eigenvalue ψ.

Proposition 4.4. *The principal connection is flat iff Ψ is integrable,* i.e., $N_\Psi = 0$.

The principal connection yields a lift $l_\omega : T(\overline{M}) \to T(P)$ if following condition holds

$$N_F(l_\omega \mathcal{U}, l_\omega \mathcal{V}) = [l_\omega \mathcal{U}, l_\omega \mathcal{V}] - l_\omega [\mathcal{U}, \mathcal{V}]$$

where $\mathcal{U}$ and $\mathcal{V}$ are the vector fields on $\overline{M}$ [60].

Proposition 4.5. *The lift l_ω is a morphism iff Ψ is integrable,* i.e. $N_\Psi = 0$.

4.1.5.2 Connection in tangent bundles

Suppose $\overline{M}$ be a differentiable manifold of n-dimension and $(T\overline{M}, \pi, \overline{M})$ denote the tangent bundle associated with the base space $\overline{M}$. Suppose $(L, x^i)_{1 \le i \le n}$ represents a local coordinate system on $\overline{M}$ and $(\pi^{-1}(L), x^i, y^i)_{1 \le i \le n}$ be the induced local coordinate system on $T\overline{M}$ defined by $y^i(u) = dx^i(u)$ and $x^i(u) = x^i(\pi(u))$ for every $u \in \pi^{-1}(L)$. The kernel of π_* is denoted by $K(\overline{M})$, *i.e.*, $K = \{\mathcal{U} \in T\overline{M} : \pi_*(\mathcal{U}) = 0\}$ and is known as vertical distribution of $\overline{M}$.

Let T denotes a $(1,1)$-tensor field on the base $\overline{M}$ and an almost tangent structure is given by $T = \frac{\partial}{\partial y^i} \otimes dx^i$, that is, $T^2 = 0$.

Now parallel to [60], we have the following results.

Definition 4.1.7. *[60] Let ν denote a $(1,1)$-tensor field and is said to be a vertical projector if following condition is satisfied:*

$$\begin{cases} \nu \circ T = T, \\ T \circ \nu = 0. \end{cases} \tag{4.1.18}$$

Definition 4.1.8. *[60] Suppose N be the complementary distribution to the vertical distribution K is called a normalization or non-linear connection, if*

$$T(\overline{M}) = N \oplus K. \tag{4.1.19}$$

Since ν is $C^\infty(\overline{M})$-linear associated with $K = im\ \nu$, we have the following statement:

Proposition 4.6. *[60] Let $N(\nu)$ be a non-linear connection induced by the vertical vector ν, and is given by the relation $ker\ \nu = N(\nu)$.*

Consider N be a non-linear connection. Suppose h_N and ν_N represents the horizontal as well as vertical projections related with the decomposition of equation (4.1.19). Consequently, we have

Proposition 4.7. *[60] Suppose h_N and v_N, respectively, be the corresponding projections of N and K. Let N be a non-linear-connection $\Rightarrow$ ν_N is a vertical projector with $N = N(\nu_N)$.*

Definition 4.1.9. *[60] Let ζ denote a tensor field of type $(1,1)$, and is said to be a non-linear connection of an almost product type if following equation is satisfied*

$$\begin{cases} T \circ \zeta = T, \\ \zeta \circ T = -T. \end{cases}$$

Proposition 4.8. *[60] "Let ζ represents a non-linear connection of an almost product type, then* *(i)* $(I_{T(\overline{M})} - \zeta) = 2\nu_\zeta$ *is a vertical vector.*

(ii) $N(\nu_\zeta)$ *is the* $(+1)$*-eigenspace of* ζ *and* $K(\overline{M})$ *is the* (-1)*-eigenspace of* ζ*".*

Proposition 4.9. *[60] "Let* ζ *denotes a non-linear connection of an almost product type,* i.e., $\zeta = I_{T(\overline{M})} - 2\nu$*, where* ν *denotes a vertical vector. Then* ζ *defines an almost product structure on manifold* $\overline{M}$*."*

Hence, we have the following proposition for a Bronze structure.

Proposition 4.10. *Suppose* N *denotes a non-linear connection on the manifold* $\overline{M}$*, which is induced by* ν*. Thus* N *can be defined by a Bronze structure* Ψ*,* i.e.,

$$\Psi = \psi I_{T(\overline{M})} - \sqrt{13}\,\nu,$$

with K*, the* $(3-\psi)$*-eigenspace and* N*, the* ψ*-eigenspace.*

4.1.6 Integrability and Parallelism of Bronze Structures

Let N_Ψ represents the Nijenhuis tensor of $(1,2)$-type tensor field and Ψ is a Bronze structure on $\overline{M}$. From [49], we have

$$N_\Psi(\mathcal{U},\mathcal{V}) = \Psi^2[\mathcal{U},\mathcal{V}] + [\Psi\mathcal{U},\Psi\mathcal{V}] - \Psi[\Psi\mathcal{U},\mathcal{V}] - \Psi[\mathcal{U},\Psi\mathcal{V}], \tag{4.1.20}$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$.

Suppose R and S be the complementary distributions on the manifold $\overline{M}$, respectively associated with the eigenvalues ψ and $3-\psi$. Let r and s denote the corresponding projections. Thus, we have the following results:

$$\begin{cases} s^2 = s,\ r^2 = r, \\ s + r = I,\ rs = 0 = sr. \end{cases} \tag{4.1.21}$$

A direct computation from (4.1.5), we get

$$\begin{cases} r = \frac{1}{\sqrt{13}}\Psi - \frac{3-\psi}{\sqrt{13}}I, \\ s = -\frac{1}{\sqrt{13}}\Psi + \frac{\psi}{\sqrt{13}}I. \end{cases} \tag{4.1.22}$$

From [49], it follows that

(i) If $N_\Psi = 0$, then the Bronze structure is said to be integrable.

(ii) If $s[r\mathcal{U}, r\mathcal{V}] = 0$ and $r[s\mathcal{U}, s\mathcal{V}] = 0$, then R and S both are integrable distributions, respectively.

From the equations (4.1.3) and (4.1.22), we get

$$\begin{cases} \Psi r = r\Psi = \psi r = \frac{\psi}{\sqrt{13}}\Psi + \frac{1}{\sqrt{13}}I, \\ \Psi s = s\Psi = (3-\psi)\, s = \frac{(\psi-3)}{\sqrt{13}}\Psi - \frac{1}{\sqrt{13}}I. \end{cases} \tag{4.1.23}$$

Now

$$\begin{aligned} \psi r &= \psi(\frac{1}{\sqrt{13}}\Psi - \frac{3-\psi}{\sqrt{13}}I) \\ &= \frac{\psi}{\sqrt{13}}\Psi + \frac{\psi^2 - 3\psi}{\sqrt{13}}I \\ &= \frac{\psi}{\sqrt{13}}\Psi + \frac{1}{\sqrt{13}}I. \end{aligned}$$

and

$$\begin{aligned} (3-\psi)\, s &= (3-\psi)\,(-\frac{1}{\sqrt{13}}\Psi + \frac{\psi}{\sqrt{13}}I) \\ &= \frac{(\psi-3)}{\sqrt{13}}\Psi - \frac{\psi^2 - 3\psi}{\sqrt{13}}I \\ &= \frac{(\psi-3)}{\sqrt{13}}\Psi - \frac{1}{\sqrt{13}}I. \end{aligned}$$

Thus for a Bronze structure Ψ, we obtain

$$\begin{cases} sN_\Psi(r\mathcal{U}, r\mathcal{V}) = 13\ s\left[r\mathcal{U}, r\mathcal{V}\right], \\ rN_\Psi(s\mathcal{U}, s\mathcal{V}) = 13\ r\left[s\mathcal{U}, s\mathcal{V}\right]. \end{cases} \tag{4.1.24}$$

Proposition 4.11. *An almost product structure $P = \frac{1}{\sqrt{13}}(2\Psi - 3I)$ is said to be integrable iff a Bronze structure is integrable.*

Proposition 4.12. *Let $\mathcal{U}$ and $\mathcal{V}$ be vector fields on $\overline{M}$. The distribution S is integrable iff $rN_\Psi(s\mathcal{U}, s\mathcal{V})$ vanishes and the distribution R is integrable iff $sN_\Psi(r\mathcal{U}, r\mathcal{V})$ vanishes. Both of the distributions R and S are integrable, if a Bronze structure is integrable.*

Suppose $\overline{\nabla}$ is a linear connection on the manifold $\overline{M}$. Thus, the linear connections for $(\Psi, \overline{\nabla})$ is given by the following [3]:

(i) "The connection
$$\overset{Sc}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{\mathcal{U}} r\mathcal{V}) + s(\overline{\nabla}_{\mathcal{U}} s\mathcal{V}) \tag{4.1.25}$$
is known as a *Schouten* connection.

(ii) The connection
$$\overset{Vr}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{r\mathcal{U}} r\mathcal{V}) + s(\overline{\nabla}_{s\mathcal{U}} s\mathcal{V}) + r[s\mathcal{U}, r\mathcal{V}] + s[r\mathcal{U}, s\mathcal{V}] \tag{4.1.26}$$
is known as a *Vrănceanu* connection."

Proposition 4.13. *Let $\overline{\nabla}$ be a linear connection on $\overline{M}$, the projectors s and r are parallels in terms of Schouten $\overset{Sc}{\nabla}$ and Vrănceanu $\overset{Vr}{\nabla}$ connections. Furthermore, the Bronze structure Ψ is also parallel with respect to $\overset{Sc}{\nabla}$ and $\overset{Vr}{\nabla}$.*

Proof. *Suppose $\mathcal{U}$ and $\mathcal{V}$ be vector fields on $\overline{M}$. Then, from equation* (4.1.21)*, we have*
$$\begin{aligned}(\overset{Sc}{\nabla}_{\mathcal{U}} r)\mathcal{V} &= \overset{Sc}{\nabla}_{\mathcal{U}} r\mathcal{V} - r(\overset{Sc}{\nabla}_{\mathcal{U}}\mathcal{V}) \\ &= r(\overline{\nabla}_{\mathcal{U}} r\mathcal{V}) - r(\overline{\nabla}_{\mathcal{U}} r\mathcal{V}) = 0.\end{aligned}$$

$$\begin{aligned}(\overset{Vr}{\nabla}_{\mathcal{U}} r)\mathcal{V} &= \overset{Vr}{\nabla}_{\mathcal{U}} r\mathcal{V} - r(\overset{Vr}{\nabla}_{\mathcal{U}}\mathcal{V}) \\ &= r(\overline{\nabla}_{r\mathcal{U}} r\mathcal{V}) + r[s\mathcal{U}, r\mathcal{V}] \\ &\quad - r(\overline{\nabla}_{r\mathcal{U}} r\mathcal{V}) - r[s\mathcal{U}, r\mathcal{V}] = 0.\end{aligned}$$

Therefore, the projection r is parallel corresponding to Schouten and Vrănceanu connection. Analogously, it can be shown that the above equations are valid for the projector s.
Also, the result immediately follows for a Bronze structure Ψ from equation (4.1.23).

Now analogous to [53], we have

Definition 4.1.10. *If $\overline{\nabla}_{\mathcal{U}}\mathcal{V} \in R$, then the distribution R is said to be parallel corresponding to linear connection $\overline{\nabla}$, for all $\mathcal{U} \in T(\overline{M})$ and $\mathcal{V} \in R$.*

Definition 4.1.11. *If $(\Delta\Psi)(\mathcal{U},\mathcal{V}) \in R$, then the distribution R is said to be $\overline{\nabla}$-half parallel, for all $\mathcal{U} \in R$, $\mathcal{V} \in T(\overline{M})$, where*

$$(\Delta\Psi)(\mathcal{U},\mathcal{V}) = \Psi\overline{\nabla}_{\mathcal{U}}\mathcal{V} - \Psi\overline{\nabla}_{\mathcal{V}}\mathcal{U} - \overline{\nabla}_{\Psi\mathcal{U}}\mathcal{V} + \overline{\nabla}_{\mathcal{V}}(\Psi\mathcal{U}).$$

Definition 4.1.12. *If $(\Delta\Psi)(\mathcal{U},\mathcal{V}) \in S$, then the distribution R is known as $\overline{\nabla}$-anti half parallel, for all $\mathcal{U} \in R$, $\mathcal{V} \in T(\overline{M})$.*

Proposition 4.14. *Suppose $\overline{\nabla}$ be a linear connection on $\overline{M}$. Then, the distributions R and S are parallel corresponding to Schouten as well as Vrănceanu connections.*

Proof. *Let $\mathcal{V} \in R$ and $\mathcal{U} \in T(\overline{M})$. Hence, $s\mathcal{V} = 0$ as well as $r\mathcal{V} = \mathcal{V}$.*
From equations (4.1.25) and (4.1.26), we have

$$\overset{Sc}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{\mathcal{U}}\mathcal{V}),$$

$$\overset{Vr}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{r\mathcal{U}}\mathcal{V}) + r[s\mathcal{U},\mathcal{V}].$$

Thus, both $\overset{Sc}{\nabla}$ and $\overset{Vr}{\nabla}$ belongs to R. Therefore, the distribution R is parallel with respect to $\overset{Sc}{\nabla}$ and $\overset{Vr}{\nabla}$. Analogously, S also satisfies above conditions.

4.1.7 On the Bronze Riemannian Manifolds

"Suppose g represents a Riemannian metric and P denotes an almost product structure on manifold $\overline{M}$, and is related by the following expression

$$g(\mathcal{U},\mathcal{V}) = g(P(\mathcal{U}),P(\mathcal{V})),$$

for any vector fields $\mathcal{U}$ and $\mathcal{V}$ on manifold $\overline{M}$. In other words, P is a g-symmetric endomorphism and is defined by

$$g(P(\mathcal{U}),\mathcal{V}) = g(\mathcal{U},P(\mathcal{V})),$$

and the ordered pair (g, P) is called a Riemannian almost product structure" [8, 47].

By making use of equations (4.1.5) and (4.1.6), we have the following results.

Proposition 4.15. *The Bronze structure Ψ defines a g-symmetric endomorphism iff an operator P is a g-symmetric endomorphism.*

Definition 4.1.13. *Let g denote a Riemannian metric on the manifold $\overline{M}$. Then, (g, Ψ) is called a Bronze Riemannian structure, if the following equality is satisfied:*

$$g(\Psi(\mathcal{U}), \mathcal{V}) = g(\mathcal{U}, \Psi(\mathcal{V})),$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on manifold $\overline{M}$ and the triplet $(\overline{M}, g, \Psi)$ is said to be a Bronze Riemannian manifold.

Corollary 4.1.1. *Suppose $(\overline{M}, g, \Psi)$ denotes a Bronze Riemannian manifold, then the following conditions hold:*

(i) The Bronze structure Ψ on manifold $\overline{M}$ is N_Ψ-symmetric, i.e.,

$$N_\Psi(\Psi(\mathcal{U}), \mathcal{V}) = N_\Psi(\mathcal{U}, \Psi(\mathcal{V})).$$

(ii) The projectors r and s are g-symmetric, i.e.,

$$\begin{cases} g(s(\mathcal{U}), \mathcal{V}) = g(\mathcal{U}, s(\mathcal{V})), \\ g(r(\mathcal{U}), \mathcal{V}) = g(\mathcal{U}, r(\mathcal{V})). \end{cases}$$

(iii) The distributions R and S are g-orthogonal, i.e.,

$$g(r(\mathcal{U}), s(\mathcal{V})) = 0.$$

Proposition 4.16. *[60] "Let $\overline{\nabla}^g$ be a Levi-Civita connection of g. If P is parallel corresponding to the $\overline{\nabla}^g$,* i.e., *$\overline{\nabla}^g P = 0$, then the Riemannian almost product structure is a locally product structure. Moreover, if $\overline{\nabla}$ is a linear and symmetric connection, then the Nijenhuis tensor of P satisfies*

$$N_P(\mathcal{U}, \mathcal{V}) = (\overline{\nabla}_{P\mathcal{U}} P)\mathcal{V} - (\overline{\nabla}_{P\mathcal{V}} P)\mathcal{U} - P(\overline{\nabla}_{\mathcal{U}} P)\mathcal{V} + P(\overline{\nabla}_{\mathcal{V}} P)\mathcal{U}.$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$."

Thus for the case of Bronze structure Ψ, we have

Proposition 4.17. *If $(\overline{M}, g, \Psi)$ defines a locally product Bronze Riemannian manifold then Ψ is said to be integrable.*

4.2 Copper Differential Geometry

4.2.1 Introduction

In this section, we have investigated Copper structure on the manifold $\overline{M}$, described by a $(1,1)$-tensor field $\overline{\Psi}$ satisfying $\overline{\Psi}^2 - 4\overline{\Psi} - I = 0$. The concept of the Copper structure is inspired by the Copper proportion $\overline{\psi} = 2 + \sqrt{5} \approx 4.236\ldots$, which is a root of the algebraic equation $x^2 - 4x - 1 = 0$. The Copper proportion is also known as Copper mean, Copper ratio and Copper number.

4.2.2 Preliminaries of Copper Structure

Analogous to the Bronze structure, we have the following;

Definition 4.2.1. *Suppose $\overline{M}$ denotes a differentiable manifold and $\overline{\Psi}$ represents the tensor field of type $(1,1)$ on $\overline{M}$. A tensor field $\overline{\Psi}$ is said to be a Copper structure if it satisfies*

$$\overline{\Psi}^2 = 4\overline{\Psi} + I, \tag{4.2.1}$$

where I denotes the identity tensor field of type $(1,1)$ on $\overline{M}$ and $(\overline{M}, \overline{\Psi})$ is called a Copper manifold.

Proposition 4.18. *(i) The eigenvalues of the $\overline{\Psi}$ are $4 - \overline{\psi}$ and the Copper proportion $\overline{\psi}$.*

(ii) For every $p \in \overline{M}$, $\overline{\Psi}$ is an isomorphism on $T_p\overline{M}$.

(iii) Therefore, the Copper structure is invertible and its inverse $\overline{\Psi}^{-1}$ satisfies the following equation:

$$(\overline{\Psi}^{-1})^2 = -4\overline{\Psi}^{-1} + I$$

4.2.3 Geometry of the Copper Structure

From [60], "if J, P and T is considered as an almost complex (ac)-structure, almost product (ap)-structure and almost tangent (at)-structure then $-J$, $-P$ and $-T$ is also considered as an (ac)-structure, (ap)-structure and (at)-structure."

The relationship between almost product structure P and the Copper structure $\overline{\Psi}$ on the manifold $\overline{M}$ is given by the following theorem:

Theorem 4.2.1. *Let P be an almost product structure, then P gives a Copper structure $\overline{\Psi}$ on $\overline{M}$ as*

$$\overline{\Psi} = 2I + \sqrt{5}\,P. \tag{4.2.2}$$

On the other hand, if $\overline{\Psi}$ be a Copper structure on $\overline{M}$, then $\overline{\Psi}$ induces an almost product structure on $\overline{M}$ in the following manner

$$P = \frac{1}{\sqrt{5}}(\overline{\Psi} - 2I). \tag{4.2.3}$$

On the pattern of structure (4.2.2), we introduce the following definitions.

Definition 4.2.2. *Let $\overline{M}$ be endowed with an almost tangent structure T and $(\overline{M}, T)$ denotes the almost tangent manifold. Let $\overline{\Psi}_t$ be a tensor field and is said to be a tangent Copper structure on $(\overline{M}, T)$, if*

$$\overline{\Psi}_t = 2I + \sqrt{5}\,T. \tag{4.2.4}$$

The tangent Copper structure $\overline{\Psi}_t$ satisfies the condition

$$\overline{\Psi}_t^2 - 4\overline{\Psi}_t + 4I = 0. \tag{4.2.5}$$

Over the reals $\mathbb{R}$, above equation becomes $x^2 - 4x + 4 = 0$. We get the tangent real Copper proportion $\overline{\psi}_t = 2$.

Definition 4.2.3. *Let $\overline{M}$ be endowed with an almost complex structure J and $(\overline{M}, J)$ denotes the almost complex manifold. Let $\overline{\Psi}_c$ be a tensor field and is said to be a complex Copper structure on $(\overline{M}, J)$, if*

$$\overline{\Psi}_c = 2I + \sqrt{5}J. \tag{4.2.6}$$

It follows that, the complex Copper structure $\overline{\Psi}_c$ satisfies the equation

$$\overline{\Psi}_c^2 - 4\overline{\Psi}_c + 9I = 0.$$

For $\overline{M} = \mathbb{R}$, we have

$$x^2 - 4x + 9 = 0,$$

with complex roots $x_1 = 2 + i\sqrt{5}$ and $x_2 = 2 - i\sqrt{5}$.

Definition 4.2.4. *The complex number $\overline{\psi}_c = 2 + i\sqrt{5}$ is called a complex Copper proportion.*

As a consequence, we obtain the relationship among the complex Copper proportion, Copper proportion and tangent real Copper proportion.

Tangent real Copper proportion: $\overline{\psi}_t = 2.$

Copper proportion: $\overline{\psi} = 2 + \sqrt{5}.$

Complex Copper proportion: $\overline{\psi}_c = 2 + i\sqrt{5}.$

Therefore, we have

$$\overline{\psi}_c = \overline{\psi}_t + i(\overline{\psi} - \overline{\psi}_t).$$

4.2.4 Examples

In this part, some examples have been studied for the Copper Structure.

Example 4.2.1. (Clifford Algebras).

Suppose $C'(n)$ represents the real Clifford algebra. Let $C'(n)$ be the positive definite form $\sum_{i=1}^{n}(x^i)^2$ over $\mathbb{R}^n$ and $\{e_1, e_{2,}, \ldots, e_n\}$ denotes the orthonormal basis of $\mathbb{R}^n$, then we have the following relations of $C'(n)$ [26]:

$$\begin{cases} e_i e_j = -e_j e_i, & i \neq j, \\ e_i^2 = 1. \end{cases} \tag{4.2.7}$$

Now making use of $\overline{\Psi}_i = 2 + \sqrt{5}e_i$, *we get the new relations of* $C'(n)$:

$$\begin{cases} \overline{\Psi}_i = Copper\, Structure, \\ \overline{\Psi}_i\overline{\Psi}_j + \overline{\Psi}_j\overline{\Psi}_i = 4\left(\overline{\Psi}_i + \overline{\Psi}_j\right) - 8, \quad i \neq j. \end{cases} \tag{4.2.8}$$

The real Clifford algebra $C'(2)$ *[26] is given by*

$$1 = I_2,\ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, we have

$$\overline{\Psi}_1 = 2I_2 + \sqrt{5}e_1 = \begin{pmatrix} 2+\sqrt{5} & 0 \\ 0 & 2-\sqrt{5} \end{pmatrix} = \begin{pmatrix} \overline{\psi} & 0 \\ 0 & 4-\overline{\psi} \end{pmatrix}. \tag{4.2.9}$$

$$\overline{\Psi}_2 = 2I_2 + \sqrt{5}e_2 = \begin{pmatrix} 2 & \sqrt{5} \\ \sqrt{5} & 2 \end{pmatrix}. \tag{4.2.10}$$

Example 4.2.2. (2D Copper Matrices).

Let $\overline{\Psi}$ *be a matrix in* $\mathbb{R}^n_n$ *and is called a Copper matrix if* $\overline{\Psi}$ *satisfies the following condition*

$$\overline{\Psi}^2 = 4\overline{\Psi} + I_n, \tag{4.2.11}$$

where I_n *represents the identity matrix on* $\mathbb{R}^n_n$.

For $n = 2$, *solving* (4.2.11), *the Copper structures* $\overline{\Psi}$ *in* $\mathbb{R}^2_2$ *is given by*

(i) Let a *and* d *belongs to* $\mathbb{R}$ *and* b *belongs to* $\mathbb{R} - \{0\}$, *then we have*

$$\overline{\Psi}_{a,b} = \begin{pmatrix} a & -\frac{1}{b}(a^2-4a-1) \\ b & 4-a \end{pmatrix} \quad or \quad \overline{\Psi}_{b,d} = \begin{pmatrix} 4-d & -\frac{1}{b}(d^2-4d-1) \\ b & d \end{pmatrix}. \tag{4.2.12}$$

(ii) Let a *equals to* $\overline{\psi}$ *and* b *in* $\mathbb{R}$, *then*

$$\overline{\Psi}_{\overline{\psi},b} = \begin{pmatrix} \overline{\psi} & 0 \\ b & 4-\overline{\psi} \end{pmatrix} \quad or \quad \overline{\Psi}_{4-\overline{\psi},b} = \begin{pmatrix} 4-\overline{\psi} & 0 \\ b & \overline{\psi} \end{pmatrix}.$$

(iii) Let a equals to $\overline{\psi}$ *and b is* 0*, then*

$$\overline{\Psi}_{\overline{\psi},0} = \begin{pmatrix} \overline{\psi} & 0 \\ 0 & 4-\overline{\psi} \end{pmatrix} \qquad or \qquad \overline{\Psi}_{4-\overline{\psi},0} = \begin{pmatrix} 4-\overline{\psi} & 0 \\ 0 & \overline{\psi} \end{pmatrix}.$$

Therefore from (4.2.9), (4.2.10) *and* (4.2.12)*, we get*

$$\overline{\Psi}_1 = \lim_{b\to 0} \overline{\Psi}_{\overline{\psi},b} \quad and \quad \overline{\Psi}_2 = \overline{\Psi}_{2,\sqrt{5}}.$$

Example 4.2.3. (Copper Reflections).

Let $(E, <, >)$ *denotes the Euclidean space and the reflection associated with a hyperplane* H *with respect to the normal* $\nu \in E-\{0\}$ *in* $(E, <, >)$ *satisfies the following condition*

$$r_\nu(x) = x - \frac{2 < x, \nu >}{< \nu, \nu >}\nu, \tag{4.2.13}$$

and $r_\nu^2 = I_E$ *[60, 26].*

The Copper reflection $\overline{\Psi}_\nu$ *corresponding to v is given as*

$$\overline{\Psi}_\nu = 2I_E + \sqrt{5}\ r_\nu, \tag{4.2.14}$$

thus ν *is an eigenvector of the Copper reflection* $\overline{\Psi}_\nu$ *w.r.t. the eigenvalue* $4-\overline{\psi}$*. Also from the lemma* [[*26*] *pg.*314]*, we have*

$$\mathcal{U}\overline{\Psi}_\nu\mathcal{U}^{-1} = \overline{\Psi}_{\mathcal{U}(\nu)},$$

where $\mathcal{U}$ *in the orthogonal group of* E*,* i.e., $\mathcal{U} \in O(E, <, >)$*.*

The explicit expression of this linear transformation can be written as

$$\overline{\Psi}_\nu(x) = \overline{\psi}x - 2\sqrt{5}\frac{< x, \nu >}{< \nu, \nu >}\nu. \tag{4.2.15}$$

Example 4.2.4. (Triple structure described by Copper structures).

"Suppose P *and* F *denote two* $(1, 1)$*-tensor fields on a manifold* $\overline{M}$*. Now, we have four structures with triplet* $(P, F, J = P \circ F)$*:*

(i) $P^2 = I = F^2$ *and* $F \circ P + P \circ F = 0$*; then* $J^2 = -I$*,*

(ii) $P^2 = I = F^2$ *and* $-F \circ P + P \circ F = 0$; *then* $J^2 = I$,

(iii) $P^2 = -I = F^2$ *and* $F \circ P + P \circ F = 0$; *then* $J^2 = -I$,

(iv) $P^2 = -I = F^2$ *and* $-F \circ P + P \circ F = 0$; *then* $J^2 = I$,

are known as, almost biproduct complex (abpc), almost hyperproduct (ahp), almost hypercomplex (ahc) and almost product bicomplex (apbc), respectively" [60, 113].

From the equation (4.2.2)*, we have*

$$\overline{\Psi}_F = 2I + \sqrt{5}F, \;\; \overline{\Psi}_P = 2I + \sqrt{5}P, \;\; \overline{\Psi}_J = 2I + \sqrt{5}J.$$

Therefore, we obtain the following relation

$$\frac{\sqrt{5}}{2}\overline{\Psi}_J = \frac{1}{2}\overline{\Psi}_P\overline{\Psi}_F - \overline{\Psi}_P - \overline{\Psi}_F + \overline{\psi}I.$$

As a consequence, the triplet $(\overline{\Psi}_F, \overline{\Psi}_P, \overline{\Psi}_J)$ *is:*

(i) An almost biproduct complex (*abpc*)*-structure iff:* $\overline{\Psi}_F, \overline{\Psi}_P$ *are Copper structures and* $(\overline{\Psi}_P\overline{\Psi}_F + \overline{\Psi}_F\overline{\Psi}_P) = 4(\overline{\Psi}_P + \overline{\Psi}_F) - 8I$*; then* $\overline{\Psi}_J$ *is a complex Copper structure.*

(ii) An almost hyperproduct (*ahp*)*-structure iff:* $\overline{\Psi}_F, \overline{\Psi}_P$ *are Copper structures and* $\overline{\Psi}_F\overline{\Psi}_P = \overline{\Psi}_P\overline{\Psi}_F$*; then* $\overline{\Psi}_J$ *is a Copper structure.*

(iii) An almost hypercomplex (*ahc*)*-structure iff:* $\overline{\Psi}_F, \overline{\Psi}_P$ *are complex Copper structures and* $(\overline{\Psi}_P\overline{\Psi}_F + \overline{\Psi}_F\overline{\Psi}_P) = 4(\overline{\Psi}_P + \overline{\Psi}_F) - 8I$*; then* $\overline{\Psi}_J$ *is a complex Copper structure.*

(iv) An almost product bicomplex (*apbc*)*-structure iff:* $\overline{\Psi}_F, \overline{\Psi}_P$ *are complex Copper structures and* $\overline{\Psi}_F\overline{\Psi}_P = \overline{\Psi}_P\overline{\Psi}_F$; *then* $\overline{\Psi}_J$ *is a Copper structure.*

Example 4.2.5. (Quaternion Algebras).

Let quaternion algebra is denoted by $\mathbb{H}$ *and* $\{1, e_1, e_2, e_3\}$ *be the base of it, satisfying* $e_1^2 = e_2^2 = e_3^2 = -1$ *and*

$$e_1 = -e_3e_2 = e_2e_3, \quad e_2 = -e_1e_3 = e_3e_1, \quad e_3 = -e_2e_1 = e_1e_2.$$

The quaternion q can be written by the following expression

$$q = S_q + \vec{V}_q = a_0 + a_1e_1 + a_2e_2 + a_3e_3,$$

where $S_q = a_0$ denotes the scalar part and $\vec{\mathcal{V}}_q = a_1e_1 + a_2e_2 + a_3e_3$ denotes the vector part of the quaternion q. Since $\sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2} = ||q||$ represents the norm of q. For any $q \neq 0$, $q_0 = \frac{q}{||q||}$ is the unit quaternion.

Furthermore, $q_0 = cos\ \alpha + \vec{\varepsilon}_0\, sin\ \alpha$ be the another representation form of the every unit quaternion, where $\vec{\varepsilon}_0$ is the unit vector holds the condition $\vec{\varepsilon}_0^{\,2} = -1$. Therefore, analogous to [67] (see eg. 3.3),

(i) the Copper split quaternion structure can be define as

$$\overline{\Psi}_q = 2 + \sqrt{5}\,\vec{\varepsilon}_0,$$

with $\vec{\varepsilon}_0^{\,2} = 1$.

(ii) the Copper biquaternion structure can be define as

$$\overline{\Psi}_q = 2 + i\sqrt{5}\,\vec{\varepsilon}_0,$$

where $i^2 = -1$ and $\vec{\varepsilon}_0^{\,2} = -1$.

4.2.5 Connections as Copper Structure

4.2.5.1 Connections on principal fibre bundles

Suppose principal fibre bundle is denoted by $P(\overline{M}, \pi, G)$, where $\overline{M}$ be a base space, π be a projection, G be a structure group and P be a total space. Let K denote a vertical distribution ($K = ker\ \pi_*$) on P and H represents a horizontal distribution.

Let h and ν are the corresponding projectors of H and K, respectively. The tensor field F of type $(1, 1)$ is an almost product structure on P if

$$F = \nu - h.$$

In [60, 67], an almost product structure F denotes a principal connection iff the following conditions are satisfied:

(i) $\mathcal{U}$ be a vertical vector field $\Longleftrightarrow \mathcal{U} = F(\mathcal{U})$.

(ii) $dR_a \circ F_u = F_{ua} \circ dR_a$ for every $u \in P$ and $a \in G$.

Considering the operation (4.2.2), for a Copper structure, we have

Proposition 4.19. *Suppose $\overline{\Psi}$ represents the Copper structure on P. The Copper structure $\overline{\Psi}$ is related to a principal connection iff the following conditions are satisfied:*

(i) $\mathcal{U}$ is a vertical vector field in K iff $\mathcal{U}$ be the eigenvector of Copper structure w.r.t. an eigenvalue $\overline{\psi}$.

(ii) $dR_a \circ \overline{\Psi}_u = \overline{\Psi}_{ua} \circ dR_a$ for every $u \in P$ and $a \in G$.

Proposition 4.20. *The principal connection is flat iff $\overline{\Psi}$ is integrable,* i.e., $N_{\overline{\Psi}} = 0$. *The principal connection yields a lift $l_\omega : T(\overline{M}) \to T(P)$ if following condition holds*

$$N_F(l_\omega \mathcal{U}, l_\omega \mathcal{V}) = [l_\omega \mathcal{U}, l_\omega \mathcal{V}] - l_\omega [\mathcal{U}, \mathcal{V}] .$$

where $\mathcal{U}$ and $\mathcal{V}$ are the vector fields on $\overline{M}$ [60].

Proposition 4.21. *The lift l_ω be a morphism iff $\overline{\Psi}$ is integrable, that is, $N_{\overline{\Psi}} = 0$.*

4.2.5.2 Connection in tangent bundles

Suppose $\overline{M}$ represents a differentiable manifold of n-dimension and $(T\overline{M}, \pi, \overline{M})$ denote the tangent bundle associated with the base space $\overline{M}$. Suppose $(L, x^i)_{1\leq i\leq n}$ represents a local coordinate system on $\overline{M}$ and $(\pi^{-1}(L), x^i, y^i)_{1\leq i\leq n}$ be the induced local coordinate system on $T\overline{M}$ defined by $y^i(u) = dx^i(u)$ and $x^i(u) = x^i(\pi(u))$ for every $u \in \pi^{-1}(L)$. The kernel of π_* is denoted by $K(\overline{M})$, *i.e.*, $K = \{\mathcal{U} \in T\overline{M} : \pi_*(\mathcal{U}) = 0\}$ and is known as vertical distribution of $\overline{M}$.

Let T be a $(1,1)$-tensor field on the base space $\overline{M}$ and an almost tangent structure is given by $T = \frac{\partial}{\partial y^i} \otimes dx^i$, that is, $T^2 = 0$.

Now analogous to [60], we have

Definition 4.2.5. *[60]. Let ν denotes a $(1,1)$-tensor field and is said to be a vertical projector if following condition is satisfied:*

$$\begin{cases} \nu \circ T = T, \\ T \circ \nu = 0. \end{cases} \tag{4.2.16}$$

Definition 4.2.6. *[60]. Suppose N be the complementary distribution to the vertical distribution K is called a normalization or non-linear connection, if*

$$T(\overline{M}) = N \oplus K. \tag{4.2.17}$$

Since ν be a $C^\infty(\overline{M})$ linear associated with $K = im\,\nu$. Now, we get

Proposition 4.22. *[60]. Let $N(\nu)$ be a non-linear connection induced by the vertical vector ν, and is given by the relation ker $\nu = N(\nu)$.*

Consider N be a non-linear connection. Suppose h_N and ν_N represents the horizontal as well as vertical projections related with the decomposition of equation (4.2.17). Therefore, we have

Proposition 4.23. *[60]. Let h_N and v_N corresponding projections of N and K, respectively. Let N be a non-linear connection $\Rightarrow$ ν_N is a vertical projector with $N = N(\nu_N)$.*

Definition 4.2.7. *[60]. Let ζ denote a tensor field of type $(1,1)$ and is called a non-linear connection of an almost product type if following equation is satisfied*

$$\begin{cases} T \circ \zeta = T, \\ \zeta \circ T = -T. \end{cases}$$

Proposition 4.24. *[60]. "Let ζ denotes a non-linear connection of an almost product type, then*

(i) $(I_{T(\overline{M})} - \zeta) = 2\nu_\zeta$ is a vertical vector.

(ii) $N(\nu_\zeta)$ is the $(+1)$-eigenspace of ζ and $K(\overline{M})$ is the (-1)-eigenspace of ζ."

Proposition 4.25. *[60]. "Let ζ denotes a non-linear connection of an almost product type,* i.e., $\zeta = I_{T(\overline{M})} - 2\nu$*, where ν is a vertical vector. Then ζ defines an almost product structure on manifold $\overline{M}$."*

Hence, for a Copper structure we have

Proposition 4.26. *Suppose N denotes a non-linear connection on $\overline{M}$, which is induced by the vertical vector ν. Then N can be defined by a Copper structure $\overline{\Psi}$,* i.e.,

$$\overline{\Psi} = \overline{\psi} I_{T(\overline{M})} - 2\sqrt{5}\ \nu,$$

with K the $(4 - \overline{\psi})$-eigenspace and N the $\overline{\psi}$-eigenspace.

4.2.6 Integrability and Parallelism of Copper Structures

Let $N_{\overline{\Psi}}$ represents the Nijenhuis tensor of $(1, 2)$-type tensor field and $\overline{\Psi}$ is a Copper structure on $\overline{M}$. From [49], we have

$$N_{\overline{\Psi}}(\mathcal{U}, \mathcal{V}) = \overline{\Psi}^2[\mathcal{U}, \mathcal{V}] + [\overline{\Psi}\mathcal{U}, \overline{\Psi}\mathcal{V}] - \overline{\Psi}[\overline{\Psi}\mathcal{U}, \mathcal{V}] - \overline{\Psi}[\mathcal{U}, \overline{\Psi}\mathcal{V}], \tag{4.2.18}$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on the manifold $\overline{M}$.
Let R and S be the complementary distributions on the manifold $\overline{M}$, respectively associated with $\overline{\psi}$ and $4 - \overline{\psi}$. Let r and s denote the corresponding projections. Thus we have

$$\begin{cases} s^2 = s,\ r^2 = r, \\ rs = sr = 0,\ r + s = I. \end{cases} \tag{4.2.19}$$

Now using (4.2.2), a direct calculation yields

$$\begin{cases} r = \frac{1}{2\sqrt{5}}\overline{\Psi} - \frac{4-\overline{\psi}}{2\sqrt{5}} I, \\ s = -\frac{1}{2\sqrt{5}}\overline{\Psi} + \frac{\overline{\psi}}{2\sqrt{5}} I. \end{cases} \tag{4.2.20}$$

From [49], it follows that

(i) If $N_{\overline{\Psi}} = 0$, then the Copper structure is said to be integrable.

(ii) If $s[r\mathcal{U}, r\mathcal{V}] = 0$ and $r[s\mathcal{U}, s\mathcal{V}] = 0$, then R and S both are integrable distributions, respectively.

From the equations (4.2.1) and (4.2.20), we obtain

$$\begin{cases} \overline{\Psi}r = r\overline{\Psi} = \overline{\psi}r = \frac{\overline{\psi}}{2\sqrt{5}}\overline{\Psi} + \frac{1}{2\sqrt{5}}I, \\ \overline{\Psi}s = s\overline{\Psi} = (4-\overline{\psi})s = \frac{(\overline{\psi}-4)}{2\sqrt{5}}\overline{\Psi} - \frac{1}{2\sqrt{5}}I. \end{cases} \tag{4.2.21}$$

Then, for a Copper structure we have

$$\begin{cases} rN_{\overline{\Psi}}(s\mathcal{U}, s\mathcal{V}) = 20\ r[s\mathcal{U}, s\mathcal{V}], \\ sN_{\overline{\Psi}}(r\mathcal{U}, r\mathcal{V}) = 20\ s[r\mathcal{U}, r\mathcal{V}]. \end{cases} \tag{4.2.22}$$

Proposition 4.27. *An almost product structure* $P = \frac{1}{\sqrt{5}}(\overline{\Psi} - 2I)$ *said to be integrable iff a Copper structure* $\overline{\Psi}$ *is integrable.*

Proposition 4.28. *Suppose* $\mathcal{U}$ *and* $\mathcal{V}$ *be vector fields on* $\overline{M}$*. The distribution* S *is integrable iff* $rN_{\overline{\Psi}}(s\mathcal{U}, s\mathcal{V})$ *vanishes and the distribution* R *is integrable iff* $sN_{\overline{\Psi}}(r\mathcal{U}, r\mathcal{V})$ *vanishes. Both of the distributions* R *and* S *are integrable, if a Copper structure is integrable.*

Consider a connection $\overline{\nabla}$ which is linear on $\overline{M}$. Now, the linear connections for $(\overline{\Psi}, \overline{\nabla})$ is given by the following [3]:

(i) "The connection

$$\overset{Sc}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{\mathcal{U}} r\mathcal{V}) + s(\overline{\nabla}_{\mathcal{U}} s\mathcal{V}) \tag{4.2.23}$$

is called a *Schouten* connection.

(ii) The connection

$$\overset{Vr}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{r\mathcal{U}} r\mathcal{V}) + s(\overline{\nabla}_{s\mathcal{U}} s\mathcal{V}) + r[s\mathcal{U}, r\mathcal{V}] + s[r\mathcal{U}, s\mathcal{V}] \tag{4.2.24}$$

is called a *Vrănceanu* connection."

Proposition 4.29. *Suppose* $\overline{\nabla}$ *be a linear connection on* $\overline{M}$*, the projectors* s *and* r *are parallels in terms of Schouten* $\overset{Sc}{\nabla}$ *and Vrănceanu* $\overset{Vr}{\nabla}$ *connections. Furthermore, the Copper structure* $\overline{\Psi}$ *is also parallel with respect to* $\overset{Sc}{\nabla}$ *and* $\overset{Vr}{\nabla}$*.*

Proof. *Suppose $\mathcal{U}$ and $\mathcal{V}$ be vector fields on $\overline{M}$. From* (4.2.19), *we have*

$$(\overset{Sc}{\nabla}_{\mathcal{U}} r)\mathcal{V} = \overset{Sc}{\nabla}_{\mathcal{U}} r\mathcal{V} - r(\overset{Sc}{\nabla}_{\mathcal{U}}\mathcal{V})$$
$$= r(\overline{\nabla}_{\mathcal{U}} r\mathcal{V}) - r(\overline{\nabla}_{\mathcal{U}} r\mathcal{V}) = 0.$$

$$(\overset{Vr}{\nabla}_{\mathcal{U}} r)\mathcal{V} = \overset{Vr}{\nabla}_{\mathcal{U}} r\mathcal{V} - r(\overset{Vr}{\nabla}_{\mathcal{U}}\mathcal{V})$$
$$= r(\overline{\nabla}_{r\mathcal{U}} r\mathcal{V}) + r[s\mathcal{U}, r\mathcal{V}]$$
$$- r(\overline{\nabla}_{r\mathcal{U}} r\mathcal{V}) - r[s\mathcal{U}, r\mathcal{V}] = 0.$$

Therefore, the projection r is parallel corresponding to $\overset{Sc}{\nabla}$ and $\overset{Vr}{\nabla}$. Analogously, it can be shown that the above equations are also valid for the projection s.
Also, the result immediately follows for a Copper structure $\overline{\Psi}$ from equation (4.2.21).

Proposition 4.30. *Let $\overline{\nabla}$ be a linear connection on $\overline{M}$. Then, the distributions R and S are parallel corresponding to Schouten $\overset{Sc}{\nabla}$ as well as Vrănceanu $\overset{Vr}{\nabla}$ connections.*

Proof. *Suppose $\mathcal{V} \in R$ and $\mathcal{U} \in T(\overline{M})$. Hence, $s\mathcal{V} = 0$ as well as $r\mathcal{V} = \mathcal{V}$.*

From equations (4.2.23) *and* (4.2.24), *we have*

$$\overset{Sc}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{\mathcal{U}}\mathcal{V}),$$
$$\overset{Vr}{\nabla}_{\mathcal{U}}\mathcal{V} = r(\overline{\nabla}_{r\mathcal{U}}\mathcal{V}) + r[s\mathcal{U}, \mathcal{V}].$$

Thus, both $\overset{Sc}{\nabla}$ and $\overset{Vr}{\nabla}$ belongs to R. Therefore, the distribution R is parallel with respect to $\overset{Sc}{\nabla}$ and $\overset{Vr}{\nabla}$. Analogously, S also satisfies above conditions.

4.2.7 Copper Riemannian Manifolds

"Suppose g represents a Riemannian metric and P denotes an almost product structure on manifold $\overline{M}$, and is related by the following expression

$$g(\mathcal{U}, \mathcal{V}) = g(P(\mathcal{U}), P(\mathcal{V})),$$

for any vector fields $\mathcal{U}$ and $\mathcal{V}$ on manifold $\overline{M}$. In other words, P is a g-symmetric endomorphism and is defined by

$$g(P(\mathcal{U}), \mathcal{V}) = g(\mathcal{U}, P(\mathcal{V})),$$

For this case, the Riemannian almost product structure is represented by the ordered pair (g, P)" [8].
By making use of (4.2.2) and (4.2.3), we have

Proposition 4.31. *The Copper structure $\overline{\Psi}$ defines a g-symmetric endomorphism iff an operator P is a g-symmetric endomorphism.*

Definition 4.2.8. *Let g represents the Riemannian metric on the manifold $\overline{M}$. Then, the ordered pair $(g, \overline{\Psi})$ is called a Copper Riemannian structure if the following equality is satisfied:*

$$g(\overline{\Psi}(\mathcal{U}), \mathcal{V}) = g(\mathcal{U}, \overline{\Psi}(\mathcal{V})),$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on $\overline{M}$ and the triplet $(\overline{M}, g, \overline{\Psi})$ is said to be a a Copper Riemannian manifold.

Corollary 4.2.1. *Suppose $(\overline{M}, g, \overline{\Psi})$ denotes a Copper Riemannian manifold, then the following conditions hold:*

(i) The projectors s and r are g-symmetric, i.e.,

$$\begin{cases} g(s(\mathcal{U}), \mathcal{V}) = g(\mathcal{U}, s(\mathcal{V})), \\ g(r(\mathcal{U}), \mathcal{V}) = g(\mathcal{U}, r(\mathcal{V})). \end{cases}$$

(ii) The Copper structure $\overline{\Psi}$ on $\overline{M}$ is $N_{\overline{\Psi}}$-symmetric, i.e.,

$$N_{\overline{\Psi}}(\overline{\Psi}(\mathcal{U}), \mathcal{V}) = N_{\overline{\Psi}}(\mathcal{U}, \overline{\Psi}(\mathcal{V})).$$

(iii) The distributions R and S are g-orthogonal, i.e.,

$$g(s(\mathcal{V}), r(\mathcal{U})) vanishes.$$

Proposition 4.32. *[60] "Let $\overline{\nabla}^g$ be a Levi-Civita connection of g. If P is parallel corresponding to the $\overline{\nabla}^g$,* i.e., *$\overline{\nabla}^g P = 0$, then the Riemannian almost product structure is a locally product structure. Moreover, if $\overline{\nabla}$ is a linear and symmetric connection, then the Nijenhuis tensor of P satisfies*

$$N_P(\mathcal{U}, \mathcal{V}) = (\overline{\nabla}_{P\mathcal{U}} P)\mathcal{V} - (\overline{\nabla}_{P\mathcal{V}} P)\mathcal{U} - P(\overline{\nabla}_{\mathcal{U}} P)\mathcal{V} + P(\overline{\nabla}_{\mathcal{V}} P)\mathcal{U}.$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$."

Then, for a Copper structure, we have

Corollary 4.2.2. *If* $(\overline{M}, g, \overline{\Psi})$ *defines a locally product Copper Riemannian manifold then* $\overline{\Psi}$ *is said to be integrable.*

4.3 Conclusion

In this chapter, we have investigated the Bronze structure and the Copper structure on manifolds. We have obtained some examples of the Bronze and Copper structure on manifolds. Further, we studied connections, integrability and parallelism for Bronze and Copper structures. Besides, some results have been obtained for the Bronze Riemannian manifold and Copper Riemannian manifold.

Chapter 5

Adapted Connections on Kaehler-Norden Silver Manifolds and Harmonicity

5.1 Introduction

In 1970, the notion of the polynomial structure on a manifold was introduced by Goldberg and Yano [99]. Suppose $\overline{M}$ denote a C^∞-differentiable real manifold. A tensor field F of type $(1,1)$ on $\overline{M}$ is said to define a polynomial structure of degree n, if F satisfies

$$Q(F) = F^n + a_n F^{n-1} + \cdots + a_2 F + a_1 I = 0,$$

and F has constant rank on $\overline{M}$. Here I denote $(1,1)$ identity tensor field and $a_1, a_2, \ldots, a_n$ are real numbers.

Gezer *et al.* [6] studied the properties of Riemannian manifolds equipped with the Hessian metric h and a complex Golden structure. In [5], Gezer *et al.* obtained a new sufficient condition of integrability for a Golden Riemannian structure. They also investigated some properties of twin Golden Riemannian metrics and the curvature properties of locally decomposable Golden Riemannian manifolds. Primo *et al.* [9]

obtained some algebraic and geometric characterizations of the Silver ratio.

In [10], Salimov *et al.* investigated Norden metrics of Hessian type $h = \overline{\nabla}^2 f$. Salimov [11] introduced anti-Hermitian metric connections of type I and type II, respectively. Moreover, Salimov considered the classes of anti-Hermitian manifolds associated with these connections. Sahin *et al.* [15] introduced the notion of a Golden map between Golden Riemannian manifolds and proved that such maps are harmonic. In [24], Hreţcanu *et al.* introduced a structure on a class of Riemannian manifolds, known as a Golden structure. Also, they established some interesting properties of the Golden structure.

Etayo *et al.* [34] investigated adapted connections and obtained two special connections on almost Golden Riemannian structure, which measure the integrability of $(1,1)$-tensor field φ and the integrability of G-structure corresponding to an almost Golden Riemannian manifold (φ, g). Bilen *et al.* [50] studied the curvature properties of a pseudo-Riemannian manifold equipped with a Kaehler-Norden-Codazzi Golden structure, and defined special connections of type I and type II.

In [60], Crăşmăreanu *et al.* investigated the geometry of Golden structure on a manifold by using a corresponding almost product structure. Iscan *et al.* [62] investigated the geometry of Kaehler-Norden manifolds. Moreover, Iscan *et al.* used Tachibana operators to study the properties of curvature scalars and Riemannian curvature tensors of Kaehler-Norden manifolds. In [89], Kumar *et al.* studied the adapted connections on Kaehler-Norden Golden manifolds and almost complex Norden Golden manifolds.

Inspired by [89], we investigate Kaehler-Norden Silver manifolds and almost complex Norden Silver manifolds. Analogous to [50, 89], we define adapted connections of first, second and third type to an almost complex Norden Silver manifold and prove that a complex Norden Silver map is a harmonic map between Kaehler-Norden Silver manifolds.

5.2 Kaehler-Norden Silver Manifold

Suppose Θ_c is a tensor field of type $(1,1)$ on manifold $\overline{M}^{2n}$. A tensor field Θ_c is said to be an *almost complex Silver structure*, if it satisfies

$$\Theta_c^2 - 2\Theta_c + 3I = 0, \tag{5.2.1}$$

and the pair $(\overline{M}^{2n}, \Theta_c)$ is called an *almost complex Silver manifold*. The complex number $1 + i\sqrt{2}$, which is a root of the equation $x^2 - 2x + 3 = 0$, is known as a *complex Silver ratio* [67].

Let J be an almost complex structure on $\overline{M}^{2n}$, then

$$\Theta_c^J = (I \mp \sqrt{2}J), \tag{5.2.2}$$

is called an *almost complex Silver structure* on $\overline{M}^{2n}$.

Conversely, let Θ_c denote an almost complex Silver structure on $\overline{M}^{2n}$, then

$$J^{\Theta_c} = \mp\frac{1}{\sqrt{2}}\left(\Theta_c - I\right), \tag{5.2.3}$$

is said to be an *almost complex structure* induced by Θ_c. Therefore, an almost complex Silver structure Θ_c defines a Θ_c-associated almost complex structure J^{Θ_c} and vice-versa. Obviously, $J^{\Theta_c^J} = J$ and $\Theta_c^{J^{\Theta_c}} = \Theta_c$.

Hence, there exist a one-to-one correspondence between almost complex structures and almost complex Silver structures on $\overline{M}^{2n}$.

It is well known that the almost complex Silver manifold is integrable if Nijenhuis tensor N_{Θ_c} vanishes and is given by

$$N_{\Theta_c} = \Theta_c^2\left[\mathcal{U}, \mathcal{V}\right] + \left[\Theta_c\mathcal{U}, \Theta_c\mathcal{V}\right] - \Theta_c\left[\Theta_c\mathcal{U}, \mathcal{V}\right] - \Theta_c\left[\mathcal{U}, \Theta_c\mathcal{V}\right]. \tag{5.2.4}$$

If the almost complex Silver structure Θ_c is integrable, then this structure is said to be complex Silver structure and $(\overline{M}^{2n}, \Theta_c)$ is known as a *complex Silver manifold*.

Let $(\overline{M}^{2n}, g)$ denote the pseudo-Riemannian manifold associated with an almost complex structure J. If the pseudo-Riemannian metric g is pure w.r.t. an almost complex structure J, *i.e.*,

$$g(J\mathcal{U},\mathcal{V})=g(\mathcal{U},J\mathcal{V}),$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on $\overline{M}$, then $(\overline{M}^{2n},J,g)$ is called an *almost complex Norden manifold* and $\overline{M}^{2n}$ is known as *Norden manifold* if J is integrable.

Suppose g be a pseudo-Riemannian metric equipped with an almost complex Silver structure Θ_c, then the triplet $(\overline{M}^{2n},\Theta_c,g)$ is said to be an almost complex Norden Silver manifold if it satisfies the following equation

$$g\left(\Theta_c\mathcal{U},\mathcal{V}\right)=g\left(\mathcal{U},\Theta_c\mathcal{V}\right), \tag{5.2.5}$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on $\overline{M}$.

Thus, for an almost complex Norden Silver manifold $(\overline{M}^{2n},\Theta_c,g)$, we have

$$g\left(\Theta_c\mathcal{U},\Theta_c\mathcal{V}\right)=2g\left(\Theta_c\mathcal{U},\mathcal{V}\right)-3g(\mathcal{U},\mathcal{V}). \tag{5.2.6}$$

Suppose φ be a tensor field of type $(1,1)$ and $\Im^p_q(\overline{M})$ denote the set of all (p,q)-tensor fields on the smooth manifold $\overline{M}$. A tensor field ω of type $(0,s)$ is known as pure tensor field associated with φ, if

$$\omega\left(\varphi\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_s\right)=\omega\left(\mathcal{U}_1,\varphi\mathcal{U}_2,\ldots,\mathcal{U}_s\right)=\cdots=\omega\left(\mathcal{U}_1,\mathcal{U}_2,\ldots,\varphi\mathcal{U}_s\right),$$

for any $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_s\in\Im^1_0(\overline{M})$.

"Let φ be a tensor field of type $(1,1)$, then consider an operator

$$\phi_\varphi:\ \Im^0_s(\overline{M})\rightarrow\Im^0_{s+1}(\overline{M}),$$

operated on the pure tensor field ω of type $(0,s)$ with respect to φ and is given by

$$\begin{aligned}\left(\phi_\varphi\omega\right)\left(\mathcal{U},\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_s\right)=&\left(\varphi\mathcal{U}\right)\left(\omega(\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_s)\right)\\&-\mathcal{U}\left(\omega(\varphi\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_s)\right)\\&+\omega\left((L_{\mathcal{V}_1}\varphi)\mathcal{U},\mathcal{V}_2,\ldots,\mathcal{V}_s\right)\\&\qquad\cdots\\&+\omega\left(\mathcal{V}_1,\mathcal{V}_2,\ldots,(L_{\mathcal{V}_s}\varphi)\mathcal{U}\right),\end{aligned} \tag{5.2.7}$$

for any $\mathcal{U}, \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s \in \Im_0^1(\overline{M})$, where $L_{\mathcal{V}}$ denotes the Lie differentiation with respect to $\mathcal{V}$" [108].

If J is integrable then an almost complex structure J is said to be a *complex structure*. Suppose $\varphi = J$ represents the complex structure on $\overline{M}^{2n}$ then a tensor field ω is called as a *holomorphic tensor field*, if

$$(\phi_J \omega)(\mathcal{U}, \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_s) = 0.$$

Let $(\overline{M}^{2n}, J, g)$ be the Norden manifold. If $\phi_J g = 0$ then a Norden metric g is called holomorphic and the Norden manifold is said to be a holomorphic Norden manifold [10, 62]. In some sense, holomorphic Norden manifolds are analogous to Kaehler manifolds due to the following theorem by Iscan *et al.* [62].

Theorem 5.2.1. *"An almost Norden manifold is holomorphic Norden manifold if and only if the almost complex structure is parallel with respect to the Levi-Civita connection* $\overline{\nabla}^g$*."*

If $\overline{M}^{2n}$ is associated with a pseudo-Riemannian metric g and an almost complex structure J such that $\overline{\nabla}^g J = 0$, then the triplet $(\overline{M}^{2n}, J, g)$ is said to be a *Kaehler-Norden manifold*, where $\overline{\nabla}^g$ is the Levi-Civita connection of g [62]. Therefore, there exists a one-one correspondence between Norden manifolds and Kaehler-Norden manifolds with holomorphic metrics.

In the study of almost complex structure, the ϕ-operator method can be used for almost complex Silver structures because the almost complex structure J and almost complex Silver structure Θ_c are related to each other. Therefore, for the integrability of Θ_c on pseudo-Riemannian manifolds, we have the following result.

Theorem 5.2.2. *Suppose* $(\overline{M}^{2n}, \Theta_c, g)$ *is an almost complex Silver Norden manifold and* $\overline{\nabla}^g$ *denotes the Levi-Civita connection of* g*. Then*

(i) if $\phi_{\Theta_c} g = 0$*, then* Θ_c *is integrable,*

(ii) the equality $\phi_{\Theta_c} g = 0$ *is equivalent to* $\overline{\nabla}^g \Theta_c = 0$*.*

Proof. *From* (5.2.5) *and* $\overline{\nabla}^g g = 0$, *it follows that*

$$g(\mathcal{U}, (\overline{\nabla}^g_{\mathcal{W}}\Theta_c)\mathcal{V}) = g((\overline{\nabla}^g_{\mathcal{W}}\Theta_c)\mathcal{U}, \mathcal{V}), \tag{5.2.8}$$

where $\mathcal{U}$, $\mathcal{V}$ *and* $\mathcal{W}$ *are vector fields on* $\overline{M}$.

Now making use of (5.2.8) *and* $[\mathcal{U}, \mathcal{V}] = \overline{\nabla}^g_{\mathcal{U}}\mathcal{V} - \overline{\nabla}^g_{\mathcal{V}}\mathcal{U}$, *we can transform* (5.2.7) *as given below:*

$$(\phi_{\Theta_c} g)(\mathcal{U}, \mathcal{V}, \mathcal{W}) = -g\left(\left(\overline{\nabla}^g_{\mathcal{U}}\Theta_c\right)\mathcal{V}, \mathcal{W}\right) + g\left(\left(\overline{\nabla}^g_{\mathcal{V}}\Theta_c\right)\mathcal{U}, \mathcal{W}\right) + g\left(\mathcal{V}, \left(\overline{\nabla}^g_{\mathcal{W}}\Theta_c\right)\mathcal{U}\right). \tag{5.2.9}$$

Similarly, we have

$$(\phi_{\Theta_c} g)(\mathcal{W}, \mathcal{V}, \mathcal{U}) = -g\left(\left(\overline{\nabla}^g_{\mathcal{W}}\Theta_c\right)\mathcal{V}, \mathcal{U}\right) + g\left(\left(\overline{\nabla}^g_{\mathcal{V}}\Theta_c\right)\mathcal{W}, \mathcal{U}\right) + g\left(\mathcal{V}, \left(\overline{\nabla}^g_{\mathcal{U}}\Theta_c\right)\mathcal{W}\right). \tag{5.2.10}$$

Adding (5.2.9) *and* (5.2.10), *we obtain the following equation*

$$(\phi_{\Theta_c} g)(\mathcal{U}, \mathcal{V}, \mathcal{W}) + (\phi_{\Theta_c} g)(\mathcal{W}, \mathcal{V}, \mathcal{U}) = 2g\left(\mathcal{U}, (\overline{\nabla}^g_{\mathcal{V}}\Theta_c)\mathcal{W}\right). \tag{5.2.11}$$

Now, substituting $\phi_{\Theta_c} g = 0$ *in equation* (5.2.11), *we get* $\overline{\nabla}^g \Theta_c = 0$.

If g (pseudo-Riemannian metric) is pure associated with an almost complex Silver structure Θ_c, then g is pure along with the almost complex structure J. From equation (5.2.2), we have

$$\phi_{\Theta_c} g = \sqrt{2}\,\phi_J g.$$

Consequently, from the above equality and theorem (5.2.2), we have

Theorem 5.2.3. *Suppose* J *is an almost complex structure of an almost complex Norden Silver manifold* $(\overline{M}^{2n}, \Theta_c, g)$. *If* $\phi_J g = 0$, *then almost complex Silver structure* Θ_c *is integrable.*

Now, from the Theorem (5.2.2) and Theorem (5.2.3), a Kaehler-Norden Silver manifold can be defined as

Definition 5.2.1. *Let $\overline{\nabla}^g$ be the Levi-Civita connection of g. A Kaehler-Norden Silver manifold is given by a triplet $(\overline{M}^{2n}, \Theta_c, g)$ which consists of a smooth manifold $\overline{M}^{2n}$ associated with a pseudo-Riemannian metric g and an almost complex Silver structure Θ_c such that $\overline{\nabla}^g \Theta_c = 0$. Here the metric g is supposed to be Nordenian,* i.e.,

$$g(\Theta_c \mathcal{U}, \mathcal{V}) = g(\mathcal{U}, \Theta_c \mathcal{V}).$$

Let $\tilde{g}$ be a twin Norden Silver metric for an almost complex Norden Silver manifold $(\overline{M}^{2n}, \Theta_c, g)$ defined by

$$\tilde{g}(\mathcal{U}, \mathcal{V}) = (g \circ \Theta_c)(\mathcal{U}, \mathcal{V}) = g(\Theta_c \mathcal{U}, \mathcal{V}). \tag{5.2.12}$$

Obviously $\tilde{g}(\Theta_c \mathcal{U}, \mathcal{V}) = \tilde{g}(\mathcal{U}, \Theta_c \mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$. It should be noted that both the metrics g and $\tilde{g}$ are necessarily of signature (n, n).

Moreover, by equation (5.2.6), we can write

$$\tilde{g}(\mathcal{U}, \Theta_c \mathcal{V}) = 2\tilde{g}(\mathcal{U}, \mathcal{V}) - 3g(\mathcal{U}, \mathcal{V}). \tag{5.2.13}$$

Now, we have the following theorem.

Theorem 5.2.4. *An almost complex Silver structure Θ_c is an isomorphism on the tangent space $T_u\overline{M}$, $\forall\ u \in \overline{M}$.*

Proof. *Let $\mathcal{U} \in ker\, \Theta_c$* i.e. *$\Theta_c \mathcal{U} = 0$. Consequently, we have $\Theta_c^2 \mathcal{U} = 0$. Now, from equation* (5.2.1)*, it follows that $\mathcal{U} = 0$,* i.e. *$ker\, \Theta_c = \{0\}$. Therefore, Θ_c is an isomorphism on $T_u\overline{M}$.*

5.3 Adapted Connections on Almost Complex Norden Silver Manifolds

Etayo *et al.* [34] and Kumar *et al.* [89] investigated the connections which parallelize the Riemannian metric and almost Golden structures. Motivated by the work of Etayo *et al.* and Kumar *et al.*, we study the connections which parallelize the pseudo-Riemannian metric and the almost complex Silver structures.

Definition 5.3.1. *Let* $\overline{\nabla}^{\alpha}$ *be a linear connection on* $\overline{M}$ *and* $(\overline{M}^{2n}, \Theta_c, g)$ *is an almost complex Norden Silver manifold. If* $\overline{\nabla}^{\alpha}$ *parallelizes together* g *and* Θ_c, i.e., $\overline{\nabla}^{\alpha} g = 0$ *and* $\overline{\nabla}^{\alpha}\Theta_c = 0$, *then* $\overline{\nabla}^{\alpha}$ *is an adapted connection to the almost complex Norden Silver structure* (Θ_c, g).

Theorem 5.3.1. *Suppose* $\overline{\nabla}^{\alpha}$ *is a linear connection on* $\overline{M}$ *and* $(\overline{M}^{2n}, \Theta_c, g)$ *denotes the almost complex Norden Silver manifold. Then* $\overline{\nabla}^{\alpha}$ *is the adapted connection to the almost complex Norden structure* (J^{Θ_c}, g), *induced by* Θ_c *if and only if* $\overline{\nabla}^{\alpha}$ *be the adapted connection to almost complex Norden Silver structure* (Θ_c, g).

Proof. *Let* $\overline{\nabla}^{\alpha}$ *be the adapted connection to almost complex Silver structure* Θ_c, i.e., $\overline{\nabla}^{\alpha}\Theta_c = 0$, *then using equation* (5.2.3), *we have*

$$\begin{aligned}\overline{\nabla}^{\alpha}_{\mathcal{U}} J^{\Theta_c}\mathcal{V} &= \mp\frac{1}{\sqrt{2}}\overline{\nabla}^{\alpha}_{\mathcal{U}}\Theta_c\mathcal{V} \pm \frac{1}{\sqrt{2}}\overline{\nabla}^{\alpha}_{\mathcal{U}}\mathcal{V}\\ &= \mp\frac{1}{\sqrt{2}}\left[\Theta_c\overline{\nabla}^{\alpha}_{\mathcal{U}}\mathcal{V} - \overline{\nabla}^{\alpha}_{\mathcal{U}}\mathcal{V}\right]\\ &= J^{\Theta_c}\overline{\nabla}^{\alpha}_{\mathcal{U}}\mathcal{V},\end{aligned}$$

where $\mathcal{U}$ *and* $\mathcal{V}$ *are vector fields on* $\overline{M}$ *and hence this implies that* $\overline{\nabla}^{\alpha} J^{\Theta_c} = 0$. *Similarly, taking* $\overline{\nabla}^{\alpha} J^{\Theta_c} = 0$ *and using* (5.2.2), *we get* $\overline{\nabla}^{\alpha}\Theta_c = 0$.

Suppose $(\overline{M}^{2n}, \Theta_c, g)$ is an almost complex Norden Silver manifold. Let $\overline{\nabla}^{\alpha}$ is an adapted connection to the almost complex Norden Silver structure (Θ_c, g) and $\overline{\nabla}^{g}$ denotes the Levi-Civita connection of g. Now, let $\mathcal{S} \in \Im^1_2(\overline{M})$ denote the potential tensor of $\overline{\nabla}^{\alpha}$ associated with $\overline{\nabla}^{g}$ and be given by

$$\mathcal{S}(\mathcal{U}, \mathcal{V}) = \overline{\nabla}^{\alpha}_{\mathcal{U}}\mathcal{V} - \overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V}, \tag{5.3.1}$$

where $\mathcal{U}$ and $\mathcal{V}$ are vector fields on $\overline{M}$. Thus, the adapted connections for an almost complex Norden Silver manifold $(\overline{M}^{2n}, \Theta_c, g)$ are given as follows:

Theorem 5.3.2. *Let* $(\overline{M}^{2n}, \Theta_c, g)$ *be an almost complex Norden Silver manifold. Then the set of linear connections adapted to the* (Θ_c, g) *is*

$$\left\{\overline{\nabla}^{g} + \mathcal{S} : \quad \begin{array}{l}\left(\overline{\nabla}^{g}_{\mathcal{U}}\Theta_c\right)\mathcal{V} = \Theta_c\mathcal{S}\left(\mathcal{U}, \mathcal{V}\right) - \mathcal{S}(\mathcal{U}, \Theta_c\mathcal{V}),\\ g\left(\mathcal{S}(\mathcal{U}, \mathcal{V}), \mathcal{W}\right) + g\left(\mathcal{S}(\mathcal{U}, \mathcal{W}), \mathcal{V}\right) = 0,\end{array}\right\}$$

where $\mathcal{U}$, $\mathcal{V}$ *and* $\mathcal{W}$ *are any vector fields on* $\overline{M}$.

Proof. *Let $\mathcal{S}$ denote the potential tensor of the adapted connection $\overline{\nabla}^{\alpha}$ w.r.t. $\overline{\nabla}^{g}$, the equation* (5.3.1) *yields*

$$\overline{\nabla}^{\alpha}_{\mathcal{U}}\mathcal{V} = \overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V} + \mathcal{S}(\mathcal{U},\mathcal{V}).$$

By using above equation, we get

$$\left(\overline{\nabla}^{\alpha}_{\mathcal{U}}\Theta_c\right)\mathcal{V} = \left(\overline{\nabla}^{g}_{\mathcal{U}}\Theta_c\right)\mathcal{V} - \Theta_c\mathcal{S}\left(\mathcal{U},\mathcal{V}\right) + \mathcal{S}(\mathcal{U},\Theta_c\mathcal{V}),$$

and

$$\left(\overline{\nabla}^{\alpha}_{\mathcal{U}}g\right)\mathcal{V} = -g\left(\mathcal{S}(\mathcal{U},\mathcal{V}),\mathcal{W}\right) - g\left(\mathcal{S}(\mathcal{U},\mathcal{W}),\mathcal{V}\right).$$

Since $\overline{\nabla}^{\alpha}$ be the adapted connection to the almost complex Norden Silver structure (Θ_c, g). Hence, we get the desired result.

For an arbitrary connection D from (5.2.12), we have

$$(D_{\mathcal{U}}\tilde{g})(\mathcal{V},\mathcal{W}) = (D_{\mathcal{U}}g)(\Theta_c\mathcal{V},\mathcal{W}) + g\left((D_{\mathcal{U}}\Theta_c)\mathcal{V},\mathcal{W}\right). \tag{5.3.2}$$

Thus, we have the following result.

Theorem 5.3.3. *Suppose $(\overline{M}^{2n}, \Theta_c, g)$ is an almost complex Norden Silver manifold. Then $\overline{\nabla}^{\alpha}g = 0$ and $\overline{\nabla}^{\alpha}\tilde{g} = 0$, if and only if $\overline{\nabla}^{\alpha}$ is the adapted connection to the almost complex Norden Silver structure (Θ_c, g).*

Bilen *et al.* [50] introduced a special connection of the first type and second type for an almost complex Norden Golden manifold. Similarly, in case of an almost complex Norden Silver manifold $(\overline{M}^{2n}, \Theta_c, g)$ the special connection of the first type and second type can be defined as follows:

Definition 5.3.2. *A linear connection $\overline{\nabla}^{i}_{\mathcal{U}}\mathcal{V} = \overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V} + \mathcal{S}^{i}(\mathcal{U},\mathcal{V})$ on an almost complex Norden Silver manifold $(\overline{M}^{2n}, \Theta_c, g)$ satisfying the conditions $g\left(\mathcal{S}^{i}(\mathcal{U},\mathcal{V}),\Theta_c\mathcal{W}\right) = g\left(\mathcal{S}^{i}(\mathcal{U},\mathcal{W}),\Theta_c\mathcal{V}\right)$ and $\overline{\nabla}^{i}\tilde{g} = 0$ is said to be a special connection of the first type, where $\mathcal{S}^{i}$ denotes a $(1,2)$-tensor field.*

Taking covariant derivative of the twin Norden Silver metric $\tilde{g}$ with respect to $\overline{\nabla}^i$, we obtain

$$(\overline{\nabla}^i_{\mathcal{U}}\tilde{g})(\mathcal{V},\mathcal{W}) = \left(\overline{\nabla}^g_{\mathcal{U}}\tilde{g}\right)(\mathcal{V},\mathcal{W}) - \tilde{g}\left(\mathcal{S}^i(\mathcal{U},\mathcal{V}),\mathcal{W}\right) - \tilde{g}\left(\mathcal{S}^i(\mathcal{U},\mathcal{W}),\mathcal{V}\right).$$

Now, using the definition of $\tilde{g}$ and special connection of the first type, we get the following equation

$$\left(\overline{\nabla}^g_{\mathcal{U}}\tilde{g}\right)(\mathcal{V},\mathcal{W}) = 2\,g\left(\mathcal{S}^i(\mathcal{U},\mathcal{V}),\Theta_c\mathcal{W}\right)$$

or,

$$\Theta_c\mathcal{S}^i\left(\mathcal{U},\mathcal{V}\right) = \frac{1}{2}\left(\overline{\nabla}^g_{\mathcal{U}}\Theta_c\right)\mathcal{V}.$$

By a direct calculation, we have

$$(\overline{\nabla}^i_{\mathcal{U}}g)(\mathcal{V},\mathcal{W}) = -\frac{1}{3}g\left(\mathcal{V},\left(\overline{\nabla}^g_{\mathcal{U}}\Theta_c\right)\mathcal{W}\right) \neq 0. \tag{5.3.3}$$

Therefore from Definition (5.2.1), Definition (5.3.2), Theorem (5.3.3) and equation (5.3.3), we have the following result.

Theorem 5.3.4. *Suppose $(\overline{M}^{2n},\Theta_c,g)$ is an almost complex Norden Silver manifold. Then the special connection of the first type $\overline{\nabla}^i$ is not an adapted connection to the almost complex Norden Silver structure (Θ_c,g) of $\overline{M}^{2n}$. Furthermore, the special connection of the first type $\overline{\nabla}^i$ is an adapted connection to (Θ_c,g) if almost complex Norden Silver manifold $(\overline{M}^{2n},\Theta_c,g)$ is a Kaehler-Norden Silver manifold.*

Now, we give the definition of special connection of the second type.

Definition 5.3.3. *A linear connection $\overline{\nabla}^{ii}_{\mathcal{U}}\mathcal{V} = \overline{\nabla}^g_{\mathcal{U}}\mathcal{V}+\mathcal{S}^{ii}(\mathcal{U},\mathcal{V})$ on an almost complex Norden Silver manifold $(\overline{M}^{2n},\Theta_c,g)$ satisfying the conditions $g\left(\mathcal{S}^{ii}(\mathcal{U},\mathcal{V}),\Theta_c\mathcal{W}\right) = g\left(\mathcal{S}^{ii}(\mathcal{W},\mathcal{V}),\Theta_c\mathcal{U}\right)$ and $\overline{\nabla}^{ii}\tilde{g} = 0$ is said to be a special connection of the second type, where $\mathcal{S}^{ii}$ denotes a tensor field of type $(1,2)$.*

From the definition (5.3.3), we have

$$(\overline{\nabla}^{ii}_{\mathcal{U}}g)(\mathcal{V},\mathcal{W}) = \left(\overline{\nabla}^g_{\mathcal{U}}g\right)(\mathcal{V},\mathcal{W}) - g\left(\mathcal{S}^{ii}(\mathcal{U},\mathcal{V}),\mathcal{W}\right) - g\left(\mathcal{S}^{ii}(\mathcal{U},\mathcal{W}),\mathcal{V}\right).$$

Since $\overline{\nabla}^g$ represents the Levi-Civita connection of g, above equation yields

$$(\overline{\nabla}^{ii}_{\mathcal{U}} g)(\mathcal{V}, \mathcal{W}) = -g\left(\mathcal{S}^{ii}(\mathcal{U}, \mathcal{V}), \mathcal{W}\right) - g\left(\mathcal{S}^{ii}(\mathcal{U}, \mathcal{W}), \mathcal{V}\right). \tag{5.3.4}$$

For a special connection of the second type, by taking the covariant derivative of the twin Norden Silver metric $\tilde{g}$ with respect to $\overline{\nabla}^{ii}$, we obtain

$$2g\left(\Theta_c \mathcal{S}^{ii}(\mathcal{U}, \mathcal{V}), \mathcal{W}\right) = \left(\overline{\nabla}^g_{\mathcal{U}} \tilde{g}\right)(\mathcal{V}, \mathcal{W}) - \left(\overline{\nabla}^g_{\mathcal{V}} \tilde{g}\right)(\mathcal{W}, \mathcal{U}) + \left(\overline{\nabla}^g_{\mathcal{W}} \tilde{g}\right)(\mathcal{U}, \mathcal{V}). \tag{5.3.5}$$

From equation (5.3.2), we get

$$\left(\overline{\nabla}^g_{\mathcal{U}} \tilde{g}\right)(\mathcal{V}, \mathcal{W}) = \left(\overline{\nabla}^g_{\mathcal{U}} g\right)(\Theta_c \mathcal{V}, \mathcal{W}) + g\left(\left(\overline{\nabla}^g_{\mathcal{U}} \Theta_c\right) \mathcal{V}, \mathcal{W}\right).$$

Since $\overline{\nabla}^g$ is the Levi-Civita connection of g, it follows that

$$\left(\overline{\nabla}^g_{\mathcal{U}} \tilde{g}\right)(\mathcal{V}, \mathcal{W}) = g\left(\left(\overline{\nabla}^g_{\mathcal{U}} \Theta_c\right) \mathcal{V}, \mathcal{W}\right).$$

Using above equation in equation (5.3.5), so we have

$$2g\left(\Theta_c \mathcal{S}^{ii}(\mathcal{U}, \mathcal{V}), \mathcal{W}\right) = g\left(\left(\overline{\nabla}^g_{\mathcal{U}} \Theta_c\right) \mathcal{V}, \mathcal{W}\right) - g\left(\left(\overline{\nabla}^g_{\mathcal{V}} \Theta_c\right) \mathcal{W}, \mathcal{U}\right) + g\left(\left(\overline{\nabla}^g_{\mathcal{W}} \Theta_c\right) \mathcal{U}, \mathcal{V}\right). \tag{5.3.6}$$

Suppose $(\overline{M}^{2n}, \Theta_c, g)$ represents the Kaehler-Norden Silver manifold. Now, making use of $\overline{\nabla}^g \Theta_c = 0$ in equations (5.3.4) and (5.3.6), we get

$$\overline{\nabla}^{ii} g = 0 \text{ and } \mathcal{S}^{ii}(\mathcal{U}, \mathcal{V}) = 0.$$

Therefore from Theorem (5.3.3) and the Definition (5.3.3), we have

Theorem 5.3.5. *Let $(\overline{M}^{2n}, \Theta_c, g)$ be a Kaehler-Norden Silver manifold. Then the special connection of the second type $\overline{\nabla}^{ii}$ is an adapted connection to the almost complex Norden Silver structure (Θ_c, g).*

Suppose $\tilde{g}$ is a twin Norden Silver metric and $\overline{\nabla}^{\tilde{g}}$ denote the Levi-Civita connection of $\tilde{g}$. Let $\overline{\nabla}^{\tilde{\alpha}}$ is an adapted connection to (Θ_c, g) and then $\tilde{\mathcal{S}}$ is the potential tensor of $\overline{\nabla}^{\tilde{\alpha}}$ associated with $\overline{\nabla}^{\tilde{g}}$ given by

$$\tilde{\mathcal{S}}(\mathcal{U}, \mathcal{V}) = \overline{\nabla}^{\tilde{\alpha}}_{\mathcal{U}} \mathcal{V} - \overline{\nabla}^{\tilde{g}}_{\mathcal{U}} \mathcal{V},$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$ and $\tilde{\mathcal{S}} \in \Im_2^1(\overline{M})$.

If $\overline{\nabla}^{\tilde{\alpha}}$ be the adapted connection to the almost complex Norden Silver structure (Θ_c, g) of $\overline{M}$, then by the Theorem (5.3.3), we have $\overline{\nabla}^{\tilde{\alpha}}\Theta_c = 0$ and $\overline{\nabla}^{\tilde{\alpha}}\tilde{g} = 0$. Now, analogous to the Theorem (5.3.2), the following result is given as

Theorem 5.3.6. *Let $(\overline{M}^{2n}, \Theta_c, g)$ be an almost complex Norden Silver manifold. Then the set of linear connections adapted to the (Θ_c, g) is*

$$\left\{ \overline{\nabla}^{\tilde{g}} + \tilde{\mathcal{S}} : \begin{array}{c} (\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{V} = \Theta_c \tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}) - \tilde{\mathcal{S}}\left(\mathcal{U},\Theta_c\mathcal{V}\right), \\ g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\Theta_c\mathcal{W}) + g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{W}),\Theta_c\mathcal{V}) = 0, \end{array} \right\}$$

where $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are vector fields on $\overline{M}$.

Now, operating the ϕ_{Θ_c}-operator to $\tilde{g}$ and making use of equation (5.2.7), we get

$$\begin{aligned} (\phi_{\Theta_c}\tilde{g})\left(\mathcal{U},\mathcal{V},\mathcal{W}\right) &= \left(\Theta_c\mathcal{U}\right)\left(\tilde{g}(\mathcal{V},\mathcal{W})\right) - \mathcal{U}\left(\tilde{g}\left(\Theta_c\mathcal{V},\mathcal{W}\right)\right) + \tilde{g}\left(\left(L_{\mathcal{V}}\Theta_c\right)\mathcal{U},\mathcal{W}\right) \\ &\quad + \tilde{g}\left(\mathcal{V},\left(L_{\mathcal{W}}\Theta_c\right)\mathcal{U}\right) \\ &= \left(L_{\Theta_c\mathcal{U}}\tilde{g} - L_{\mathcal{U}}\left(\tilde{g}\circ\Theta_c\right)\right)\left(\mathcal{V},\mathcal{W}\right) + \tilde{g}\left(\mathcal{V},\Theta_c L_{\mathcal{U}}\mathcal{W}\right) \\ &\quad - \tilde{g}\left(\Theta_c\mathcal{V}, L_{\mathcal{U}}\mathcal{W}\right) \\ &= \left(\phi_{\Theta_c}g\right)\left(\mathcal{U},\Theta_c\mathcal{V},\mathcal{W}\right) + g\left(N_{\Theta_c}\left(\mathcal{U},\mathcal{V}\right),\mathcal{W}\right). \end{aligned} \tag{5.3.7}$$

Now, we have the following theorem.

Theorem 5.3.7. *Suppose $(\overline{M}^{2n}, \Theta_c, g)$ is an almost complex Norden Silver manifold and $\overline{\nabla}^{\tilde{g}}$ denotes the Levi-Civita connection of twin Norden Silver metric $\tilde{g}$. Then*

(i) if $\phi_{\Theta_c}\tilde{g} = 0$, then Θ_c is integrable,

(ii) the equality $\phi_{\Theta_c}\tilde{g} = 0$ is equivalent to $\overline{\nabla}^{\tilde{g}}\Theta_c = 0$.

Therefore from the second condition of the Theorem (5.3.7) and equation (5.3.7), we infer that; if $\overline{\nabla}^{\tilde{g}}\Theta_c = 0$, then an almost complex Norden Silver manifold $(\overline{M}^{2n}, \Theta_c, g)$ becomes a Kaehler-Norden Silver manifold.

Theorem 5.3.8. *Let $(\overline{M}^{2n}, \Theta_c, g)$ be an almost complex Norden Silver manifold. The linear connection $\overline{\nabla}^{\tilde{\alpha}}_{\mathcal{U}}\mathcal{V} = \tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}) + \overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\mathcal{V}$ on $\overline{M}$ satisfying $g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\mathcal{W}) = g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{W}),\mathcal{V})$ and $\overline{\nabla}^{\tilde{\alpha}} g = 0$. Then, $\overline{\nabla}^{\tilde{\alpha}}$ is not an adapted connection on the almost complex Norden Silver structure (Θ_c, g). Furthermore, $\overline{\nabla}^{\tilde{\alpha}}$ be an adapted connection on the almost complex Norden Silver structure if $(\overline{M}^{2n}, \Theta_c, g)$ is a Kaehler-Norden Silver manifold.*

Proof. *Suppose $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be any vector fields on $\overline{M}$, then by the definition of $\overline{\nabla}^{\tilde{\alpha}}$, we get*

$$(\overline{\nabla}^{\tilde{\alpha}}_{\mathcal{U}}\tilde{g})(\mathcal{V},\mathcal{W}) = (\overline{\nabla}^{\tilde{g}}\tilde{g})(\mathcal{V},\mathcal{W}) - \tilde{g}(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\mathcal{W}) - \tilde{g}(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{W}),\mathcal{V}),$$

Since $\overline{\nabla}^{\tilde{g}}$ is the Levi-Civita connection of $\tilde{g}$, above equation yields

$$\begin{aligned}(\overline{\nabla}^{\tilde{\alpha}}_{\mathcal{U}}\tilde{g})(\mathcal{V},\mathcal{W}) &= -\tilde{g}(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\mathcal{W}) - \tilde{g}(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{W}),\mathcal{V}) \\ &= -g(\Theta_c\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\mathcal{W}) - g(\Theta_c\tilde{\mathcal{S}}(\mathcal{U},\mathcal{W}),\mathcal{V}). \end{aligned} \tag{5.3.8}$$

Taking covariant derivative of the metric g corresponding to $\overline{\nabla}^{\tilde{\alpha}}$, we obtain

$$(\overline{\nabla}^{\tilde{\alpha}}_{\mathcal{U}}g)(\mathcal{V},\mathcal{W}) = (\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}g)(\mathcal{V},\mathcal{W}) - g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\mathcal{W}) - g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{W}),\mathcal{V}). \tag{5.3.9}$$

By the assumptions in theorem, above equation becomes

$$(\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}g)(\mathcal{V},\mathcal{W}) = 2g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\mathcal{W}),$$

Now replacing $\mathcal{W}$ by $\Theta_c\mathcal{W}$ in above equation, yields

$$(\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}g)(\mathcal{V},\Theta_c\mathcal{W}) = 2g(\tilde{\mathcal{S}}(\mathcal{U},\mathcal{V}),\Theta_c\mathcal{W}). \tag{5.3.10}$$

By definition of $\tilde{g}$ and equation (5.3.2), we have

$$(\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\tilde{g})(\mathcal{V},\mathcal{W}) = (\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}g)(\mathcal{V},\Theta_c\mathcal{W}) + g((\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{V},\mathcal{W}),$$

Since $\overline{\nabla}^{\tilde{g}}$ be the Levi-Civita connection of $\tilde{g}$, we get

$$(\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}g)(\mathcal{V},\Theta_c\mathcal{W}) = -g((\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{V},\mathcal{W}),$$

Using above equality in (5.3.10), *we have*

$$-g((\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{V},\mathcal{W}) = 2g(\Theta_c\tilde{\mathcal{S}}\left(\mathcal{U},\mathcal{V}\right),\mathcal{W}),$$

or,

$$\Theta_c\tilde{\mathcal{S}}\left(\mathcal{U},\mathcal{V}\right) = -\frac{1}{2}(\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{V}. \tag{5.3.11}$$

Now, putting equation (5.3.11) *in equation* (5.3.8), *we have*

$$\begin{aligned}(\overline{\nabla}^{\tilde{\alpha}}_{\mathcal{U}}\tilde{g})\left(\mathcal{V},\mathcal{W}\right) &= \frac{1}{2}g((\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{V},\mathcal{W}) + \frac{1}{2}g((\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{W},\mathcal{V})\\ &= g((\overline{\nabla}^{\tilde{g}}_{\mathcal{U}}\Theta_c)\mathcal{V},\mathcal{W}) \neq 0. \end{aligned} \tag{5.3.12}$$

Hence, the result follows.

Kumar *et al.* [89] established a special connection of the type three for an almost complex Norden Golden manifold. Analogously, we introduce a special connection of the type three for an almost complex Norden Silver manifold as given below:

Definition 5.3.4. *Suppose* $\overline{\nabla}^g$ *denotes the Levi-Civita connection of* g *and* $(\overline{M}^{2n},\Theta_c,g)$ *is an almost complex Norden Silver manifold. Then, we define the special connection of the type three* $\overline{\nabla}^{iii}$ *of* $(\overline{M}^{2n},\Theta_c,g)$ *by the following relation*

$$\overline{\nabla}^{iii}_{\mathcal{U}}\mathcal{V} = \overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V} + \frac{1}{2}\left(\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\right)J^{\Theta_c}\mathcal{V}, \tag{5.3.13}$$

where $\mathcal{U}$, $\mathcal{V}$ *are any vector fields on* $\overline{M}$ *and* J^{Θ_c} *denote the almost complex structure induced from the Silver structure* Θ_c.

Theorem 5.3.9. *Let* $(\overline{M}^{2n},\Theta_c,g)$ *is an almost complex Norden Silver manifold. Then the special connection of the type three* $\overline{\nabla}^{iii}$ *is an adapted connection to the almost complex Norden Silver structure* (Θ_c,g).

Proof. *Let* $\mathcal{U}$, $\mathcal{V}$ *and* $\mathcal{W}$ *be any vector fields on* $\overline{M}$, *by using* (5.3.13), *we get*

$$\overline{\nabla}^{iii}_{\mathcal{U}}J^{\Theta_c}\mathcal{V} = \frac{1}{2}\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\mathcal{V} + \frac{1}{2}J^{\Theta_c}\overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V}, \tag{5.3.14}$$

and

$$\begin{aligned}J^{\Theta_c}\overline{\nabla}^{iii}_{\mathcal{U}}\mathcal{V} &= J^{\Theta_c}\overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V} + \frac{1}{2}J^{\Theta_c}\left[-\overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V} - J^{\Theta_c}\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\mathcal{V}\right]\\ &= \frac{1}{2}\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\mathcal{V} + \frac{1}{2}J^{\Theta_c}\overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V}. \end{aligned} \tag{5.3.15}$$

From (5.3.14) *and* (5.3.15)*, we have* $\overline{\nabla}^{iii}J^{\Theta_c}=0.$

By using the Theorem (5.3.1)*, we obtain*

$$\overline{\nabla}^{iii}\Theta_c=0.$$

Now, from (5.3.13)*, we have*

$$\left(\overline{\nabla}^{iii}_{\mathcal{U}}g\right)(\mathcal{V},\mathcal{W})=\left(\overline{\nabla}^{g}_{\mathcal{U}}g\right)(\mathcal{V},\mathcal{W})-\frac{1}{2}\left[g\left(\left(\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\right)J^{\Theta_c}\mathcal{V},\mathcal{W}\right)+g\left(\left(\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\right)J^{\Theta_c}\mathcal{W},\mathcal{V}\right)\right].$$

As we know, $\overline{\nabla}^{g}$ *be the Levi-Civita connection of* g*, above equation yields*

$$\begin{aligned}\left(\overline{\nabla}^{iii}_{\mathcal{U}}g\right)(\mathcal{V},\mathcal{W})&=\frac{1}{2}\Big[g\left(\overline{\nabla}^{g}_{\mathcal{U}}\mathcal{V},\mathcal{W}\right)+g\left(\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\mathcal{V},J^{\Theta_c}\mathcal{W}\right)+g\left(\overline{\nabla}^{g}_{\mathcal{U}}\mathcal{W},\mathcal{V}\right)\\&\quad+g\left(\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c}\mathcal{W},J^{\Theta_c}\mathcal{V}\right)\Big]\\&=\frac{1}{2}\left[\mathcal{U}g\left(\mathcal{V},\mathcal{W}\right)+\mathcal{U}g\left(J^{\Theta_c}\mathcal{V},J^{\Theta_c}\mathcal{W}\right)\right]\\&=0.\end{aligned}$$

Hence, the Proof.

As we know, if the Nijenhuis tensor $N_{J^{\Theta_c}}$ vanishes, then the almost complex structure J^{Θ_c} is integrable and is given by

$$N_{J^{\Theta_c}}(\mathcal{U},\mathcal{V})=[J^{\Theta_c}\mathcal{U},J^{\Theta_c}\mathcal{V}]-J^{\Theta_c}[J^{\Theta_c}\mathcal{U},\mathcal{V}]-J^{\Theta_c}[\mathcal{U},J^{\Theta_c}\mathcal{V}]+\left(J^{\Theta_c}\right)^2[\mathcal{U},\mathcal{V}], \quad (5.3.16)$$

where $\mathcal{U}$ and $\mathcal{V}$ are any vector fields on $\overline{M}$. It is well-known that Nijenhuis tensor $N_{J^{\Theta_c}}$ is also given by

$$N_{J^{\Theta_c}}(\mathcal{U},\mathcal{V})=(\overline{\nabla}^{g}_{\mathcal{U}}J^{\Theta_c})J^{\Theta_c}\mathcal{V}+(\overline{\nabla}^{g}_{J^{\Theta_c}\mathcal{U}}J^{\Theta_c})\mathcal{V}-(\overline{\nabla}^{g}_{\mathcal{V}}J^{\Theta_c})J^{\Theta_c}\mathcal{U}-(\overline{\nabla}^{g}_{J^{\Theta_c}\mathcal{V}}J^{\Theta_c})\mathcal{U}. \quad (5.3.17)$$

Theorem 5.3.10. *Let* $N_{J^{\Theta_c}}$ *and* N_{Θ_c} *are the Nijenhuis tensors of* J^{Θ_c} *and* Θ_c*, respectively. Suppose* $(\overline{M}^{2n},\Theta_c,g)$ *is an almost complex Norden Silver manifold, then the following relation holds*

$$N_{J^{\Theta_c}}=\frac{1}{2}N_{\Theta_c}, \quad (5.3.18)$$

where $\mathcal{U}$ *and* $\mathcal{V}$ *are any vector fields on* $\overline{M}$*.*

Proof. *Using* (5.2.3) *in equation* (5.3.16) *and then making use of* (5.2.4)*, the result follows.*

Suppose T^{iii} be the torsion tensor of the special connection of the type three $\overline{\nabla}^{iii}$ on $(\overline{M}^{2n}, \Theta_c, g)$, then using equation (5.3.13), we get

$$T^{iii}\left(\mathcal{U}, \mathcal{V}\right) = \frac{1}{2}\left(\overline{\nabla}^{g}_{\mathcal{U}} J^{\Theta_c}\right) J^{\Theta_c}\mathcal{V} - \frac{1}{2}\left(\overline{\nabla}^{g}_{\mathcal{V}} J^{\Theta_c}\right) J^{\Theta_c}\mathcal{U}, \tag{5.3.19}$$

and

$$T^{iii}\left(J^{\Theta_c}\mathcal{U}, J^{\Theta_c}\mathcal{V}\right) = -\frac{1}{2}\left(\overline{\nabla}^{g}_{J^{\Theta_c}\mathcal{U}} J^{\Theta_c}\right)\mathcal{V} + \frac{1}{2}(\overline{\nabla}^{g}_{J^{\Theta_c}\mathcal{V}} J^{\Theta_c})\mathcal{U}. \tag{5.3.20}$$

Subtracting equation (5.3.20) from equation (5.3.19) and using (5.3.17), we obtain

$$T^{iii}\left(\mathcal{U}, \mathcal{V}\right) - T^{iii}\left(J^{\Theta_c}\mathcal{U}, J^{\Theta_c}\mathcal{V}\right) = \frac{1}{2} N_{J^{\Theta_c}}\left(\mathcal{U}, \mathcal{V}\right). \tag{5.3.21}$$

Therefore from equation (5.3.18) and (5.3.21), we have the following result.

Theorem 5.3.11. *Suppose* $(\overline{M}^{2n}, \Theta_c, g)$ *is an almost complex Norden Silver manifold. Then* Θ_c *is integrable if and only if*

$$T^{iii}\left(\mathcal{U}, \mathcal{V}\right) = T^{iii}\left(J^{\Theta_c}\mathcal{U}, J^{\Theta_c}\mathcal{V}\right),$$

where $\mathcal{U}$ *and* $\mathcal{V}$ *are any vector fields on* $\overline{M}$.

5.4 Complex Silver Maps

Suppose $\theta : \overline{M} \to \overline{N}$ represents the smooth mapping from $\overline{M}$ to $\overline{N}$, and $(\overline{M}, g)$, $(\overline{N}, g')$ be two pseudo-Riemannian manifolds. Then the differential $d\theta$ of θ is a section of the bundle $Hom(T\overline{M}, \theta^{-1}T\overline{N}) \to \overline{M}$, where $\theta^{-1}T\overline{N}$ is the pullback bundle with fibres $\left(\theta^{-1}T\overline{N}\right)_u = T_{\theta(u)}\overline{N},\ u \in \overline{M}$.

The second fundamental form of θ is given by

$$\overline{\nabla} d\theta(\mathcal{U}, \mathcal{V}) = \overline{\nabla}^{\theta}_{\mathcal{U}}\, d\theta(\mathcal{V}) - d\theta(\overline{\nabla}^{M}_{\mathcal{U}}\mathcal{V}), \tag{5.4.1}$$

for any vector fields $\mathcal{U}$ and $\mathcal{V}$ on $\overline{M}$. It is also known that the second fundamental form is symmetric. Here $\overline{\nabla}^{\overline{M}}$ be the Levi-Civita connection on $T\overline{M}$, $\overline{\nabla}^{\theta}$ denotes the pull-back of the Levi-Civita connection on $\overline{N}$ to the bundle $\theta^{-1}T\overline{N}$, and $\overline{\nabla}$ be the connection on $Hom(T\overline{M}, \theta^{-1}T\overline{N})$ induced from these connections.

If $trace\overline{\nabla}d\theta = 0$, then the mapping θ is said to be *harmonic*. Let $\tau(\theta)$ be the $tension\ field$ of θ and is defined by

$$\tau(\theta) = trace\overline{\nabla}\, d\theta = \sum_{i=1}^{m} \overline{\nabla} d\theta(e_i, e_i), \tag{5.4.2}$$

where $e_1, e_2, \ldots, e_m$ be the local orthonormal frame on $\overline{M}$ and $\tau(\theta) = 0$ if and only if θ is harmonic [80].

Definition 5.4.1. *Suppose $\theta : (\overline{M}, \Theta_c, g) \to (\overline{N}, \Theta'_c, g')$ is a smooth map from a complex Norden Silver manifold $(\overline{M}, \Theta_c, g)$ to a complex Norden Silver manifold $(\overline{N}, \Theta'_c, g')$. Then, θ is said to be a complex Norden Silver map if*

$$d\theta\, \Theta_c = \Theta'_c\, d\theta. \tag{5.4.3}$$

From above definition, we have the following result.

Theorem 5.4.1. *Let θ is a complex Norden Silver map between Kaehler-Norden Silver manifolds $(\overline{M}, \Theta_c, g)$ and $(\overline{N}, \Theta'_c, g')$. Then θ is a harmonic map.*

Proof. *Let $\mathcal{U}, \mathcal{V} \in \Gamma(T\overline{M})$ then by (5.2.1), (5.4.1) and (5.4.3), we have*

$$\left(\overline{\nabla} d\theta\right)(\mathcal{U}, \Theta_c\mathcal{V}) = \overline{\nabla}^{\theta}_{\mathcal{U}}\Theta'^{2}_c d\theta\,(\mathcal{V}) - 2d\theta(\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\Theta_c\mathcal{V}) + 3\overline{\nabla}^{\theta}_{\mathcal{U}} d\theta\,(\mathcal{V}),$$

Now, using (5.2.1) in above equation, yields

$$\left(\overline{\nabla} d\theta\right)(\mathcal{U}, \Theta_c\mathcal{V}) = 2\overline{\nabla}^{\theta}_{\mathcal{U}}\Theta'_c d\theta\,(\mathcal{V}) - 2d\theta(\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\Theta_c\mathcal{V}). \tag{5.4.4}$$

Since, $(\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\Theta_c)\mathcal{V} = 0$ for a Kaehler-Norden Silver manifold $(\overline{M}, \Theta_c, g)$. i.e.

$$\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\Theta_c\mathcal{V} = \Theta_c\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\mathcal{V}.$$

Analogously, for a Kaehler-Norden Silver manifold $(\overline{N}, \Theta'_c, g')$*, it follows that*

$$\overline{\nabla}^{\theta}_{\mathcal{U}}\Theta'_c\mathcal{V} = \Theta'_c\overline{\nabla}^{\theta}_{\mathcal{U}}\mathcal{V}.$$

Substituting (5.4.3) *in* (5.4.4) *and using* $\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\Theta_c\mathcal{V} = \Theta_c\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\mathcal{V}$ *and* $\overline{\nabla}^{\theta}_{\mathcal{U}}\Theta'_c\mathcal{V} = \Theta'_c\overline{\nabla}^{\theta}_{\mathcal{U}}\mathcal{V}$*, we get*

$$\left(\overline{\nabla}d\theta\right)(\mathcal{U}, \Theta_c\mathcal{V}) = 2\Theta'_c(\overline{\nabla}^{\theta}_{\mathcal{U}}d\theta(\mathcal{V}) - d\theta(\overline{\nabla}^{\overline{M}}_{\mathcal{U}}\mathcal{V})) = 2\Theta'_c(\overline{\nabla}d\theta)(\mathcal{U}, \mathcal{V}). \tag{5.4.5}$$

By the symmetry of the second fundamental form, we obtain

$$(\overline{\nabla}d\theta)(\mathcal{U}, \Theta_c\mathcal{V}) = (\overline{\nabla}d\theta)(\Theta_c\mathcal{U}, \mathcal{V}). \tag{5.4.6}$$

From (5.4.5) *and* (5.4.6)*, we get*

$$\left(\overline{\nabla}d\theta\right)(\Theta_c\mathcal{U}, \Theta_c\mathcal{V}) = 2\Theta'^2_c(\overline{\nabla}d\theta)(\mathcal{U}, \mathcal{V}). \tag{5.4.7}$$

Suppose $\{e_1, e_2, \ldots, e_{2n}\}$ *is an orthonormal basis of* $T_u\overline{M}$*,* $u \in \overline{M}$*, then by the Theorem* (5.2.4)*,* Θ_c *is an isomorphism. Then* $\{\Theta_c e_1, \Theta_c e_2, \ldots, \Theta_c e_{2n}\}$ *is also an orthonormal basis of* $T_u\overline{M}$*. From* (5.2.1) *and* (5.4.7)*, it follows that*

$$\sum_{i=1}^{2n}\left(\overline{\nabla}d\theta\right)(\Theta_c e_i, \Theta_c e_i) = 2\left[2\sum_{i=1}^{2n}\Theta'_c(\overline{\nabla}d\theta)(e_i, e_i) - 3\sum_{i=1}^{2n}(\overline{\nabla}d\theta)(e_i, e_i)\right],$$

a simple calculation gives

$$\tau(\theta) = 4\ \Theta'_c\,\tau(\theta) - 6\,\tau(\theta),$$

or

$$\Theta'_c\,\tau(\theta) = \frac{7}{4}\tau(\theta). \tag{5.4.8}$$

Now, operating Θ'_c *on equation* (5.4.8) *and making use of equation* (5.2.1)*, we get*

$$\Theta'_c\tau(\theta) = \frac{97}{32}\tau(\theta). \tag{5.4.9}$$

Subtracting (5.4.8) *from* (5.4.9)*, we have* $\tau(\theta) = 0$*. Hence the assertion follows.*

5.5 Conclusion

The purpose of this chapter is to explore Kaehler-Norden Silver manifolds as well as almost complex Norden Silver manifolds. We define the adapted connections of first type, second type, and third type to an almost complex Norden Silver manifold. Further, the necessary and sufficient condition for the integrability have been established for an almost complex Norden Silver structure. Moreover, it has been accomplished that a complex Norden Silver map is a harmonic map between Kaehler-Norden Silver manifolds. Analogous to this work, we can explore the adapted connections on Kaehler-Norden Bronze manifolds.

Chapter 6

Differential Equations for Indicatrices, spacelike and timelike curves

6.1 Introduction

In differential geometry, the curve is among one of the fascinating topics. Helices, spherical curves, and rectifying curves are a few important types of curves that appear in many important applications. For example, helical structures arise in seashells, vines, carbon nanotubes, DNA double, and nano-springs etc. Though many authors [13, 14, 21, 58, 59, 85, 95, 117, 74] studied curves from the last several decades nevertheless curves are still a relevant and significant area of the research. In the study of curves, the notion of associated curves is pretty exciting. If there exist a mathematical relation between two or more curves, then the curves are known as associated curves.

Izumiya *et al.* [100] introduced some special curves which are known as a slant helix and conical geodesic curves in Euclidean 3-space. Besides, Izumiya and Takeuchi gave some classifications of the special developable surfaces and obtained an example of a slant helix. In [52], Kula *et al.* studied the spherical images of the tangent indicatrix and binormal indicatrix of a slant helix. Besides, they obtained that the spherical images of the slant helices are spherical helices and a curve of constant precession is a slant helix.

In [12], Ali obtained the position vector of a general helix ($\tau/\kappa = m$) associated with Frenet frame and represented the general helix in terms of curvature (κ) and torsion (τ) through a standard frame of Euclidean 3-space, where m is a constant given by $m = cos[\phi]$, here ϕ denotes the angle between the axis of a general helix and the tangent of the curve. In [13], Ali *et al.* extended the concept of a slant helix to Euclidean space of dimension n, and gave the necessary and sufficient conditions for a curve in Euclidean n-space to be a slant helix. Moreover, Ali also gave an example of a slant helix in Euclidean Euclidean space of 5-dimension.

Recently, Sahiner [14] defined the associated curves as integral curves of a vector field produced by Frenet vectors of the tangent indicatrix of a curve in Euclidean 3-space and obtained some relations between curvatures and Frenet vectors. Besides, he gave a few techniques to obtain helices and slant helices from special spherical curves and constructed some examples of it. In [17], B. Y. Chen investigated the characterization and classification of the rectifying curves. On the other hand in [18], B. Y. Chen studied via rectifying curves that all geodesics on an arbitrary cone in Euclidean space of dimension 3, are not necessarily a circular cone.

Yilmaz *et al.* [19] used the system of linear ordinary differential equations to construct the slant helices. Also, from the integration according to alternative moving frame in Minkowski 3-space, they obtained the position vectors for slant helices. In [21], Camci *et al.* studied and obtained a spherical slant helix and gave some examples of the spherical slant helices in Euclidean 3-space. Arroyo *et al.* [36] investigated the unit speed curves contained in a real space form of arbitrary dimension m. Moreover, they gave a classification of semi-Riemannian Hopf cylinders of $H_1^3(-1)$ and Hopf cylinders of S^3 with proper mean curvature function.

In [37], Choi *et al.* introduced the concept of the principal-direction curve and principal-donor curve of a Frenet curve in Euclidean space of dimension 3. Moreover, they constructed a canonical method for associated curves and characterized some associated curves in Euclidean 3-space. Kula *et al.* [51] obtained a relationship between a slant helix and a general helix. Furthermore, they deduced some differential equations by characterizing of a slant helix and gave a few examples of slant helices in Euclidean space of dimension 3.

In [86], Lucas *et al.* studied a weaker version of the classic slant helices in Minkowski 3-space and Euclidean 3-space which are known as general slant helices. Furthermore, Lucas showed that the classic slant helix is a general helix but the converse is not true. Also, he obtained equations that involve the torsion and curvature.

In [95], Deshmukh *et al.* investigated the rectifying curves via the dilation of the unit speed curve on S^2 (unit sphere) in Euclidean 3-space and obtained a necessary and sufficient condition for centrode of a unit speed curve in Euclidean 3-space. Moreover, Deshmukh and Chen proved that if a unit speed curve is neither a helix nor a planar curve, then its dilated centrode is always a rectifying curve. Deshmukh *et al.* [96] shown that for every Frenet curve in Euclidean 3-space, the distance function satisfies a 4th-order differential equation and using this they derived a new characterization of helices. In [121], Ozdemir *et al.* introduced the notion of type-3 slant helix according to the parallel transport frame in Euclidean 4-space.

6.2 Preliminaries of Frenet Curves

In this section, we recall some basic concepts of the curves and indicatrices in the Euclidean 3-space. Let $\beta : I \to \mathbb{R}^3$ represents the unit speed curve in the Euclidean 3-space and $\mathcal{T}'$, $\mathcal{N}'$, $\mathcal{B}'$ be the three orthonormal vectors of the Frenet frame $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$, given by

$$\mathcal{T} = \frac{d\beta}{ds}, \qquad \mathcal{N} = \frac{\mathcal{T}'}{\kappa}, \qquad \mathcal{B} = \mathcal{T} \times \mathcal{N},$$

where $\mathcal{T}$, $\mathcal{N}$, $\mathcal{B}$ represent the *unit tangent vector field*, *unit principal normal vector field* and *unit binormal vector field*, respectively.

The Serret-Frenet formulas are given by

$$\begin{cases} \mathcal{T}'(s) = \kappa(s)\mathcal{N}(s) \\ \mathcal{N}'(s) = -\kappa(s)\mathcal{T}(s) + \tau(s)\mathcal{B}(s) \\ \mathcal{B}'(s) = -\tau(s)\mathcal{N}(s) \end{cases} \tag{6.2.1}$$

where $\kappa(s) = \|\mathcal{T}'(s)\|$ denote the *curvature* and $\tau(s) = -\langle \mathcal{B}'(s), \mathcal{N}(s)\rangle$ denote the *torsion* of the curve β. Here the curve β is parameterized in terms of the arc-length parameter s [94].

If the position vector of the curve β lies in the rectifying plane then it is known as a *rectifying curve*. The distance function $d(s) = \|\beta(s)\|$ of a rectifying curve β satisfies the following equation

$$d(s) = \sqrt{s^2 + c_1 s + c_2},$$

here c_1 and c_2 denote the arbitrary constants.

Furthermore, it can be shown that the unit speed curve β is also a *rectifying curve* if and only if the ratio of torsion τ and curvature κ satisfies

$$\frac{\tau}{\kappa} = as + b,$$

where $a \neq 0$ and b are constants [17].

Choi and Kim investigated the relationship between curvature and torsion of the principal-direction curve and principal-donor curve in [37].

Theorem 6.2.1. *[37] Let β be a Frenet curve in Euclidean 3-space with the curvature κ and the torsion τ and $\overline{\beta}$ be the principal-direction curve of the curve β. Then the curvature $\overline{\kappa}$ and torsion $\overline{\tau}$ of the principal-direction curve $\overline{\beta}$ are given by*

$$\overline{\kappa} = \sqrt{\kappa^2 + \tau^2} \qquad \text{and} \qquad \overline{\tau} = \frac{\kappa^2}{\kappa^2 + \tau^2}\left(\frac{\tau}{\kappa}\right)'.$$

Theorem 6.2.2. *[37] Let β be a principal-donor curve of the curve $\overline{\beta}$ in Euclidean 3-space with the curvature $\overline{\kappa}$ and torsion $\overline{\tau}$. Then the curvature κ and torsion τ of the principal-donor curve β are given by*

$$\kappa = \overline{\kappa}\ \left|\cos\left(\int \overline{\tau} ds\right)\right| \quad \text{and} \quad \tau = \overline{\kappa}\sin\left(\int \overline{\tau} ds\right).$$

A curve β is said to be *general helix* if unit tangent $\mathcal{T}(s)$ makes a constant angle with a fixed straight line. Likewise, if unit principal normal $\mathcal{N}(s)$ makes a constant angle with a fixed straight line then a curve β is said to be *slant helix*.

Let β be a unit speed curve in Euclidean space with Frenet vectors $\mathcal{T}$, $\mathcal{N}$ and $\mathcal{B}$. The unit tangent vectors along the curve β generate a curve β_t on the unit sphere centered at the origin, called the *tangent indicatrix* of curve β. Similarly, we have the *binormal indicatrix* β_b and *principal normal indicatrix* β_n [51].

Deshmukh and B. Y. Chen shown that for every Frenet curve in Euclidean 3-space, the distance function satisfies a general differential equation. We recall the following proposition from [96].

Proposition 6.1. *[96] If β be a unit speed curve then every unit speed Frenet curve satisfies the following equation:*

$$\rho\sigma h''' + (\rho\sigma' + 2\rho'\sigma)h'' + \left\{(\sigma\rho')' + \frac{\rho}{\sigma} + \frac{\sigma}{\rho}\right\} h' + \left(\frac{\sigma}{\rho}\right)' h = (\sigma\rho')' + \frac{\rho}{\sigma}, \qquad (6.2.2)$$

where $\rho = \kappa^{-1}$, $\sigma = \tau^{-1}$, $h(s) = d(s)d'(s)$.

The Minkowski 3-space $\mathbb{E}^3_1$ is the Euclidean 3-space provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$.

Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}^3_1$ can have one of the three causal characters; it can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and lightlike (null) if $g(v, v) = 0$ and $v \neq 0$. Analogously, an arbitrary curve $\beta = \beta(s)$ in $\mathbb{E}^3_1$ can locally be spacelike, timelike or lightlike (null), if all of its velocity vectors $\beta'(s)$ are respectively spacelike, timelike or lightlike. The norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$ and the spacelike or (timelike) curve $\beta(s)$ is said to be of unit speed if $g(\beta'(s), \beta'(s)) = \pm 1$ [73].

6.3 Derivation of the differential equations

In this section, first we give some propositions for indicatrices, Serret-Frenet formulae, and a few useful results for spacelike and timelike curves. Finally, we obtain the 4th-order differential equations for spacelike and timelike curves.

Proposition 6.2. *If β be a unit speed curve then tangent indicatrix β_t of the curve β does not form a non-trivial differential equation.*

Proof. Since the tangent indicatrix β_t has constant norm equal to one. By differentiating the distance function $d(s) = \|\beta_t(s)\|$, we get $d'(s) = 0$. □

Proposition 6.3. *If β be a unit speed curve then binormal indicatrix β_b of curve β does not form a non-trivial differential equation.*

Proof. Since the binormal indicatrix β_b has constant norm equal to one. By differentiating the distance function $d(s) = \|\beta_b(s)\|$, we get $d'(s) = 0$. □

Proposition 6.4. *If β be a unit speed curve then principal normal indicatrix β_n of curve β does not form a non-trivial differential equation.*

Proof. Since the principal normal indicatrix β_n has constant norm equal to one. By differentiating the distance function $d(s) = \|\beta_n(s)\|$, we get $d'(s) = 0$. □

Remark 6.3.1. *Suppose β denote a spacelike curve with a spacelike principal normal $\mathcal{N}$ and β' be the tangent vector field, then the Serret-Frenet formulae are given by*

$$\begin{cases} \mathcal{T}' = \kappa\mathcal{N} \\ \mathcal{N}' = -\kappa T + \ \tau\mathcal{B} \\ \mathcal{B}' = \tau\mathcal{N} \end{cases} \tag{6.3.1}$$

where $\langle\mathcal{T},\mathcal{T}\rangle = 1$, $\langle\mathcal{N},\mathcal{N}\rangle = 1$, $\langle\mathcal{B},\mathcal{B}\rangle = -1$, $\langle\mathcal{T},\mathcal{N}\rangle = \langle\mathcal{T},\mathcal{B}\rangle = \langle\mathcal{N},\mathcal{B}\rangle = 0$.

From the above formula, we have the following

$$\begin{cases} \langle \beta, \mathcal{T} \rangle' = 1 + \kappa \langle \beta, \mathcal{N} \rangle \\ \langle \beta, \mathcal{N} \rangle' = -\kappa \langle \beta, \mathcal{T} \rangle + \tau \langle \beta, \mathcal{B} \rangle \\ \langle \beta, \mathcal{B} \rangle' = \tau \langle \beta, \mathcal{N} \rangle \end{cases} \tag{6.3.2}$$

Theorem 6.3.1. *Suppose β denote a spacelike curve with a spacelike principal normal $\mathcal{N}$, then the function $f(s) = d(s)d'(s)$ satisfies the following differential equation*

$$\frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} + \frac{\kappa}{\tau} - \frac{\tau}{\kappa}\right] f' + \left[\frac{\kappa}{\tau'} + \frac{\kappa'}{\tau}\right] f = \left[\frac{1}{\tau\kappa'}\right]' - \frac{\tau}{\kappa} \tag{6.3.3}$$

where $d(s) = \|\beta(s)\|$ is the distance function of β.

Proof. Differentiating $d^2(s) = \langle \beta(s), \beta(s) \rangle$ and making use of equation (6.3.1), we get

$$f = \langle \beta, \mathcal{T} \rangle \tag{6.3.4}$$

Now, differentiating above equation and using (6.3.2), we get

$$f' - 1 = \kappa \langle \beta, \mathcal{N} \rangle \tag{6.3.5}$$

Further, differentiating equation (6.3.5), yields

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' + \frac{\kappa}{\tau} f - \frac{1}{\tau\kappa'} = \langle \beta, \mathcal{B} \rangle \tag{6.3.6}$$

Now, differentiating equation (6.3.6) and using (6.3.2), (6.3.5), we get the desired result. □

Remark 6.3.2. *Suppose β denote a spacelike curve with a timelike principal normal $\mathcal{N}$ and β' be the tangent vector field, then the Serret-Frenet formulae are given by*

$$\begin{cases} \mathcal{T}' = \kappa \mathcal{N} \\ \mathcal{N}' = \kappa \mathcal{T} + \tau \mathcal{B} \\ \mathcal{B}' = \tau \mathcal{N} \end{cases} \tag{6.3.7}$$

where $\langle \mathcal{T}, \mathcal{T} \rangle = 1$, $\langle \mathcal{N}, \mathcal{N} \rangle = -1$, $\langle \mathcal{B}, \mathcal{B} \rangle = 1$, $\langle \mathcal{T}, \mathcal{N} \rangle = \langle \mathcal{T}, \mathcal{B} \rangle = \langle \mathcal{N}, \mathcal{B} \rangle = 0$.

From the above equation, we get the following

$$\begin{cases} \langle \beta, \mathcal{T} \rangle' = 1 + \kappa \langle \beta, \mathcal{N} \rangle \\ \langle \beta, \mathcal{N} \rangle' = \kappa \langle \beta, \mathcal{T} \rangle + \tau \langle \beta, \mathcal{B} \rangle \\ \langle \beta, \mathcal{B} \rangle' = \tau \langle \beta, \mathcal{N} \rangle \end{cases} \tag{6.3.8}$$

Theorem 6.3.2. *Suppose β denote a spacelike curve with a timelike principal normal $\mathcal{N}$, then the function $f(s) = d(s)d'(s)$ satisfies the following differential equation*

$$\frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} - \frac{\kappa}{\tau} - \frac{\tau}{\kappa}\right] f' - \left[\frac{\kappa}{\tau'} + \frac{\kappa'}{\tau}\right] f = \left[\frac{1}{\tau\kappa'}\right]' - \frac{\tau}{\kappa} \tag{6.3.9}$$

where $d(s) = \|\beta(s)\|$ is the distance function of β.

Proof. Differentiating $d^2(s) = \langle \beta(s), \beta(s) \rangle$ and making use of equation (6.3.7), we get

$$f = \langle \beta, \mathcal{T} \rangle \tag{6.3.10}$$

Using (6.3.10) and (6.3.8), a simple computation gives

$$f' - 1 = \kappa \langle \beta, \mathcal{N} \rangle \tag{6.3.11}$$

Now, differentiating (6.3.11), we get

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' - \frac{\kappa}{\tau} f - \frac{1}{\tau\kappa'} = \langle \beta, \mathcal{B} \rangle \tag{6.3.12}$$

Finally, differentiating equation (6.3.12) and using (6.3.8), (6.3.11), we get the desired result. □

Remark 6.3.3. *Suppose β denote a spacelike curve with a lightlike principal normal $\mathcal{N}$ and β' be the tangent vector field, then the Serret-Frenet formulae are given by*

$$\begin{cases} \mathcal{T}' = \kappa \mathcal{N} \\ \mathcal{N}' = \tau \mathcal{N} \\ \mathcal{B}' = -\kappa \mathcal{T} - \tau \mathcal{B} \end{cases} \tag{6.3.13}$$

where $\langle \mathcal{T}, \mathcal{T} \rangle = 1, \langle \mathcal{N}, \mathcal{B} \rangle = 1, \langle \mathcal{N}, \mathcal{N} \rangle = \langle \mathcal{B}, \mathcal{B} \rangle = \langle \mathcal{T}, \mathcal{N} \rangle = \langle \mathcal{T}, \mathcal{B} \rangle = 0.$

From the above formula, we have the following

$$\begin{cases} \langle \beta, \mathcal{T} \rangle' = 1 + \kappa \langle \beta, \mathcal{N} \rangle \\ \langle \beta, \mathcal{N} \rangle' = \tau \langle \beta, \mathcal{N} \rangle \\ \langle \beta, \mathcal{B} \rangle' = -\kappa \langle \beta, \mathcal{T} \rangle - \tau \langle \beta, \mathcal{B} \rangle \end{cases} \tag{6.3.14}$$

Theorem 6.3.3. *Suppose β denote a spacelike curve with a lightlike principal normal $\mathcal{N}$, then the function $f(s)$ satisfies the following differential equation*

$$\frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'} - \frac{1}{\kappa}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} - \frac{1}{\kappa'}\right] f' = \left[\frac{1}{\tau\kappa'}\right]' - \frac{1}{\kappa'} \tag{6.3.15}$$

where $f(s) = d(s)d'(s)$, and $d(s) = \|\beta(s)\|$ is the distance function of the curve β.

Proof. Differentiating $d(s) = \|\beta(s)\|$ and using equation (6.3.13), we get

$$f = \langle \beta, \mathcal{T} \rangle \tag{6.3.16}$$

From equations (6.3.16) and (6.3.14), we have

$$f' - 1 = \kappa \langle \beta, \mathcal{N} \rangle \tag{6.3.17}$$

Now, differentiating (6.3.17), yields

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' - \frac{1}{\tau\kappa'} = \langle \beta, \mathcal{N} \rangle \tag{6.3.18}$$

Differentiating equation (6.3.18) and using (6.3.16), we get the result. □

Remark 6.3.4. *Suppose β denote a timelike curve and β' be the tangent vector field, then the Serret-Frenet formulae are given by*

$$\begin{cases} \mathcal{T}' = \kappa \mathcal{N} \\ \mathcal{N}' = \kappa \mathcal{T} + \tau \mathcal{B} \\ \mathcal{B}' = -\tau \mathcal{N} \end{cases} \tag{6.3.19}$$

where $\langle \mathcal{T}, \mathcal{T} \rangle = -1$, $\langle \mathcal{N}, \mathcal{N} \rangle = 1$, $\langle \mathcal{B}, \mathcal{B} \rangle = 1$, $\langle \mathcal{T}, \mathcal{N} \rangle = \langle \mathcal{T}, \mathcal{B} \rangle = \langle \mathcal{N}, \mathcal{B} \rangle = 0$.

From the above formula, we have the following

$$\begin{cases} \langle \beta, \mathcal{T} \rangle' = -1 + \kappa \langle \beta, \mathcal{N} \rangle \\ \langle \beta, \mathcal{N} \rangle' = \kappa \langle \beta, \mathcal{T} \rangle + \tau \langle \beta, \mathcal{B} \rangle \\ \langle \beta, \mathcal{B} \rangle' = -\tau \langle \beta, \mathcal{N} \rangle \end{cases} \tag{6.3.20}$$

Theorem 6.3.4. *Suppose β denote a timelike curve, then the function $f(s) = d(s)d'(s)$ satisfies the following differential equation*

$$\frac{f'''}{\tau\kappa} + \left[\frac{1}{\tau'\kappa} + \frac{2}{\tau\kappa'}\right] f'' + \left[\frac{1}{\tau'\kappa'} + \frac{1}{\tau\kappa''} - \frac{\kappa}{\tau} + \frac{\tau}{\kappa}\right] f' - \left[\frac{\kappa}{\tau'} + \frac{\kappa'}{\tau}\right] f = -\left[\frac{1}{\tau\kappa'}\right]' - \frac{\tau}{\kappa} \tag{6.3.21}$$

where $d(s) = \|\beta(s)\|$ is the distance function of β.

Proof. Differentiating $d^2(s) = \langle \beta(s), \beta(s) \rangle$ and making use of equation (6.3.19), we get

$$f = \langle \beta, \mathcal{T} \rangle \tag{6.3.22}$$

Now, differentiating above equation and using (6.3.20), we get

$$f' + 1 = \kappa \langle \beta, \mathcal{N} \rangle \tag{6.3.23}$$

Further, differentiating equation (6.3.23), yields

$$\frac{1}{\tau\kappa} f'' + \frac{1}{\tau\kappa'} f' - \frac{\kappa}{\tau} f + \frac{1}{\tau\kappa'} = \langle \beta, \mathcal{B} \rangle \tag{6.3.24}$$

Now, differentiating equation (6.3.24) and using (6.3.20), (6.3.23), the result follows. □

6.4 Conclusion

In this chapter, we have investigated the distance function which satisfies the 4th-order differential equation of the Frenet curve in Euclidean 3-space. We have shown that Tangent, Binormal, and Principal Normal indicatrices do not form non-trivial differential equations. Finally, we obtain the 4th-order differential equations for spacelike and timelike curves.

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