GENERALIZATION OF VARMA AND TSALLIS ENTROPIES AND THEIR PROPERTIES

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DECLARATION BY THE SCHOLAR

I hereby declare that the work reported in the Ph.D. thesis entitled, **"GENERALIZATION OF VARMA AND TSALLIS ENTROPIES AND THEIR PROPERTIES"** submitted at **Jaypee University of Information Technology, Waknaghat, India**, is an authentic record of my work carried out under the supervision of **Prof. Harinder Singh and Dr. Nitin Gupta**. I have not submitted this work elsewhere for any other degree or diploma.

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CERTIFICATE

This is to certify that the thesis entitled, **"GENERALIZATION OF VARMA AND TSALLIS ENTROPIES AND THEIR PROPERTIES"** which is being submitted by **Madan Mohan Sati** in fulfillment for the award of degree of **Doctor of Philosophy** in **Mathematics** by the **Jaypee University of Information Technology, Waknaghat, India** is the record of candidate's own work carried by him under our supervision. This work has not submitted partially or wholly to any other University or Institute for the award of this or any other degree or diploma.

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Abstract

Information Theory was introduced by C. E. Shannon (1948) in his landmark paper "A Mathematical Theory of Communication". It is a branch of mathematics and computer science which studies the quantification of information. It was introduced to study and improve the efficiency of transmission of information through a channel. In Communication theory the basic questions can be answered through Information theory such as: what is the ultimate transmission rate of communication (answer: the channel capacity), and what is the ultimate data compression (answer: the entropy). Important applications of information theory were found in different areas of Cryptography, Coding theory, Thermal Physics, Neurobiology, Quantum Computing etc.

The objective of this thesis entitled, "Generalization of Varma and Tsallis Entropies and Their Properties" is to study the monotonic behaviour, convolution results, characterizations and reliability properties of residual life time of Generalized Shannon information measures.

In chapter 1, we present the literature survey related to the Shannon information measure, Generalized information measure, the reliability properties of residual life time and well known life time distributions. In addition to this, the basic fundamental background is also provided.

The entropy measures is the uncertainty about the outcomes of a random experiment. In case the outcome is captured in an interval which is contracting, the measure of entropy should be decreasing. In the present chapter, Varma entropy of order α and type β has been studied for this monotonic behaviour. For an absolutely continuous type random variable, necessary and sufficient conditions on the distribution function have been provided so that the conditional Varma entropy is a monotonic on an interval. Further, the results on the convolutions of Varma entropy have also been provided.

In chapter 3, we propose a generalized cumulative residual information measure based on Tsallis entropy and its dynamic version. We study the charac-

terizations of the proposed information measure and define new classes of life distributions based on this measure. Some applications are provided in relation to weighted and equilibrium probability models. Finally the empirical cumulative Tsallis entropy is proposed to estimate the new information measure.

In chapter 4, we extend the definition of dynamic cumulative residual Tsallis entropy (DCRTE) into the bivariate setup and study its properties in the context of reliability theory. Earlier, Sati and Gupta (2015) proposed two measures of uncertainty based on non-extensive entropy, called the dynamic cumulative residual Tsallis entropy (DCRTE) and the empirical cumulative Tsallis entropy. We also define a new class of life distributions based on proposed Bivariate DCRTE.

In chapter 5, we present the conclusions.

Publications Based on Present Work

1. Sati M. M. and Gupta N. "On Partial Monotonic Behaviour of Varma Entropy and Its Application in Coding Theory," *Journal of the Indian Statistical Association*, Vol.53 No. 1 and 2, pp.135-152, 2015.
   [**MathSci Net, STMA, CIS**]

2. Sati M. M. and Gupta N. "Some Characterization Results on Dynamic Cumulative Residual Tsallis Entropy," *Journal of Probability and Statistics*, vol. 2015, Article ID 694203 (*Hindawi Publishing Corporation*), 8 pages, 2015.
   [**MathSci Net, STMA, CIS, Scopus**]

3. Sati M. M. and Singh H. "Bivariate Dynamic Cumulative Residual Tsallis Entropy," *Journal of Applied Mathematics and Informatics*, Vol. 35, No. 1-2, pp. 45-58, 2017.
   [**MathSci Net, ZbMATH**]

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Chapter 1

Introduction

1.1 Background and Motivation

All the structures on the planet Earth are normally under two types of order - visible order and invisible order. In invisible order every bit is complex. This order makes us realize that nature has been harnessing for billions of the years to control and make use of resources, especially those producing energy. Our minds and surroundings are full of such information and we constantly exchange these with each other.

In communication systems, Information Theory identifies, characterizes and computes the basic limits of performance measures. For example, in order to answer about the information rates in the multi-user setting that can be transmitted reliably over a given noisy channel which involves various transmitter as well as various receivers.

Long distance message transmission gets depleted and is eventually unreadable. The actual problem of communication was resolved with the help of amplification. This problem was unpredictable perturbation of the message called noise. It is this noise which prevents a message from getting transferred. Even though the noise is small, we amplify the message over and over; and eventually the too noise amplified with the message. And if the noise is more amplified than the message, then the message cannot be read.

Shannon gave an amazingly simple solution to the above communication problem by observing that all messages can be converted into binary digits or a sequence of binary digits, better known as bits. Shannon formulated a theory which aimed to quantify the communication of information and tackled the problem of how to transmit information most efficiently through a given channel. It has diverse applications in different areas of Coding theory, Decision theory, Cryptography, Thermal Physics, Sampling theory, Neurobiology, Psychology, Economics, Quantum computing, Biology etc.

1.2 Shannon's Entropy

Let X be a random variable which is of discrete type can take the finite numbers of values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ in an experiment then the Shannon entropy is defined as

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i, \ 0 \leq p_i \leq 1 \tag{1.2.1}$$

unit of the quantity $H(P)$ is called bit, Nat or Hartley according to whether the base of the logarithm in equation (1.2.1) is taken to be 2, e or 10 respectively.

Shannon entropy defined by (1.2.1) satisfies the following properties:

(i) Non-negativity: The entropy is always non-negative, i.e.,

$$H(p_1, p_2, \ldots, p_n) \geq 0$$

(ii) Continuity: $H(p_1, p_2, \ldots, p_n)$ is a continuous function of their probabilities

(iii) Symmetry: $H(p_1, p_2)$ is a symmetric function of its arguments, that is,

$$H(p_1, p_2) = H(p_2, p_1)$$

(iv) Normality: For two equally probable events entropy becomes unity, that is,

$$H\left(\frac{1}{2}, \frac{1}{2}\right) = 1$$

(v) Maximality: The entropy is maximum when the probabilities of all events are equal

$$H(p_1, p_2, \ldots, p_n) \leq H\left(\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n}\right)$$

(vi) Additivity: If two independent probability distributions are $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$, then

$$H(p_1q_1, \ldots, p_1q_n, p_2q_1, \ldots, p_2q_n, \ldots, p_nq_1, \ldots, p_nq_n) = H(p_1, p_2, \ldots, p_n) + H(q_1, q_2, \ldots, q_n)$$

Let X be a continuous random variable with the probability density function $f(x)$ then the Shannon entropy is defined as

$$H(X) = -E[\log f(x)], \quad \text{or}$$

$$H(X) = -\int_{-\infty}^{\infty} f(x) \log\left(f(x)\right) dx. \tag{1.2.2}$$

Equation (1.2.2) is also known as the Shannon's differential entropy of random variable X.

1.3 Generalization of Shannon's Entropy

The Shanon entropy measures have been generalized in a number of different ways by different pioneer researchers such as Renyi (1961), Varma (1966), Kapur (1967) and Tsallis (1961). Varma entropy is a generalized two parameter Shannon entropy and plays vital role as a measure of complexity and uncertainty in different areas such as Physics, Electronics and Coding Theory. Whereas, Tsallis entropy is defined as generalization of the standard Boltzmann-Gibbs entropy. It is generally referred as nonextensive statistics dealing with one-parameter generalization of Shannon entropy.

However, Varma entropy was not defined for conditional entropy and for its monotonic nature. In case of Tsallis entropy, generalized cumulative residual entropy and its dynamic form was not defined. Therefore, we formulated conditional Varma's entropy of X given A, where A is an event, $X \in (a; b)$ and studied their monotonic and convolution properties. We also formulated and studied the dynamic cumulative residual Tsallis entropy for univariate and bivariate setup. In the sequel $H(X|A)$ should be read as $H(X|X \in A)$.

1.3.1 Varma Entropy

Varma (1966) proposed the generalized entropy of order α and type β for discrete case

$$H_{\alpha}^{\beta}(P) = \frac{1}{\beta - \alpha} \log \left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} \right), \ \beta - 1 < \alpha < \beta, \ \ \beta \geq 1, \tag{1.3.1}$$

and in continuous case

$$H_{\alpha}^{\beta}(X) = \frac{1}{\beta - \alpha} \log \left(\int_0^{\infty} (f(x))^{\alpha+\beta-1} dx \right), \ \beta - 1 < \alpha < \beta, \ \ \beta \geq 1. \tag{1.3.2}$$

As $\beta = 1, \alpha \to 1$, the measure (1.3.1) and (1.3.2) reduces to (1.2.1) and (1.2.2) respectively.

1.3.2 Tsallis Entropy

Tsallis (1961) proposed the generalized entropy of order α for discrete case

$$S_{\alpha}(P) = \frac{1}{\alpha - 1} \left(1 - \sum_{i=1}^{n} p_i^{\alpha} \right), \ \alpha > 0, \, \alpha \neq 1, \tag{1.3.3}$$

and in continuous case

$$S_{\alpha}(X) = \frac{1}{\alpha - 1} \left(1 - \int_0^{\infty} (f(x))^{\alpha} dx \right), \ \alpha > 0, \, \alpha \neq 1. \tag{1.3.4}$$

As $\alpha \to 1$, the measure (1.3.3) and (1.3.4) reduces to (1.2.1) and (1.2.2) respectively.

1.4 Some Basic Concepts in Reliability

To describe the distribution of a random variable,various functions are used such as distribution functions, survival functions, density functions, hazard rates, mean residual lives. None of these can be called as best, it so happens that for a particular problem one function gives a very simple form of distribution while other functions may be awkward to work with. But in another problem the situation may be reversed. The most important is the fact that characteristic of a distribution may be more clearly revealed by a particular function than any other. The use of a function also varies person to person.

1.4.1 Distribution Function

If X is a continuous random variable with probability density function $f(x)$, then the function

$$F(x) \;=\; P(X \leq x) \;=\; \int_{-\infty}^{x} f\,(t)\,dt, \quad \forall x$$

is called distribution function or cumulative distribution functions of the random variable X. The Distribution function has the following properties

(i) F is non-decreasing and right continuous function.

(ii) $F(-\infty) = 0$ and $F(+\infty) = 1$.

1.4.2 Survival Function

If X is a continuous random variable with density function $f(x)$, then the survival function is denoted by $\bar{F}\,(x)$ and defined as

$$\bar{F}\,(x) \;=\; P\{X > x\} \;=\; \int_{x}^{\infty} f\,(t)\,dt$$

The survival function is also known as the "Reliability function". It is a non-increasing continuous function with $\bar{F}(0) = 1$ and $\bar{F}(\infty) = 0$. The relationship between survival function and distribution function is defined as follows

$$\bar{F}(x) = 1 - F(x)$$

Differentiating both sides with respect to x, we get

$$f(x) = -\frac{d}{dx}\bar{F}(x)$$

1.4.3 Probability Mass and Density Functions

For any random variable, distribution function and survival function always exist but this is not true in case of probability mass function and probability density function.

If X is a discrete random variable which can take the values $x_1, x_2, \ldots$ and $P(X = x_i) = p(x_i), \ i = 1, 2, \ldots$ then

$$F(x) = \sum_{x_i \leq x} p(x_i)$$

and p is called the probability mass function of X.

If X is continuous random variable such that

$$F(x) = \int_{-\infty}^{x} f(z)\,dz, \ \forall x \in \mathbb{R}$$

then f is called the probability density function of X.
The density functions are not unique, in most of the cases distribution function is differentiable except at countable isolated points. The derivative of distribution function, if it exists, is the density function of the distribution and it is preferred to describe the distribution. For further details one can refer to Albert W. Marshall & Ingram Olkin (2007).

1.4.4 Hazard Functions and Hazard Rates

The function R defined on $(-\infty,\infty)$ by $R(x) = -\log \bar{F}(x)$ is called the hazard function of F, or of X. For a non-negative random variable, $R(0-) = 0$, R is increasing, and $\lim_{x\to\infty} R(x) = \infty$; any function with these properties is a hazard function.

If F is an absolutely continuous distribution function with probability density function f, then the function r defined on $(-\infty,\infty)$ by

$$\begin{aligned} r(x) &= \tfrac{f(x)}{\bar{F}(x)}, &\quad \text{if } \bar{F}(x) > 0 \\ &= \infty, &\quad \text{if } \bar{F}(x) = 0 \end{aligned}$$

is called a hazard rate or failure rate function of F, or of X.

The hazard rate or failure rate function is also defined as the conditional probability of failure between $(x, x+\Delta x)$ given that there is no failure up to time x,

$$r(x) = \lim_{\Delta x\to 0} \frac{P\{x < X \le x+\Delta x | X > x\}}{\Delta x}$$

Thus, it is the probability that the item will fail in the next Δx time unit given that the item is functioning properly in time x.

1.4.5 The Residual Life Distribution

The distribution of remaining life for an unfailed item of age t is often of interest. Let F be a distribution function such that $F(0) = 0$. the residual life distribution F_t of F at t is defined for all $t \ge 0$ such that $\bar{F}(t) > 0$ by

$$\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}, \quad x \ge 0.$$

If F has density f, then F_t has density f_t and hazard rate r_t given by

$$f_t(x) = \frac{f(x+t)}{\bar{F}(t)}, \quad x \ge 0,$$

$$r_t(x) = \frac{f(x+t)}{\bar{F}(x+t)} = r(x+t), \quad x \ge 0.$$

Thus, the residual life distribution F_t is a conditional distribution of the remaining life given survival up to time t.

1.4.6 The Mean Residual Life Function

The mean residual life function is the average life time remaining for a system, which has survived up to an age t. The mean residual life function for a continuous random variable X, is given by

$$m(t) = E(X - t \,|\, X \geq t)$$

or

$$m(t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}.$$

For the various properties and application of mean residual life function can refer to Barlow and Proschan (1975), Marshall and Olkin (2007).

The relationship between mean residual life function and hazard rate is given by

$$r(t) = \frac{1 + m'(t)}{m(t)}.$$

The mean residual life function $m(t)$ is the mean of residual life distribution F_t as a function of t. More explicitly, when F has finite mean μ and $F(x) = 0$, for $x < 0$, the mean residual life function is given by

$$m(t) = \int_0^\infty \frac{\bar{F}(x+t)}{\bar{F}(t)}\,dx = \int_t^\infty \frac{\bar{F}(z)}{\bar{F}(t)}\,dz = \int_t^\infty \frac{(t-z)}{\bar{F}(t)}\,dF(z)$$

for t such that $\bar{F}(t) > 0$, and is equal to 0 if $\bar{F}(t) = 0$.

1.5 Proportional Hazards model

Cox (1972) has introduced and studied a dependence structure among two distributions, which is referred as the proportional hazards model (PHM) and also known as Cox PH Model. Let X and Y are two non-negative continuous random variables with survival functions $\bar{F}(x)$ and $\bar{G}(x)$, and hazard rate $r_F(x)$ and $r_G(x)$ respectively. If

$$r_G(x) \;=\; \theta\, r_F(x), \tag{1.5.1}$$

where θ is a positive real constant. This PH model defined by equation (1.5.1) is equivalent to

$$\bar{G}(x) \;=\; \left[\bar{F}(x)\right]^{\theta},\; \theta > 0.$$

This model has many application in different fields such as reliability, survival analysis, economics, medicine etc. Ebrahimi and Kirmani (1996) and Nair and Gupta (2007) gave some results of characterization of probability distribution based on the proportional hazards model.

1.6 Parametric Families of Life Distributions

We recall the following definitions:

I. Increasing Function

Let I be an open interval in $\mathbb{R}$. The function $f : I \to \mathbb{R}$ is said to be an increasing function on I if

$$f(x_1) \leq f(x_2), \quad \forall\, x_1 < x_2\,.$$

II. Decreasing Function

Let I be an open interval in $\mathbb{R}$. The function $f : I \to \mathbb{R}$ is said to be an decreasing function on I if

$$f(x_1) \geq f(x_2), \quad \forall\, x_1 < x_2\,.$$

III. Concave and Convex Functions

Let $g(x)$ be a real valued function defined on interval $I = (a, b)$ is said to be concave (convex) function if for all x_1, $x_2 \in I$ and for all $\alpha \in [0, 1]$, we have

$$g(\alpha x_1 + (1-\alpha)x_2) \geq (\leq)\, \alpha\, g(x_1) + (1-\alpha)\, g(x_2)$$

IV. Log-Concave and Log-Convex Functions

Let $g(x)$ be a real valued function defined on interval $I = (a, b)$ is said to be log-concave (log-convex) function if for all x_1, $x_2 \in I$ and for all $\alpha \in [0, 1]$, we have

$$g(\alpha x_1 + (1-\alpha)x_2) \geq (\leq)\, (g(x_1))^{\alpha}\, (g(x_2))^{(1-\alpha)}$$

V. Hölder's Inequality

If x_i, $y_i > 0$, $i = 1, 2, \ldots, n$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p > 0$, $q > 0$ then the following inequality holds

$$\left(\sum_{i=1}^{n} x_i y_i\right) \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}$$

Some of the probability distributions describe the failure process and reliability of a component or a system more satisfactorily than others. They are the exponential, Weibull, Gamma, Pareto distributions. The reliability characteristics relating to these failure distributions are as under.

The Exponential Distribution:

For exponential distribution, the parameter $\lambda > 0$ is a scale parameter and

$$\bar{F}(x) = e^{-\lambda x}, \quad x \geq 0,$$

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

and

$$r(x) = \lambda, \quad x \geq 0.$$

For the exponential distribution, we have

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} = \bar{F}(x).$$

Thus, exponential distribution is the conditional probability of surviving an additional period of x, given survival up to time t, is the same as the unconditional probability of survival to time x.

The Gamma Distribution:

For gamma distribution, we have the scale parameter $\lambda > 0$ and shape parameter $\alpha > 0$ and

$$f(x|\lambda, \alpha) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0.$$

For $\alpha = 1$, above equation reduces to the exponential distribution.

The Weibull Distribution:

For Weibull distribution, we have the scale parameter $\alpha > 0$ and shape parameter $\lambda > 0$. The survival function for Weibull distribution has a simple form given by

$$\bar{F}(x) = e^{-(\lambda x)^{\alpha}}, \quad x \geq 0.$$

Therefore, the density function for Weibull distribution is given by

$$f(x) = \alpha\lambda (\lambda x)^{\alpha-1} e^{-(\lambda x)^{\alpha}}, \quad x \geq 0.$$

The Pareto Distribution:

For Pareto distribution, we have the shape parameter $a > 0$ and scale parameter $b > 0$

$$\bar{F}(x) = \frac{b}{x}^{a}, \quad x \in [b, \infty),$$

and

$$f(x) = \frac{ab^{a}}{x^{a+1}}, \quad x \in [b, \infty).$$

1.7 Literature Survey

The concept of entropy has been widely used in different areas, e.g., Information theory, Probability and Statistics, Communication theory, Physics, Economics and Computer Science etc. Shannon (1948) was the first to introduce entropy, known as Shannon's entropy or Shannon Information measure. A large numbers of generalization of Shannon entropy are available in the literature, among them some important generalization are given by Renyi (1961), Varma (1966), Kapur (1967), Tsallis (1988), Sharma and Taneja (1975), Sharma and Mittal (1977), Boekee and Lubbe (1980).

Shannon's differential entropy defined by equation (1.2.2) is not useful for such a system which has survived to an age t, Ebrahimi and Pellery (1995) and Ebrahimi (1996) proposed a modified form of Shannon's differential entropy known as residual entropy function. This new measure of uncertainty for the residual life time $X_t = [X - t | X > t]$, where $t > 0$ given by

$$H(X;t) = -\int_{t}^{\infty} \frac{f(x)}{\bar{F}(t)} \log\left(\frac{f(x)}{\bar{F}(t)}\right) dx.$$

For the study of the properties and applications of residual entropy we refer to Asadi and Ebrahimi (2000), Asadi et al. (2005), Nanda and Paul (2006) and Sunoj et al. (2009).

Shannon entropy play an important role in different area of research and it is well defined for discrete and continuous random variables. Shannon entropy of a discrete random variable is always non-negative, while it is not always non-negative for a continuous random variable . Rao et al. (2004) identified some limitations of the use of Shannon's differential entropy and introduced an alternate measure of uncertainty called cumulative residual entropy (CRE). This new measure is based on the survival function $\bar{F}(x)$ rather than the density function of a random variable X. The distribution function is more regular than the density function, because the density is computed as the derivative of the distribution. The cumulative residual entropy of a non-negative random variable X is defined as

$$\xi(X) = -\int_0^\infty \bar{F}(x) \log\left(\bar{F}(x)\right) dx.$$

Cumulative residual entropy has many important properties as follows:

(1) Cumulative residual entropy has consistent definitions in both the continuous and discrete domains.

(2) Cumulative residual entropy is always non-negative.

(3) Cumulative residual entropy can be easily computed from sample data and these computations asymptotically converge to the true values.

For the study of the properties and applications of CRE, we refer to Asadi and Ebrahimi (2000), Asadi et al. (2005), Rao (2005), Drissi et al. (2008), Gupta (2009), Di Crescenzo and Longobardi (2009), Navarro et al. (2010), Gupta and Taneja (2012), Taneja and Kumar (2012).

Asadi and Zohrevand (2007) further studied the cumulative residual entropy for the residual lifetime and proposed a new measure of uncertainty called a dynamic cumulative residual entropy (DCRE), given by

$$\xi(X;t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right) dx.$$

Abbasnejad et al. (2010) proposed dynamic survival entropy of order α and gave the relation of dynamic survival entropy with the mean residual life function. Sunoj and Linu (2010) defined cumulative residual Renyi entropy of order β and its dynamic version. Kumar and Taneja (2011) defined generalized cumulative residual information measure and its dynamic version based on Varma's entropy function. For more details of the properties and applications of DCRE, we refer Asadi and Zohervand (2007), Di Crescenzo and Longobardi (2009), Abbasnejad et al. (2010), Navarro et al. (2010), Sunoj and Linu (2010), Kumar and Taneja (2011).

The reliability characteristics can be extended to higher dimensions. Although, a lot of work have been done on information measures in the univariate case, but very limited works have been done in higher dimensions. For more details, we refer [Sunoj and Linu (2010), Rajesh and Nair (2000), Nadarajah and Zografos (2005), Ebrahimi et al. (2007), Sathar et al. (2009), Rajesh et al. (2009), (2014a), (2014b)].

Chapter 2

On Partial Monotonic Behaviour of Varma Entropy and its application in coding theory

2.1 Introduction

The concept of entropy, originally introduced by Shannon (1948), has been widely used in the fields of Information Theory, Physics, Probability and Statistics, Economics and Communication Theory. Entropy is a measure of the average amount of uncertainty associated with the outcomes of the random experiment. Consider a random variable X which is of discrete type (or of absolutely continuous type) with probability mass function $p_i,\ i = 1, 2 \ldots, n$ (or probability density function $f_X(x)$ with support set $\mathfrak{D} = \{x \in \mathbb{R} : f_X(x) > 0\}$). The Shannon entropy in discrete version is defined as:

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i,$$

and its differential form is

$$H(X) = -\int_{\mathfrak{D}} f_X(x) \log\left(f_X(x)\right) dx. \tag{2.1.1}$$

Various generalizations of entropy are proposed by different researchers in order to quantify uncertainty.

Renyi (1961) proposed a generalized entropy of order α as

$$H_\alpha(X) = \frac{1}{1-\alpha} \log \left(\int_{\mathfrak{D}} (f_X(x))^\alpha dx \right), \ \alpha > 0, \alpha \neq 1. \tag{2.1.2}$$

Tsallis (1988) defined the generalized entropy as

$$S_\alpha(X) = \frac{1}{\alpha-1} \left(1 - \int_{\mathfrak{D}} (f_X(x))^\alpha dx \right), \ \alpha > 0, \alpha \neq 1. \tag{2.1.3}$$

Kapur (1967) defined the measure of entropy of order α and type β as

$$H_{\alpha,\beta}(X) = \frac{1}{\beta-\alpha} \log \left(\frac{\int_{\mathfrak{D}} (f_X(x))^\alpha dx}{\int_{\mathfrak{D}} (f_X(x))^\beta dx} \right), \ \alpha \neq \beta, \ \alpha > 0, \beta > 0.$$

Varma (1966) also proposed the generalized entropy of order α and type β as

$$H_\alpha^\beta(X) = \frac{1}{\beta-\alpha} \log \left(\int_{\mathfrak{D}} (f_X(x))^{\alpha+\beta-1} dx \right), \quad \beta - 1 < \alpha < \beta, \ \beta \geq 1. \tag{2.1.4}$$

Entropies are important in Probability and Statistics because of their role in large deviations theory and in the study of likelihood-based inference principles. Shannon, Renyi and Tsallis entropies have operational meaning in terms of data compression. Varma entropy plays a vital role as a measure of complexity and uncertainty in different areas such as Physics, Electronics and Coding Theory. We refer reader to Cover and Thomas (2006) for applications of entropies in Information Theory.

The conditional Shannon entropy of X given A, where $A = (a, b)$, is given by

$$H(X|A) = -\int_a^b f_{X|A}(x) \log \left(f_{X|A}(x) \right) dx,$$

where

$$f_{X|A}(x) = \frac{f_X(x)}{F_X(b) - F_X(a)}, \ \ a < x < b.$$

Here $F_X(x) = P(X \leq x),\ x \in (-\infty, \infty)$, denotes the cumulative distribution function of X.

The conditional Shannon entropy is interpreted as the average entropy of the regular conditional distribution when averaged over different possible outcomes of the variable being conditioned on interval A. Sunoj et. al. (2009) provided an excellent review of the conditional Shannon entropy. The conditional Shannon entropy, Renyi entropy, Tsallis entropy and Kapur's entropy have been studied for monotonicity properties and convolution in the literature [see Shangari and Chen (2012), Chen (2013) and Gupta and Bajaj (2013)]. The Varma entropy (Generalized entropy) properties of records has been studied by Kayal and Vellaisamy (2011).

As described by Shangari and Chen (2012), Chen (2013) for an interval A, the conditional Shannon entropy $H(X|X \in A)$ may serve as indicator of uncertainty. When the interval provides the information about the outcome, the measure of uncertainty shrinks/expands as the interval shrinks/expands. For an interval A and B such that $B \subseteq A$, if $H(X|X \in B) \leq (\geq) H(X|X \in A)$ then entropy function H is said to be partially increasing (decreasing). Shangari and Chen (2012) proved that conditional Shannon entropy $H(X|A)$ of X given $A = (a, b)$ is a partially increasing function in the interval A if $F_X(x)$ is log-concave. They also proved that the conditional Renyi entropy $H_\alpha(X|A)$ of X given $A = (a, b)$ is a partially increasing function in the interval A for $\alpha \geq 0$, $\alpha \neq 1$ if $F_X(x)$ is log-concave. Gupta and Bajaj (2013) proved that conditional Kapur entropy $H_{\alpha,\beta}(X|A)$ of X given $A = (a, b)$ is a partially increasing function in the interval A if $F_X(x)$ is concave. They also proved that if $F(x)$ is log-concave then the conditional Tsallis entropy $S_\alpha(X|A)$ of X given A is a partially increasing function in the interval A where $A = (a, b)$. One may also refer to Ash (1990), Yeung (2002) and Cover and Thomas (2006) for details on various properties, which play an important role in information theory.

In the present chapter we propose a new conditional entropy which is based on Varma's entropy of order α and type β and also we define the monotonic nature of conditional Varma entropy.

In Section 2, we prove that if $F_X(x)$ is log concave and $\alpha+\beta > (<)2$ then the conditional Varma's entropy $H_\alpha^\beta(X|A)$ is partially decreasing (increasing) in $A=(a,b)$. If $F_X(x)$ is log convex and $\alpha+\beta < (>)2$ then the conditional Varma's entropy $H_\alpha^\beta(X|A)$ is decreasing (increasing) in b, for fixed a.

Consider an experiment X that is repeated to measure its reproducibility or precision or both, then the function $U=|X_1-X_2|$ measures the uncertainty of the experiment; where X_1 and X_2 are two independent and identically distributed copies of an experiment X. We prove that if random variables X_1 and X_2 have log concave probability functions then the conditional Varma's entropy of $U=|X_1-X_2|$ given $B=\{a\le X_1, X_2\le b\}$, is partially increasing (decreasing) function of B if $\alpha+\beta<2(\alpha+\beta>2)$.

In Section 3, we provide applications of Varma entropy in coding theory.

2.2 Monotonic Behaviour and convolution of Varma entropy

The conditional Varma entropy of X given $A=(a,b)$ is given as

$$\begin{aligned} H_\alpha^\beta(X|A) &= \frac{1}{\beta-\alpha}\log\left(\int_a^b \left(f_{X|A}(x)\right)^{\alpha+\beta-1}dx\right), \ \beta-1<\alpha<\beta,\ \ \beta\ge 1 \\ &= \frac{1}{\beta-\alpha}\log\left(\int_a^b \left(\frac{f_X(x)}{F_X(b)-F_X(a)}\right)^{\alpha+\beta-1}dx\right), \ \beta-1<\alpha<\beta,\ \ \beta\ge 1. \end{aligned}$$

The following theorem provides the conditions under which the conditional Varma's entropy $H_\alpha^\beta(X|A)$ is partially decreasing/increasing in interval $A=(a,b)$.

Theorem 2.2.1:

Let A be the event $a < X < b$. If $F_X(x)$ is

(a) log-concave and $\alpha+\beta > (<) 2$, then the conditional Varma entropy $H_\alpha^\beta(X|A)$ is partially decreasing (increasing) in interval $A = (a, b)$.

(b) log-convex and $\alpha + \beta < (>) 2$, then the conditional Varma entropy $H_\alpha^\beta(X|A)$ is decreasing (increasing) in b, for fixed a.

Proof:

(a) We will only show that if $F_X(x)$ is log-concave and $\alpha + \beta > 2$, then the conditional Varma entropy $H_\alpha^\beta(X|A)$ is partially decreasing in interval $A = (a, b)$. When $F_X(x)$ is log-concave and $\alpha+\beta < 2$, then the conditional Varma entropy $H_\alpha^\beta(X|A)$ is partially increasing in interval $A = (a, b)$ may be shown in a similar fashion. Define

$$\phi(a, b) = \int_a^b \left(\frac{f_X(x)}{F_X(b) - F_X(a)}\right)^{\alpha+\beta-1} dx.$$

For fixed a,

$$\begin{aligned}\frac{\partial \phi(a, b)}{\partial b} &= -\frac{(\alpha + \beta - 1) f_X(b)}{(F_X(b) - F_X(a))^{\alpha+\beta}} \int_a^b f_X^{\alpha+\beta-1}(x) dx + \left(\frac{f_X(b)}{F_X(b) - F_X(a)}\right)^{\alpha+\beta-1} \\ &= \frac{f_X(b)}{(F_X(b) - F_X(a))^{\alpha+\beta}} \psi_1(b), \qquad (2.2.1)\end{aligned}$$

where

$$\psi_1(b) = -(\alpha + \beta - 1) \int_a^b f_X^{\alpha+\beta-1}(x) dx + f_X^{\alpha+\beta-2}(b)(F_X(b) - F_X(a)),$$
$$\psi_1(b)|_{b=a} = 0,$$

and

$$\begin{aligned}\psi_1'(b) &= -(\alpha + \beta - 1) f_X^{\alpha+\beta-1}(b) + (\alpha + \beta - 2) f_X^{\alpha+\beta-3}(b) f_X'(b) \\ &\qquad (F_X(b) - F_X(a)) + f_X^{\alpha+\beta-1}(b) \\ &= (2 - \alpha - \beta) f_X^{\alpha+\beta-3}(b) \left(f_X^2(b) - f_X'(b)(F_X(b) - F_X(a))\right). \qquad (2.2.2)\end{aligned}$$

Note that $F_X(x)$ is log-concave, implies that
$(F_X(x) - F_X(a))/(F_X(b) - F_X(a))$ is log-concave. Hence we have

$$f_X^2(b) - f_X'(b)(F_X(b) - F_X(a)) \geq 0, \ \forall \ b$$

Consider the following cases:

Case-I: Since $\alpha + \beta > 2$, then from (2.2.2) we have $\psi_1'(b) \leq 0$, i.e. $\psi_1(b)$ is decreasing in b. Now for $b > a$ we have $\psi_1(b) \leq \psi_1(a)$, i.e. $\psi_1(b) \leq 0$. Therefore from (2.2.1) we have, $\frac{\partial \phi(a,b)}{\partial b} \leq 0$. Hence $\phi(a, b)$ is decreasing in b, for fixed a. Therefore the conditional Varma entropy $H_\alpha^\beta(X|A)$ is decreasing in b.

Case II: Since $\alpha + \beta < 2$, then from (2.2.2) we have $\psi_1'(b) \geq 0$, i.e. $\psi_1(b)$ is increasing in b. Now for $b > a$ we have $\psi_1(b) \geq \psi_1(a)$, i.e. $\psi_1(b) \geq 0$. Therefore from (2.2.1) we have, $\frac{\partial \phi(a,b)}{\partial b} \geq 0$. Hence $\phi(a, b)$ is increasing in b, for fixed a. Therefore the conditional Varma entropy $H_\alpha^\beta(X|A)$ is increasing in b.

Now for fixed b, consider

$$\frac{\partial \phi(a,b)}{\partial a} = \frac{f_X(a)}{(F_X(b) - F_X(a))^{\alpha+\beta}} \psi_2(a), \tag{2.2.3}$$

where

$$\psi_2(a) = (\alpha + \beta - 1) \int_a^b f_X^{\alpha+\beta-1}(x)dx - f_X^{\alpha+\beta-2}(a)(F_X(b) - F_X(a)),$$
$$\psi_2(a)|_{a=b} = 0,$$

and

$$\psi_2'(a) = (2 - \alpha - \beta) f_X^{\alpha+\beta-3}(a) \left(f_X^2(a) + f_X'(a)(F_X(b) - F_X(a)) \right). \tag{2.2.4}$$

Note that $F_X(x)$ is log-concave, implies that
$(F_X(b) - F_X(x))/(F_X(b) - F_X(a))$ is log-concave. Hence we have

$$f_X^2(a) + f_X'(a)(F_X(b) - F_X(a)) \geq 0, \ \forall \ a$$

Consider the following cases:

Case-III: Since $\alpha+\beta>2$, then from (2.2.4) we have $\psi_2'(a)\le 0$, i.e. $\psi_2(a)$ is decreasing in a. Now for $b>a$ we have $\psi_2(a)\le\psi_2(b)$, i.e. $\psi_2(a)\le 0$. Therefore from (2.2.3) we have, $\frac{\partial\phi(a,b)}{\partial a}\le 0$. Hence $\phi(a,b)$ is decreasing in a, for fixed b. Therefore the conditional Varma entropy $H_\alpha^\beta(X|A)$ is decreasing in a.

Case IV: Since $\alpha+\beta<2$, then from (2.2.4) we have $\psi_2'(a)\ge 0$, i.e. $\psi_2(a)$ is increasing in a. Now for $b>a$ we have $\psi_2(a)\ge\psi_2(b)$, i.e. $\psi_2(a)\ge 0$. Therefore from (2.2.3) we have, $\frac{\partial\phi(a,b)}{\partial a}\ge 0$. Hence $\phi(a,b)$ is increasing in a, for fixed b. Therefore the conditional Varma entropy $H_\alpha^\beta(X|A)$ is increasing in a.

Hence the conditional Varma entropy $H_\alpha^\beta(X|A)$ is partially decreasing (increasing) function in interval $A=(a,b)$

(b) Define

$$\phi(a,b)=\int_a^b\left(\frac{f_X(x)}{F_X(b)-F_X(a)}\right)^{\alpha+\beta-1}dx.$$

For fixed a,

$$\begin{aligned}\frac{\partial\phi(a,b)}{\partial b}&=-\frac{(\alpha+\beta-1)f_X(b)}{(F_X(b)-F_X(a))^{\alpha+\beta}}\int_a^b f_X^{\alpha+\beta-1}(x)dx+\left(\frac{f_X(b)}{F_X(b)-F_X(a)}\right)^{\alpha+\beta-1}\\&=\frac{f_X(b)}{(F_X(b)-F_X(a))^{\alpha+\beta}}\psi(b),\end{aligned}\qquad(2.2.5)$$

where

$$\psi(b)=-(\alpha+\beta-1)\int_a^b f_X^{\alpha+\beta-1}(x)dx+f_X^{\alpha+\beta-2}(b)(F_X(b)-F_X(a)),$$

$$\psi(b)|_{b=a}=0,$$

and

$$\psi'(b) = -(\alpha+\beta-1)f_X^{\alpha+\beta-1}(b) + (\alpha+\beta-2)f_X^{\alpha+\beta-3}(b)f'_X(b)$$
$$(F_X(b) - F_X(a)) + f_X^{\alpha+\beta-1}(b)$$
$$= (2-\alpha-\beta)f_X^{\alpha+\beta-3}(b)\left(f_X^2(b) - f'_X(b)(F_X(b) - F_X(a))\right). \quad (2.2.6)$$

Note that $F_X(x)$ is log-convex, implies that
$(F_X(x) - F_X(a))/(F_X(b) - F_X(a))$ is log-convex. Hence we have

$$f_X^2(b) - f'_X(b)(F_X(b) - F_X(a)) \leq 0, \ \forall \ b$$

Consider following cases:

Case-I: If $\alpha+\beta < 2$, then from (2.2.6) we have $\psi'(b) \leq 0$, i.e. $\psi(b)$ is decreasing in b. Now for $b > a$ we have $\psi(b) \leq \psi(a)$, i.e. $\psi(b) \leq 0$. Therefore from (2.2.5) we have, $\frac{\partial\phi(a,b)}{\partial b} \leq 0$. Hence $\phi(a,b)$ is decreasing in b, for fixed a. Therefore the conditional Varma entropy $H_\alpha^\beta(X|A)$ is decreasing in b, for fixed a.

Case-II: If $\alpha+\beta > 2$, then from (2.2.6) we have $\psi'(b) \geq 0$, i.e. $\psi(b)$ is increasing in b. Now for $b > a$ we have $\psi(b) \geq \psi(a)$, i.e. $\psi(b) \geq 0$. Therefore from (2.2.5) we have, $\frac{\partial\phi(a,b)}{\partial b} \geq 0$. Hence $\phi(a,b)$ is increasing in b, for fixed a. Therefore the conditional Varma entropy $H_\alpha^\beta(X|A)$ is increasing in b, for fixed a.
Hence the result follows.

□

Example 2.2.1:
Consider an exponential distribution with cumulative distribution function $F_X(x) = 1-e^{-\lambda x}, \ \lambda > 0, \ x \geq 0$, then $F_X(x)$ is log-concave. Hence using Theorem 2.2.1(a) if $\alpha+\beta > (<)\,2$, then the conditional Varma entropy $H_\alpha^\beta(X|A)$ is partially decreasing (increasing) in $A = (a, b)$.

Example 2.2.2:
Let $f(x) = c\,(-x)^{-c-1}\,,\ c > 0\,,\ x \in (-\infty, -1)$ and $F(x) = (-x)^{-c}\,,\ c > 0,\ x \in (-\infty, -1)$ are the probability density function and cumulative distribution function of Mirror-image Pareto distribution, respectively. Note that $F_X(x)$ is log-convex. Hence using Theorem 2.2.1(b) if $\alpha + \beta < (>)\,2$, then the conditional Varma entropy $H_\alpha^\beta(X|A)$ is decreasing (increasing) in b, where A is the event $-\infty < X < b$.

Next we provide a result on convolution. Consider an experiment X replicated to measure the reproducibility and precision of the experiment, i.e., the experiment represented by a random variable X have two copies X_1 and X_2 which are independent and identically distributed with common density $f_X(x)$. The uncertainty in X may be measured by the difference of two experiment X_1 and X_2. The difference $U = |X_1 - X_2|$ measures the uncertainty between two outcomes. If the additional information of the form $B = \{\, a \leq X_{1,} X_2 \leq b\,\}$ is provided then uncertainty should reduce.

Let $B = \{a < X_1, X_2 < b\}$ and $F(x) = P(X \leq x)$. If the additional information B is provided then the marginal probability density function of $U = |X_1 - X_2|$ given B, is

$$g(u; a, b) = \frac{1}{(F_X(b) - F_X(a))^2} \int_{a+u}^{b} f_X(x) f_X(x-u)dx,\ \forall\ u \in [0, b-a]. \quad (2.2.7)$$

Chen (2013) shows that if X_1 and X_2 have log-concave probability density functions which take value in B, then the conditional Shannon entropy of U given B is partially monotonic in B. Shangari and Chen (2012) claimed and Gupta and Bajaj (2013) proved that if X_1 and X_2 have log-concave probability density functions which takes value in B, then the conditional Tasalli and Renyi entropy of U given B, is partially increasing function of B if $\alpha > 0$, $\alpha \neq 1$. Here we want to study the partial monotonic behaviour of the conditional Varma entropy of U given B. The following lemma is useful in proving the next result of the section.

Lemma 2.2.1:

(a) Let X_1 and X_2 have log-concave probability density functions. If the function $\phi(u)$ is increases in u, then $E(\phi(U)|a < X_1, X_2 < b)$ is increasing in b for any a, and decreasing in a for any b; where $U = |X_1 - X_2|$.

(b) If $f_X(x)$ is log-concave, then $g(u; a, b)$ is decreasing function of u on $u \in [0, b-a]$.

Proof:

(a) For $a < b_1 < b_2$ and $0 < u < b_1 - a$, we have $g(u; a, b)$ is totally positive of order 2 according to Chen (2013), i.e.,

$$\begin{vmatrix} g(u; a, b_1) & g(u; a, b_2) \\ g(u+\delta; a, b_1) & g(u+\delta; a, b_2) \end{vmatrix} \geq 0,$$

$$g(u; a, b_1) g(u+\delta; a, b_2) \geq g(u; a, b_2) g(u+\delta; a, b_1).$$

This implies

$$g(u; a, b_1)\{g(u+\delta; a, b_2) - g(u; a, b_2)\} \geq g(u; a, b_2)\{g(u+\delta; a, b_1) - g(u; a, b_1)\}.$$

Dividing both side by δ and letting $\delta \to 0$, we get

$$g(u; a, b_1)\left\{\lim_{\delta \to 0} \frac{g(u+\delta; a, b_2) - g(u; a, b_2)}{\delta}\right\} \geq g(u; a, b_2)\left\{\lim_{\delta \to 0} \frac{g(u+\delta; a, b_1) - g(u; a, b_1)}{\delta}\right\}$$

$$\Rightarrow \qquad g(u; a, b_1) g'(u; a, b_2) \geq g(u; a, b_2) g'(u; a, b_1),$$

$$\Rightarrow \qquad \frac{g'(u; a, b_2)}{g(u; a, b_2)} \geq \frac{g'(u; a, b_1)}{g(u; a, b_1)}.$$

Integrating both sides of above inequality with respect to u, we have

$$\log g(u; a, b_2) - \log g(u; a, b_1) \geq 0.$$

Let $f(u;a,b) = \log g(u;a,b_2) - \log g(u;a,b_1)$, and differentiating both sides with respect to u, we have

$$\begin{aligned}\frac{d}{du}f(u;a,b) &= \frac{1}{g(u;a,b_2)}\frac{d}{du}g(u;a,b_2) - \frac{1}{g(u;a,b_1)}\frac{d}{du}g(u;a,b_1)\\ &= \frac{g(u;a,b_1)g'(u;a,b_2) - g(u;a,b_2)g'(u;a,b_1)}{g(u;a,b_1)g(u;a,b_2)} \geq 0.\end{aligned}$$

This implies that $\log g(u;a,b_2) - \log g(u;a,b_1)$ is an increasing function of u over $u > 0$ for all $a < b_1 < b_2$, i.e., $\frac{g(u;a,b_2)}{g(u;a,b_1)}$ is increasing in u, for $b_1 < b_2$.

$\Leftrightarrow$ $[U|X \in (a,b_1)] \leq_{lr} [U|X \in (a,b_2)] \Rightarrow [U|X \in (a,b_1)] \leq_{st} [U|X \in (a,b_2)]$

$\Leftrightarrow$ $$E(\phi(U)|a < X < b_1) \leq E(\phi(U)|a < X < b_2),$$

where ϕ is increasing function.

(b) Differentiating equation (2.2.7) with respect to u, we have

$$\begin{aligned}\frac{d}{du}g(u;a,b) &= \frac{1}{(F_X(b) - F_X(a))^2}\frac{d}{du}\left(\int_{a+u}^{b} f_X(x)\, f_X(x-u)dx\right)\\ &= \frac{-1}{(F_X(b) - F_X(a))^2}\left(\int_{a+u}^{b} f_X(x)\,\frac{d}{du}f_X(x-u)\,dx\, + f_X(a)f_X(a+u)\right)\\ &< 0.\end{aligned}$$

Hence the result.

Now, we will prove the following theorem which provides the conditions for conditional Varma entropy of U given B, to be partially increasing/decreasing function of B.

Theorem 2.2.2:

Let the random variables X_1 and X_2 have log-concave probability density functions and $B = \{a \leq X_1, X_2 \leq b\}$, then

(a) the conditional Varma entropy of U given B, is partially increasing function of B, if $\alpha + \beta < 2$.

(b) the conditional Varma entropy of U given B, is partially decreasing function of B, if $\alpha + \beta > 2$.

Proof:

(a) The conditional Varma entropy of order α and type β for U given B, is:

$$H_\alpha^\beta(U) = \frac{1}{\beta-\alpha}\log\left(\int_a^b (g(u;a,b))^{\alpha+\beta-1}\,du\right),\ \ \beta-1<\alpha<\beta,\ \ \beta\geq 1.$$

For fixed a, if we choose for any $b_1 \leq b_2$,

$$\psi_1(u) = (g(u;a,b_1))^{\alpha+\beta-1}\,(g(u;a,b_2))^{(\alpha+\beta-1)(\alpha+\beta-2)}$$

and

$$\psi_2(u) = (g(u;a,b_2))^{(\alpha+\beta-1)(2-\alpha-\beta)},$$

clearly here $\psi_1(u)$ and $\psi_2(u)$ are non-negative. Also, let $p = \frac{1}{\alpha+\beta-1}$ and $q = \frac{1}{2-\alpha-\beta}$, then $p>0,\ q>0$ and $\frac{1}{p}+\frac{1}{q}=1$. Now using Hölder's inequality we have

$$\int \psi_1(u)\psi_2(u)du \leq \left(\int(\psi_1(u))^p du\right)^{1/p}\left(\int(\psi_2(u))^q du\right)^{1/q},$$

$$\text{i.e.,} \int (g(u;a,b_1))^{\alpha+\beta-1}du$$
$$\leq \left(\int g(u;a,b_1)(g(u;a,b_2))^{\alpha+\beta-2}du\right)^{\alpha+\beta-1}\left(\int(g(u;a,b_2))^{\alpha+\beta-1}du\right)^{2-\alpha-\beta},$$
$$\Rightarrow \left(\frac{\int(g(u;a,b_1))^{\alpha+\beta-1}du}{\left(\int(g(u;a,b_2))^{\alpha+\beta-1}du\right)^{2-\alpha-\beta}}\right)^{\frac{1}{\alpha+\beta-1}} \leq \int (g(u;a,b_2))^{\alpha+\beta-2}g(u;a,b_1)du. \tag{2.2.8}$$

For fixed $b_2>0$, let

$$\phi_1(u) = (g(u;a,b_2))^{\alpha+\beta-2};$$

then,

$$\phi_1'(u) = (\alpha+\beta-2)(g(u;a,b_2))^{\alpha+\beta-3}g'(u;a,b_2) \geq 0,$$

as the probability density function $g(u;a,b)$ is decreasing in u for $0 \leq u \leq b-a$ (Using Lemma 2.2.1 (b)) and $\alpha+\beta<2$. Hence $\phi_1(u)$ increases in u. Therefore, by lemma 2.2.1 (a) for any $a<b_1<b_2$, we have

$$E(\phi(U)|a \leq X_1, X_2 \leq b_1) \leq E(\phi(U)|a \leq X_1, X_2 \leq b_2),$$
$$\Rightarrow \quad \int (g(u;a,b_2))^{\alpha+\beta-2} g(u;a,b_1)du \leq \int (g(u;a,b_2))^{\alpha+\beta-1}du, \qquad (2.2.9)$$

From (2.2.8) and (2.2.9) , we have

$$\left(\frac{\int (g(u;a,b_1))^{\alpha+\beta-1}du}{\left(\int (g(u;a,b_2))^{\alpha+\beta-1}du\right)^{2-\alpha-\beta}}\right)^{\frac{1}{\alpha+\beta-1}} \leq \int (g(u;a,b_2))^{\alpha+\beta-1}du,$$
$$\Rightarrow \quad \left(\int (g(u;a,b_1))^{\alpha+\beta-1}du)\right)^{\frac{1}{\alpha+\beta-1}} \leq \left(\int (g(u;a,b_2))^{\alpha+\beta-1}du\right)^{\frac{1}{\alpha+\beta-1}}$$
$$\Rightarrow \quad \int (g(u;a,b_1))^{\alpha+\beta-1}du \leq \int (g(u;a,b_2))^{\alpha+\beta-1}du \qquad (2.2.10)$$

Therefore we have

$$\frac{1}{\beta-\alpha}\log\left(\int (g(u;a,b_1))^{\alpha+\beta-1}du\right) \leq \frac{1}{\beta-\alpha}\log\left(\int (g(u;a,b_2))^{\alpha+\beta-1}du\right),$$
$$\text{i.e., } H_\alpha^\beta(U|a < X_1, X_2 < b_1) \leq H_\alpha^\beta(U|a < X_1, X_2 < b_2); \text{for } b_1 \leq b_2.$$

Hence, the conditional Varma entropy of U given B, is increasing in b for fixed a, if $\alpha+\beta < 2$.

Now for fixed b, if we choose for any $a_1 \leq a_2$,

$$\psi_3(u) = (g(u;a_1,b))^{\frac{\alpha+\beta-1}{\alpha+\beta-2}} (g(u;a_2,b))^{\alpha+\beta-1}$$

and

$$\psi_4(u) = (g(u;a_1,b))^{\frac{\alpha+\beta-1}{2-\alpha-\beta}},$$

clearly $\psi_3(u)$ and $\psi_4(u)$ are non-negative. Also, let $p = \frac{\alpha+\beta-2}{\alpha+\beta-1}$ and $q =$

$2-\alpha-\beta$, then $p<1,\ q<0$ and $\frac{1}{p}+\frac{1}{q}=1$. Now using Hölder's inequality we have

$$\left(\int(\psi_3(u))^p du\right)^{1/p}\left(\int(\psi_4(u))^q du\right)^{1/q}\le\int\psi_3(u)\psi_4(u)du,$$

$$i.e., \left(\int g(u;a_1,b)\,(g(u;a_2,b))^{\alpha+\beta-2}\,du\right)^{\frac{\alpha+\beta-1}{\alpha+\beta-2}}\left(\int(g(u;a_1,b))^{\alpha+\beta-1}du\right)^{\frac{1}{2-\alpha-\beta}}$$
$$\le\int(g(u;a_2,b))^{\alpha+\beta-1}du,$$

$$\Rightarrow\int g(u;a_1,b)(g(u;a_2,b))^{\alpha+\beta-2}du\le\left(\frac{\int(g(u;a_2,b))^{\alpha+\beta-1}du}{\left(\int(g(u;a_1,b))^{\alpha+\beta-1}du\right)^{\frac{1}{2-\alpha-\beta}}}\right)^{\frac{\alpha+\beta-2}{\alpha+\beta-1}}. \tag{2.2.11}$$

For fixed $a_2>0$, let

$$\phi_2(u)=(g(u;a_2,b))^{\alpha+\beta-2};$$

then,

$$\phi_2'(u)=(\alpha+\beta-2)(g(u;a_2,b))^{\alpha+\beta-3}g'(u;a_2,b)\ge 0,$$

as the probability density function $g(u;a,b)$ is decreasing in u for $0\le u\le b-a$ (Using Lemma 2.2.1 (b)) and $\alpha+\beta<2$. Hence $\phi_2(u)$ increases in u. Therefore, by lemma 2.2.1 (a) for any $a_1<a_2<b$, we have

$$E(\phi(U)|a_2\le X_1,X_2\le b)\le E(\phi(U)|a_1\le X_1,X_2\le b),$$
$$\Rightarrow\quad\int(g(u;a_2,b))^{\alpha+\beta-1}du\le\int(g(u;a_2,b))^{\alpha+\beta-2}g(u;a_1,b)du, \tag{2.2.12}$$

From (2.2.11) and (2.2.12) , we have

$$\int(g(u;a_2,b))^{\alpha+\beta-1}du\le\left(\frac{\int(g(u;a_2,b))^{\alpha+\beta-1}du}{\left(\int(g(u;a_1,b))^{\alpha+\beta-1}du\right)^{\frac{1}{2-\alpha-\beta}}}\right)^{\frac{\alpha+\beta-2}{\alpha+\beta-1}},$$
$$\Rightarrow\left(\int(g(u;a_2,b))^{\alpha+\beta-1}du\right)^{\frac{1}{\alpha+\beta-1}}\le\left(\int(g(u;a_1,b))^{\alpha+\beta-1}du\right)^{\frac{1}{\alpha+\beta-1}}$$
$$\Rightarrow\quad\int(g(u;a_2,b))^{\alpha+\beta-1}du\le\int(g(u;a_1,b))^{\alpha+\beta-1}du. \tag{2.2.13}$$

Therefore we have

$$\frac{1}{\beta-\alpha}\log\left(\int(g(u;a_2,b))^{\alpha+\beta-1}du\right) \leq \frac{1}{\beta-\alpha}\log\left(\int(g(u;a_1,b))^{\alpha+\beta-1}du\right),$$
$$\text{i.e., } H_\alpha^\beta(U|a_2 < X_1, X_2 < b) \leq H_\alpha^\beta(U|a_1 < X_1, X_2 < b); \text{for } a_1 \leq a_2.$$

Hence, the conditional Varma entropy of U given B is decreasing in a for fixed b, if $\alpha+\beta < 2$.

Therefore the conditional Varma entropy of U given B is partially increasing in B.

(b) Similarly we can prove that, the conditional Varma entropy of U given B, is partially decreasing function of B, if $\alpha+\beta > 2$.

□

The above theorem, ensures that if $\alpha+\beta < 2$, then the conditional Varma entropy of U given B, partially increases in B; hence its reasonability of being an entropy measure.

The following examples describe the densities where Theorem 2.2.2 is applicable.

Example 2.2.3:
Let X_1 and X_2 be two independent Weibull random variables with the common probability density function

$$f_\mu(x) = \mu\lambda(\lambda x)^{\mu-1}e^{-(\lambda x)^\mu},\ x \geq 0,\ \mu \geq 1, \lambda \geq 0.$$

Since Weibull density is log-concave if $\mu \geq 1$, using Theorem 2.2.2 conditional Varma entropy of U given interval B, is partially increasing (decreasing) function of B if $\alpha+\beta < 2$ $(\alpha+\beta > 2)$.

Example 2.2.4:
Let X_1 and X_2 be two independent gamma random variables with common probability density function

$$f_{\mu,\lambda}(x) = \frac{\lambda^{\mu}}{\Gamma(\mu)} x^{\mu-1} e^{-\lambda x},\ x \geq 0,\ \mu \geq 1, \lambda \geq 0.$$

Since gamma density is log-concave if $\mu \geq 1$, using Theorem 2.2.2 conditional Varma entropy of U given interval B, is partially increasing (decreasing) function of B if $\alpha + \beta < 2$ ($\alpha + \beta > 2$).

2.3 Application of Varma entropy in coding theory

Consider a source code C for a random variable X. Each realization of X as $x_i,\ i = 1, \ldots, n$, can take some values from a set of codewords $\{a_1, \ldots, a_D\}$. The length of code associated with x_i be denoted by $l_i,\ i = 1, \ldots, n$. Let L denote the average codeword length. It is known that the average codeword length L lies between

$$H_D(X) \leq L < H_D(X) + 1,$$

where $H_D(X) = -\sum p_i \log_D p_i$ (see chapter 5, pp 112, Cover and Thomas (2006)). We refer reader to Cover and Thomas (2006) for a detailed review of coding theory and information theory.

Note that minimizing a monotonic increasing function of a mean codeword length provide the same result as minimizing the mean codeword length itself. Therefore we provide a result for all uniquely decipherable codes for Varma entropy (a similar result is also stated in Tuli (2011)).

Theorem 2.2.3:
For all uniquely decipherable codes the correspondence between mean codeword length and entropy measures is

$$H_{\alpha}^{\beta}(X) \leq L(\alpha\,, \beta) < H_{\alpha}^{\beta}(X) + 1,$$

where

$$H_\alpha^\beta(X) = \frac{1}{(\beta-\alpha)} \log_D \left(\sum_{i=1}^{n} {p_i}^{\alpha+\beta-1} \right), \ \beta - 1 < \alpha < \beta, \ \ \beta \geq 1, \tag{2.3.1}$$

and

$$L(\alpha\,,\beta) = \left(\frac{\alpha+\beta-1}{\beta-\alpha} \right) \log_D \left(\sum_{i=1}^{n} p_i D^{l_i\left(\frac{\beta-\alpha}{\alpha+\beta-1}\right)} \right), where \ \ D \geq 2, \tag{2.3.2}$$

are, respectively, the Varma entropy and its mean codeword length.

Proof:

From Hölder's inequality we know that

$$\left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{n} x_i y_i \right) \ , \ \frac{1}{p} + \frac{1}{q} = 1 \,(\ p < 1, q < 0) \tag{2.3.3}$$

Let $x_i = p_i^{\left(\frac{\alpha+\beta-1}{\alpha+\beta-2}\right)} D^{-l_i}$ and $y_i = p_i^{\left(\frac{\alpha+\beta-1}{2-\alpha-\beta}\right)}$. If we choose $p = \frac{\alpha+\beta-2}{\alpha+\beta-1}$ and $q = (2-\alpha-\beta)$, then $\dfrac{1}{p} + \dfrac{1}{q} = 1(\ p < 1, q < 0)$. After substituting these values in equation (2.3.3) , we get

$$\left(\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha+\beta-2}{\alpha+\beta-1}\right)} \right)^{\frac{\alpha+\beta-1}{\alpha+\beta-2}} \left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} \right)^{\frac{1}{2-\alpha-\beta}} \leq \sum_{i=1}^{n} D^{-l_i} \leq 1.$$

Taking log of both sides, we get

$$\left(\frac{\alpha+\beta-1}{\alpha+\beta-2} \right) \log_D \left(\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha+\beta-2}{\alpha+\beta-1}\right)} \right) + \left(\frac{1}{2-\alpha-\beta} \right) \log_D \left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} \right) \leq 0,$$

$$\frac{1}{2-\alpha-\beta} \left[\log_D \left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} \right) - (\alpha+\beta-1) \log_D \left(\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha+\beta-2}{\alpha+\beta-1}\right)} \right) \right] \leq 0,$$

$$\log_D \left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} \right) \leq (\alpha+\beta-1) \log_D \left(\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha+\beta-2}{\alpha+\beta-1}\right)} \right).$$

Dividing both side by $(\beta-\alpha)$, we get

$$\left(\frac{1}{\beta-\alpha} \right) \log_D \left(\sum_{i=1}^{n} p_i^{\alpha+\beta-1} \right) \leq \left(\frac{\alpha+\beta-1}{\beta-\alpha} \right) \log_D \left(\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{\alpha+\beta-2}{\alpha+\beta-1}\right)} \right). \tag{2.3.4}$$

So that, if l_i's are to be integers, the lower bound for $L(\alpha\,,\beta)$ lies between $H_{\alpha}^{\beta}(X)$ and $H_{\alpha}^{\beta}(X)+1$. $\square$

We take the following example to illustrate the application of Varma entropy in coding theory.

Example 2.2.5:
Consider the following information transmission scenario where Alice attempts to communicate the outcome X of rolling a die experiments to her friend Bob. Assume that Alice is using an irregular five sided die for the experiment. Suppose the outcomes (source alphabets in a communication setup) follow the probability distribution as given below:

Table 2.1: Probability distribution

X	1	2	3	4	5
p_i	0.5	0.2	0.1	0.1	0.1

The problem here is to encode the source alphabets with minimum bits possible. Alice chooses an encoding method and encodes the outcomes 1, 2, 3, 4, 5 respectively with 1, 3, 3, 3, 3 bit-strings. From equations (2.3.1) and (2.3.2), we obtain the results for fixed value of β and various values of α. These results are illustrated in Table 2.2. The Table 2.2 shows that the mean codeword length of Varma entropy is greater then Varma entropy for the values of α and β considered in this example and thus demonstrating the theorem 2.2.3

Table 2.2: Entropy is lower bound of mean codeword length

β	α	$H_\alpha^\beta(X)$	$L(\alpha, \beta)$
1	0.10	2.287	2.888
1	0.15	2.270	2.823
1	0.20	2.253	2.751
1	0.25	2.235	2.674
1	0.30	2.216	2.595
1	0.35	2.199	2.517
1	0.40	2.181	2.446
1	0.45	2.163	2.376
1	0.50	2.144	2.320

2.4 Conclusion

If X is an absolutely continuous random variable and the distribution function $F_X(x)$ is log-concave then it is proved that the conditional Varma entropy $H_\alpha^\beta(X|A)$ is partially decreasing (increasing) in interval $A = (a, b)$. Further it is proved that if random variables X_1 and X_2 be independent and identically distributed copies of X and have log-concave probability density function then the conditional Varma's entropy of $U = |X_1 - X_2|$ given $B = \{a \leq X_1, X_2 \leq b\}$ is partially increasing (decreasing) function on B if $\alpha + \beta < 2(\alpha + \beta > 2)$.

Chapter 3

Some Characterization Results on Dynamic cumulative residual Tsallis entropy

3.1 Introduction

For a residual lifetime $X_t = [X - t | X > t]$, where $t > 0$, Ebrahimi (1996) defined an entropy as a dynamic measure of uncertainty which is given by

$$H(X;t) = -\int_{t}^{\infty} \frac{f(x)}{\bar{F}(t)} \log\left(\frac{f(x)}{\bar{F}(t)}\right) dx. \tag{3.1.1}$$

Alternative entropy was introduced by Rao et al. (2004) which is based on survival function instead of probability density function. Rao et al. (2004) defined entropy as

$$\xi(X) = -\int_0^\infty \bar{F}(x) \log\left(\bar{F}(x)\right) dx, \tag{3.1.2}$$

and called it as the cumulative residual entropy (CRE).

For the study of the properties and applications of CRE, we refer to Asadi and

Ebrahimi (2000), Asadi et al. (2005), Rao (2005), Drissi et al. (2008), Gupta (2009), Di Crescenzo and Longobardi (2009), Navarro et al. (2010), Gupta and Taneja (2012), Taneja and Kumar (2012).

For a residual lifetime $X_t = [X - t | X > t]$, Asadi and Zohrevand (2007) defined the dynamic measure of CRE as

$$\xi(X;t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right) dx, \tag{3.1.3}$$

and called it as dynamic cumulative residual entropy (DCRE).
Abbasnejad et al. (2010) proposed dynamic survival entropy of order α and gave the relation of dynamic survival entropy with the mean residual life function. Further Sunoj and Linu (2010) defined cumulative residual Renyi entropy of order β and its dynamic version. Recently Kumar and Taneja (2011) defined generalized cumulative residual information measure and its dynamic version based on Varma's entropy function.

Renyi (1961) defined the generalized entropy of order α as

$$H_\alpha(X) = \frac{1}{1-\alpha} \log\left(\int_0^\infty (f(x))^\alpha dx\right), \ \alpha > 0, \, \alpha \neq 1. \tag{3.1.4}$$

Tsallis (1988) defined the generalized entropy of order α as

$$S_\alpha(X) = \frac{1}{\alpha-1}\left(1 - \int_0^\infty (f(x))^\alpha dx\right), \ \alpha > 0, \, \alpha \neq 1. \tag{3.1.5}$$

Both the entropies (3.1.4) and (3.1.5) approaches the Shannon entropy (2.1.1) as $\alpha \to 1$. There is a close relationship between the Renyi entropy and the Tsallis entropy given as

$$H_\alpha(X) = \frac{1}{(\alpha-1)} \log\left[1 - (\alpha-1)S_\alpha(X)\right].$$

It may be noted that Tsallis entropy is a non-extensive entropy and it is non-logarithmic. However, Renyi entropy is an extensive entropy which is the major difference between them [cf. Beck (2009), Gupta and Bajaj (2013)].

Tsallis entropy plays a central role in different areas such as physics, chemistry, biology, medicine, economics etc. Cartwright (2014) proposed applications of Tsallis entropy in various fields such as describing the fluctuation of magnetic field in solar wind, signs of breast cancer in mammograms, atoms in optical lattices, analysis in magnetetic resonance imaging (MRI).

The aim of the chapter is to study the cumulative residual information based on non-extensive entropy measures and characterize some well known life time distributions and probability models. The empirical form of this information measure is useful for real data problems. In section 2, we propose a cumulative residual entropy based on Tsallis entropy of order α and its dynamic version. Also, we study some characterization results using the relationship of dynamic cumulative residual Tsallis entropy (DCRTE) with hazard rate function and mean residual life function. In section 3, we define new classes of life distribution based on this measures. In section 4, we propose the weighted form of DCRTE and study its various properties. In section 5 we introduce the empirical cumulative Tsallis entropy, and express it in terms of the sample spacings. In order to study the empirical cumulative Tsallis entropy, an example is also being provided.

3.2 Dynamic cumulative residual Tsallis entropy (DCRTE)

In this section, we define the cumulative residual Tsallis entropy and dynamic cumulative residual Tsallis entropy (DCRTE). We also give some characterization results of well known distributions in term of DCRTE.

Definition 3.2.1:
For a random variable X with survival function (sf) $\bar{F}(x)$, the cumulative residual entropy of order α denoted by $\eta_\alpha(X)$ is defined as

$$\eta_\alpha(X) = \frac{1}{\alpha-1}\left(1 - \int_0^\infty (\bar{F}(x))^\alpha dx\right),\ \alpha > 0,\ \alpha \neq 1. \tag{3.2.1}$$

In case $\alpha \to 1$, (3.2.1) gives

$$\lim_{\alpha\to 1} \eta_\alpha(X) = \xi(X). \tag{3.2.2}$$

Let us consider a unit whose random life is represented by random variable X. Let the unit survive up to time t, then the information based on entropy of the random variable X may not be useful. In that case we may consider dynamic (time dependent) information based on the entropy of the random variable $X_t = [X - t|X > t]$. The random variable $X_t = [X - t|X > t]$ has the survival function:

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)}, & \text{where } x > t \\ 1, & \text{otherwise.} \end{cases} \tag{3.2.3}$$

Definition 3.2.2:

For a random variable X_t with survival function $\bar{F}(x)$, the dynamic cumulative residual entropy of order α denoted by $\eta_\alpha(X;t)$ is defined as

$$\eta_\alpha(X;t) = \frac{1}{\alpha-1}\left(1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx\right),\quad \alpha > 0\,,\, \alpha \neq 1. \tag{3.2.4}$$

The following theorem shows that, the dynamic cumulative residual entropy determines the survival function $\bar{F}(x)$ uniquely.

Theorem 3.2.1:

Let the non-negative random variable X have the density function $f(x)$, the survival function $\bar{F}(x)$ and the hazard rate $r(x)$. Assume that $\eta_\alpha(X;t) < \infty$; $t \geq 0$; $\forall\, \alpha > 0\,, \alpha \neq 1$. Then for each α, $\eta_\alpha(X;t)$ (where $\eta'_\alpha(X;t) \neq 0$) uniquely determines the survival function $\bar{F}(t)$.

Proof:

Consider the dynamic cumulative residual Tsallis entropy of order α as (3.2.4). Therefore, we have

$$(\alpha-1)\,\eta_\alpha(X;t) = 1 - \frac{\int_t^\infty \left(\bar{F}(x)\right)^\alpha\, dx}{\left(\bar{F}(t)\right)^\alpha}. \tag{3.2.5}$$

Differentiating (3.2.5) with respect to t, we have

$$\begin{aligned}(\alpha-1)\,\eta'_\alpha(X;t) &= 1 - \alpha f(t)\frac{\int_t^\infty \left(\bar{F}(x)\right)^\alpha\, dx}{\left(\bar{F}(t)\right)^{\alpha+1}} \\ &= 1 - \alpha r(t)\frac{\int_t^\infty \left(\bar{F}(x)\right)^\alpha\, dx}{\left(\bar{F}(t)\right)^\alpha} \\ &= 1 + \alpha r(t)\left((\alpha-1)\eta_\alpha(X;t) - 1\right), \end{aligned} \tag{3.2.6}$$

where the last step follows from (3.2.5) and $r(x) = f(x)/\bar{F}(x)$ be the hazard rate function of random variable X.

Consider two survival functions $\bar{F}_1(t)$ and $\bar{F}_2(t)$ having dynamic entropies as $\eta_\alpha(X_1;t)$ and $\eta_\alpha(X_2;t)$, and hazard rate functions $r_1(t)$ and $r_2(t)$, respectively such that

$$(\alpha-1)\;\eta'_\alpha(X_1;t) \;=\; (\alpha-1)\;\eta'_\alpha(X_2;t),$$

using (3.2.6), we get

$$1 + \alpha\, r_1(t)\left((\alpha-1)\eta_\alpha(X_1;t)\, - 1\right) \;=\; 1 + \alpha\, r_2(t)\;\left(\,(\alpha-1)\eta_\alpha(X_2;t) - 1\,\right). \tag{3.2.7}$$

Taking $\eta_\alpha(X_1;t) \;=\; \eta_\alpha(X_2;t) \;=\; \eta_\alpha(X;t)$ in (3.2.7), we obtain $r_1(t) \;=\; r_2(t)$ or equivalently $\bar{F}_1(t) \;=\; \bar{F}_2(t)$. □

Now we provide some characterization results in terms of relationship between DCRTE and hazard rate function $r(t)$.

Theorem 3.2.2:

Let X be a non-negative continuous random variable with survival function $\bar{F}(t)$, hazard rate function $r(t)$ and dynamic cumulative residual Tsallis entropy $\eta_\alpha(X;t)$; then the relationship

$$(\alpha-1)\ \eta_\alpha'(X;t) \ = \ c\,r(t), \qquad \alpha>0\,,\alpha\neq 1, \tag{3.2.8}$$

gives survival function $\bar{F}(t) = \exp\left\{-\int\limits_0^t \frac{1}{\sqrt{K-2\alpha cx}}\,dx\right\}$, where K is the constant of integration and characterizes,

(i) an exponential distribution for $K>0$ and $c=0$ with survival function

$$\bar{F}(t)=e^{-\lambda t}, \quad \lambda=\frac{1}{\sqrt{K}}>0,\ t>0$$

and

(ii) Weibull distribution for $K=0$ and $c<0 (c=-a,\ a>0)$, with survival function

$$\bar{F}(t)=e^{-(\gamma t)^{\frac{1}{2}}}, \quad \gamma=\frac{2}{a\alpha}>0, \quad t>0.$$

Proof:

Under the assumption that the equation (3.2.8) holds; using (3.2.6), we have

$$1\,+\,\alpha\,r(t)\,\{\,(\alpha-1)\,\eta_\alpha(X;t)\,-1\,\}\;=\;c\,r(t)$$

or equivalently, using (3.2.5),

$$\frac{\bar{F}(t)}{f(t)}\;-\;c\;=\;\alpha\left\{\frac{\int\limits_t^\infty \bar{F}^\alpha(x)\,dx}{\bar{F}^\alpha(t)}\right\}. \tag{3.2.9}$$

Differentiating equation (3.2.9) with respect to t, we have

$$
\begin{aligned}
-\left[1 + \frac{\bar{F}(t)}{(f(t))^2} f'(t)\right] &= \alpha\left[-1 + \alpha\, \frac{f(t)}{\bar{F}(t)}\left(\frac{\int_t^{\infty} \bar{F}^{\alpha}(x)\, dx}{\bar{F}^{\alpha}(t)}\right)\right] \\
\Rightarrow \quad -\left[1 + \frac{1}{r(t)} \cdot \frac{f'(t)}{f(t)}\right] &= \alpha\left[-1 + r(t)\left(\frac{\bar{F}(t)}{f(t)} - c\right)\right] \\
\Rightarrow \quad \left[1 + \frac{1}{r(t)} \cdot \frac{f'(t)}{f(t)}\right] &= \alpha c\, r(t) \\
\Rightarrow \quad \alpha\, c\, (r(t))^2 - r(t) &= \frac{f'(t)}{f(t)} \\
\Rightarrow \quad \alpha\, c\, (r(t))^2 &= \frac{d}{dt}\, (\log r(t)\,). \qquad (3.2.10)
\end{aligned}
$$

Now letting $\log(r(t)) = y$, that is, $r(t) = e^y$, equation (3.2.10) reduces to

$$
\begin{aligned}
\alpha\, c\, e^{2y} &= \frac{dy}{dt}, \\
\text{i.e., } e^{-2y} dy &= \alpha\, c\, dt.
\end{aligned}
$$

Integrating on both sides, we have

$$e^{-2y} = K - 2\,\alpha\, c\, t,$$

where K is a constant. Therefore, $r(t) = e^y$ provides

$$r(t) = \frac{1}{\sqrt{K - 2\alpha c t}}.$$

Since the hazard rate uniquely determines the survival function using the relationship $\bar{F}(t) = \exp\left\{-\int_0^t r(x)\, dx\right\}$, consider the following case.

(i) For $K > 0$ and $c = 0$, $\bar{F}(t) = \exp\left\{-\int_0^t \frac{1}{\sqrt{K}}\, dx\right\}$ or equivalently $\bar{F}(t) = e^{-\lambda t}$, where $\lambda = \frac{1}{\sqrt{K}} > 0$.

(ii) For $c \neq 0$, survival function $\bar{F}(t) = e^{-\frac{\sqrt{K}}{\alpha c}}\, e^{\frac{\sqrt{K - 2\alpha c t}}{\alpha c}}$. Further, for $K = 0$ and $c < 0$ $(c = -a,\ a > 0)$, $\bar{F}(t) = e^{-(\gamma t)^{\frac{1}{2}}}$, where $\gamma = \frac{2}{a\alpha} > 0$.

Converse:

We assume that random variable X is distributed exponentially with p.d.f. $\lambda e^{-\lambda x}$, $\lambda > 0$, $x \geq 0$. Using equation (3.2.5) , we have $(\alpha - 1)\, \eta_\alpha(X;t) = \left(1 - \frac{1}{\alpha\lambda}\right)$ from which equation (3.2.8) follows with $c = 0$.

When X is distributed as Weibull with p.d.f. $\beta\gamma^\beta x^{\beta-1} e^{-(\gamma x)^\beta}$, $\gamma > 0, \beta = \frac{1}{2}$, $x \geq 0$. Using equation (3.2.5) , we have $(\alpha - 1)\, \eta_\alpha(X;t) = 1 - \frac{2}{\alpha^2\gamma}\left(1 + \alpha(\gamma t)^{\frac{1}{2}}\right)$ which on differentiation yields $(\alpha - 1)\, \eta'_\alpha(X;t) = -1/\alpha(\gamma t)^{\frac{1}{2}} = c\, r(t)$ with $r(t) = (1/2)(\gamma/t)^{1/2}$ and $c = -a < 0$. □

The following theorem characterize the distributions using relationship between DCRTE and mean residual life (MRL) $m_F(t)$.

Theorem 3.2.3:

Let X be a non-negative continuous random variable with survival function $\bar{F}(t)$, MRL $m_F(t)$ and dynamic cumulative residual Tsallis entropy of order α, $\eta_\alpha(X;t)$. Then

$$(\alpha - 1)\; \eta_\alpha(X;t) \;=\; 1 - K\, m_F(t)\,, \qquad \alpha > 0\,, \alpha \neq 1, \tag{3.2.11}$$

iff X has

(i) an exponential distribution for $K = \frac{1}{\alpha}$,

(ii) a Pareto distribution for $K < \frac{1}{\alpha}$ and

(iii) a finite range distribution for $K > \frac{1}{\alpha}$.

Proof:

(i) If random variable X denote an exponential distribution, then it has p.d.f., the survival function and the mean residual life, respectively as

$$\begin{aligned} f(t) \;&=\; \lambda\, e^{-\lambda t},\; \lambda > 0,\, t > 0, \\ \bar{F}(t) \;&=\; e^{-\lambda t} \;\text{ and }\; m_F(t) = \frac{1}{\lambda}. \end{aligned}$$

The dynamic cumulative residual Tsallis entropy of order α, $\eta_\alpha(X;t)$ is

$$\eta_\alpha(X;t) = \frac{1}{\alpha-1}\left(1-\frac{\int_t^\infty \left(\bar{F}(x)\right)^\alpha dx}{\left(\bar{F}(t)\right)^\alpha}\right),$$

$$\begin{aligned}(\alpha-1)\ \eta_\alpha(X;t) &= 1-\frac{\int_t^\infty \left(e^{-\lambda x}\right)^\alpha dx}{\left(e^{-\lambda t}\right)^\alpha}\\ &= 1-\frac{1}{\alpha\lambda}\\ &= 1-K\,m_F(t),\end{aligned}$$

where $K=\frac{1}{\alpha}$ and $m_F(t)=\frac{1}{\lambda}$.

(ii) If random variable X denote the Pareto distribution, then it has p.d.f., the survival function and the mean residual life, respectively as

$$\begin{aligned}f(t) &= \left(1+\frac{t}{a}\right)^{-a-1},\ a>1,\ t>0,\\ \bar{F}(t) &= \left(1+\frac{t}{a}\right)^{-a} \text{ and } m_F(t)=\frac{(a+t)}{(a-1)}.\end{aligned}$$

The dynamic cumulative residual Tsallis entropy of order α, $\eta_\alpha(X;t)$ is

$$\begin{aligned}(\alpha-1)\ \eta_\alpha(X;t) &= 1-\frac{\int_t^\infty \left(1+\frac{x}{a}\right)^{-a\alpha} dx}{\left(1+\frac{t}{a}\right)^{-a\alpha}}\\ &= 1-\frac{a+t}{\alpha a-1}\\ &= 1-\left(\frac{a-1}{\alpha a-1}\right)\left(\frac{a+t}{a-1}\right)\\ &= 1-K\,m_F(t),\end{aligned}$$

where $K=\frac{(a-1)}{(\alpha a-1)}<\frac{1}{\alpha}$ if $\alpha>1$ and $m_F(t)=\frac{(a+t)}{(a-1)}$.

(iii) If random variable X denote the finite range distribution, then it has p.d.f., the survival function and the mean residual life, respectively as

$$\begin{aligned} f(t) &= b\,(1-t)^{b-1}\,,\ b>0,\, 0<t<1, \\ \bar{F}(t) &= (1-t)^{b} \ \text{ and } \ m_F(t) = \frac{(1-t)}{(b+1)}. \end{aligned}$$

The dynamic cumulative residual Tsallis entropy of order α, $\eta_\alpha(X;t)$ is

$$\begin{aligned} (\alpha-1)\ \eta_\alpha(X;t) &= 1 - \frac{\int_t^\infty (1-x)^{b\alpha}\ dx}{(1-t)^{b\alpha}} \\ &= 1 - \frac{1-t}{\alpha b+1} \\ &= 1 - \left(\frac{b+1}{\alpha b+1}\right)\left(\frac{1-t}{b+1}\right) \\ &= 1 - K\, m_F(t), \end{aligned}$$

where $K = \frac{(b+1)}{(\alpha b+1)} > \frac{1}{\alpha}$ if $\alpha > 1$ and $m_F(t) = \frac{(1-t)}{(b+1)}$.

Converse:

Let equation (3.2.11) hold. Using equation (3.2.5), we get

$$K\, m_F(t) = \frac{\int_t^\infty \left(\bar{F}(x)\right)^\alpha\ dx}{\left(\bar{F}(t)\right)^\alpha}\ , \tag{3.2.12}$$

differentiating equation (3.2.12) with respect to t, we obtain

$$\begin{aligned} K\, m_F'(t) &= -1 + \alpha r(t) \frac{\int_t^\infty \left(\bar{F}(x)\right)^\alpha\ dx}{\left(\bar{F}(t)\right)^\alpha} \\ &= \ -1 + \alpha r(t) K\, m_F(t)\,. \end{aligned} \tag{3.2.13}$$

Using the relation between mean residual life and hazard rate, that is, $r(t)\, m_F(t) = 1 + m_F'(t)$, we have

$$m_F'(t) = \frac{(K\alpha-1)}{K(1-\alpha)}. \tag{3.2.14}$$

Integrating equation (3.2.14) on both sides with respect to t over $(0, x)$, we get

$$m_F(x) = \frac{(K\alpha - 1)}{K(1-\alpha)}x + m_F(0). \tag{3.2.15}$$

The equation (3.2.15) is linear in MRL function $m_F(x)$ of continuous random variable X, if and only if the underlying distribution is exponential ($K = \frac{1}{\alpha}$), Pareto ($K < \frac{1}{\alpha}$) or finite range ($K > \frac{1}{\alpha}$) refer to Hall and Wellner (1981). This completes the theorem. □

3.3 New class of life distributions

In this section, we define new class of life distributions based on the dynamic cumulative residual Tsallis entropy (DCRTE) of order α.

Definition 3.3.1:
The distribution function F is said to be increasing dynamic cumulative residual Tsallis entropy (IDCRTE), if $\eta_\alpha(X;t)$ is an increasing function of t. Similarly, the distribution function F is said to be decreasing dynamic cumulative residual Tsallis entropy (DDCRTE), if $\eta_\alpha(X;t)$ is an decreasing function of t.

The following theorem gives the necessary and sufficient condition for $\eta_\alpha(X;t)$ to be increasing (decreasing) DCRTE.

Theorem 3.3.1:
The distribution function F is increasing (decreasing) DCRTE if and only if for all $t \geq 0$.

$$\eta_\alpha(X;t) \;\geq (\leq)\; \frac{1}{(\alpha-1)}\left(1 \;-\frac{1}{\alpha r(t)}\right),\;\; \forall\; \alpha > 0\,,\, \alpha \neq 1 \tag{3.3.1}$$

Proof:

The dynamic cumulative residual Tsallis entropy of order α is

$$\eta_\alpha(X;t) = \frac{1}{\alpha-1}\left(1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx\right).$$

Differentiating the above equation with respect to t, we have

$$\begin{aligned}\eta'_\alpha(X;t) &= \frac{1}{\alpha-1}\left(1-\alpha r(t)\frac{\int_t^\infty \left(\bar{F}(x)\right)^\alpha dx}{\left(\bar{F}(t)\right)^\alpha}\right)\\ &= \frac{1}{\alpha-1}\left\{1+\alpha r(t)\left((\alpha-1)\eta_\alpha(X;t)-1\right)\right\},\end{aligned}$$

the DCRTE of order α, $\eta_\alpha(X;t)$ is increasing (decreasing) function of t, if $\eta'_\alpha(X;t) \geq (\leq)\ 0$. Therefore

$$\frac{1}{\alpha-1}\left\{1+\alpha r(t)\left((\alpha-1)\eta_\alpha(X;t)-1\right)\right\} \geq (\leq)\ 0.$$

Hence

$$\eta_\alpha(X;t) \geq (\leq) \frac{1}{(\alpha-1)}\left(1 - \frac{1}{\alpha r(t)}\right), \ \forall\ \alpha > 0\,,\ \alpha \neq 1.$$

□

In the following theorem, we give the hazard rate ordering using the DCRTE.

Theorem 3.3.2:

Let X and Y be two non-negative absolutely continuous random variables with survival functions $\bar{F}(t)$ and $\bar{G}(t)$, and hazard rate functions $r_F(t)$ and $r_G(t)$, respectively. If $X \geq_{hr} Y$, that is $r_F(t) \leq r_G(t)$ for all $t \geq 0$, then

(i) $\eta_\alpha(X;t) \ \leq\ \eta_\alpha(Y;t)$ for $\alpha > 1$.

(ii) $\eta_\alpha(X;t) \ \geq\ \eta_\alpha(Y;t)$ for $0 < \alpha < 1$.

Proof:

The assumption that $r_F(t) \leq r_G(t)$ implies $\bar{F}_{X_t}(x) \ \geq\ \bar{G}_{X_t}(x)$.

$$\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha \geq \left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^\alpha, \ \forall\ \alpha > 0,$$

$$1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx \;\le\; 1 - \int_t^\infty \left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^\alpha dx.$$

(i) For $\alpha > 1$

$$\frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx\right) \;\le\; \frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^\alpha dx\right).$$

Hence

$$\eta_\alpha(X;t) \;\le\; \eta_\alpha(Y;t).$$

(ii) For $0 < \alpha < 1$

$$\frac{1}{(1-\alpha)}\left(\int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx - 1\right) \;\ge\; \frac{1}{(1-\alpha)}\left(\int_{t_1}^\infty \left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^\alpha dx - 1\right)$$

$$\Rightarrow \quad \frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx\right) \;\ge\; \frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^\alpha dx\right).$$

Hence

$$\eta_\alpha(X;t) \;\ge\; \eta_\alpha(Y;t).$$

□

In the following lemma, we discuss the effect of linear transformation on DCRTE.

Lemma 3.3.1:

For any non-negative random variable X, let $Z = aX + b$, where $a > 0$ and $b \ge 0$, then

$$\eta_\alpha(Z;t) \;=\; \frac{(1-a)}{(\alpha-1)} \;+\; a\,\eta_\alpha\!\left(X; \frac{t-b}{a}\right), \; t \ge b.$$

Proof:

For non-negative random variable $Z = aX + b$, the DCRTE of order α is

$$\eta_\alpha(Z;t) = \frac{1}{\alpha-1}\left(1 - \int_t^\infty \left(\frac{\bar{F}_Z(x)}{\bar{F}_Z(t)}\right)^\alpha dz\right),$$

$$\begin{aligned}
\eta_\alpha(Z;t) &= \frac{1}{\alpha-1}\left(1 - \int_t^\infty \left(\frac{\bar{F}_X\left(\frac{x-b}{a}\right)}{\bar{F}_X\left(\frac{t-b}{a}\right)}\right)^\alpha a\,dx\right) \\
&= \frac{1}{\alpha-1}\left\{1 + a\left((\alpha-1)\eta_\alpha\left(X;\frac{t-b}{a}\right) - 1\right)\right\} \\
&= \frac{1}{\alpha-1}\left\{1 - a + a(\alpha-1)\eta_\alpha\left(X;\frac{t-b}{a}\right)\right\} \\
&= \frac{1-a}{\alpha-1} + a\,\eta_\alpha\left(X;\frac{t-b}{a}\right).
\end{aligned}$$

□

3.4 Weighted dynamic cumulative residual Tsallis entropy

Let X be a random variable with probability density function $f(t)$ and survival function $\bar{F}(t)$. Let X_W be weighted random variable associated to X and their probability density function and survival function denoted by $f_w(t)$ and $\bar{F}_w(t)$, given by

$$f_w(t) = \frac{w(t)\,f_X(t)}{E(w(X))}$$

and

$$\bar{F}_w(t) = \frac{E(w(X)\,|\,X > t)\;\bar{F}_X(t)}{E(w(X))};\quad 0 < E(w(X)) < \infty.$$

The weighted dynamic cumulative residual Tsallis entropy denoted by $\eta_\alpha(W;\,t)$ is proposed as

$$\eta_\alpha(W;\,t) \;=\; \frac{1}{(\alpha-1)}\;\left(1\;-\;\int\limits_t^\infty \left(\frac{\bar{F}_w(x)}{\bar{F}_w(t)}\right)^\alpha\,dt\;\right),\;\alpha>0\,,\,\alpha\neq 1. \tag{3.4.1}$$

The importance of weighted distribution can be seen in Patil and Rao (1977), Gupta and Kirmani (1990), Nair and Sunoj (2003), Di Crescenzo and Longobardi (2006), Maya and Sunoj (2008). For the weighted distribution, we obtain the following result based on MRL ordering.

Theorem 3.4.1:

(i) If $E(w(X)\,|X>x)\;\leq\;E(w(X)\,|X>t)$ for all $x\geq t$, then

  (a) $\eta_\alpha(W;\,t)\leq\;\eta_\alpha(X;\,t)$ for $0<\alpha<1$.

  (b) $\eta_\alpha(W;\,t)\geq\;\eta_\alpha(X;\,t)$ for $\alpha>1$.

(ii) If $E(w(X)\,|X>x)\;\geq\;E(w(X)\,|X>t)$for all $x\geq t$, then

  (a) $\eta_\alpha(W;\,t)\geq\;\eta_\alpha(X;\,t)$ for $0<\alpha<1$.

  (b) $\eta_\alpha(W;\,t)\leq\;\eta_\alpha(X;\,t)$ for $\alpha>1$.

Proof:

Rewriting equation (3.4.1), we have

$$\begin{aligned}
\eta_\alpha(W;t) \;&=\; \frac{1}{(\alpha-1)}\left(1\;-\;\int\limits_t^\infty \left(\frac{\bar{F}_w(x)}{\bar{F}_w(t)}\right)^\alpha\,dx\right)\\
&=\frac{1}{(\alpha-1)}\left(1\;-\;\int\limits_t^\infty \frac{[E\,(w(X)\,|X>x)\;\bar{F}(x)\,]^\alpha}{[E\,(w(X)\,|X>t)\;\bar{F}(t)\,]^\alpha}dx\right)\\
&=\frac{1}{(\alpha-1)}\left(1\;-\;\int\limits_t^\infty \left[\frac{E\,(w(X)\,|X>x)}{E\,(w(X)\,|X>t)}\right]^\alpha\left(\frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)}\right)\,dx\right).
\end{aligned} \tag{3.4.2}$$

(i) If $E(w(X)|X>x)\;\leq\;E(w(X)\,|X>t)$ for all $x\geq t$, that is,

$$\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]\leq\;1,\quad\forall x\geq t,$$

then we have

$$\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\le\;\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\,.$$

.

(a) For $0<\alpha<1$

$$\begin{aligned}
&\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\le\;\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\\
\Rightarrow\;&\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\le\;\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\\
\Rightarrow\;&\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\le\;\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\\
\Rightarrow\;&\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left[\frac{E(w(X)|X>x)}{E(w(X)|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right)\le\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right).
\end{aligned}\tag{3.4.3}$$

Now using (3.4.2) and (3.2.4) , from (3.4.3) we get

$$\eta_{\alpha}(W;t)\;\le\eta_{\alpha}(X;t)\,.$$

(b) For $\alpha>1$

$$\begin{aligned}
&\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\ge\;\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx,\\
\Rightarrow\;&\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\ge\;\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\\
\Rightarrow\;&\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\ge\;\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\\
\Rightarrow\;&\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left[\frac{E(w(X)|X>x)}{E(w(X)|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right)\ge\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right).
\end{aligned}\tag{3.4.4}$$

Now using (3.4.2) and (3.2.4) , from (3.4.4) we get

$$\eta_{\alpha}(W;t)\;\ge\eta_{\alpha}(X;t)\,.$$

(ii) If $E(\,w(X)\;|X>x)\;\geq\;E(\,w(X)\;|X>t)$ for all $x\geq t$, that is,

$$\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]\geq 1\quad\forall x\geq t,$$

then we have

$$\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\geq\;\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\,.$$

.

(a) For $0<\alpha<1$

$$\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\geq\;\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx$$

$$\Rightarrow\;\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\geq\;\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx$$

$$\Rightarrow\;\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\geq\;\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx$$

$$\Rightarrow\;\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left[\frac{E(w(X)|X>x)}{E(w(X)|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right)\geq\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right).\tag{3.4.5}$$

Now using (3.4.2) and (3.2.4) , from (3.4.5) we get

$$\eta_{\alpha}(W;t)\;\geq\eta_{\alpha}(X;t)\,.$$

(b) For $\alpha>1$

$$\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\leq\;\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx$$

$$\Rightarrow\;\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\leq\;\frac{1}{(\alpha-1)}+\frac{1}{(1-\alpha)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx$$

$$\Rightarrow\;\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left[\frac{E(w(X)\,|X>x)}{E(w(X)\,|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\;\leq\;\frac{1}{(\alpha-1)}-\frac{1}{(\alpha-1)}\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx$$

$$\Rightarrow\;\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left[\frac{E(w(X)|X>x)}{E(w(X)|X>t)}\right]^{\alpha}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right)\leq\frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}^{\alpha}(x)}{\bar{F}^{\alpha}(t)}\right)dx\right).\tag{3.4.6}$$

Now using (3.4.2) and (3.2.4) , from (3.4.6) we get

$$\eta_\alpha(W;t) \leq \eta_\alpha(X;t)\,.$$

□

Particularly, when the weight function is defined as $w(t) = \dfrac{\bar{F}(t)}{f(t)}$, the corresponding weighted distribution becomes the equilibrium distribution. Let X_E be a random variable corresponding to equilibrium distribution with probability density function $f_E(t) = \dfrac{\bar{F}(t)}{\mu}$, $t > 0$, and survival function $\bar{F}_E(t) = \dfrac{r(t)}{\mu}\bar{F}(t)$, where $\mu = E(X) < \infty$, then dynamic cumulative residual Tsallis entropy of X_E is proposed as

$$\eta_\alpha(E;t) = \frac{1}{\alpha-1}\left(1 - \int_t^\infty \left(\frac{\bar{F}_E(x)}{\bar{F}_E(t)}\right)^\alpha dx\right), \quad \alpha > 0\,, \alpha \neq 1. \tag{3.4.7}$$

Theorem 3.4.2:

(i) If $\bar{F}(t)$ has decreasing hazard rate, then

- (a) $\eta_\alpha(E;t) \geq \eta_\alpha(X;t)$ for $0 < \alpha < 1$.
- (b) $\eta_\alpha(E;t) \leq \eta_\alpha(X;t)$ for $\alpha > 1$.

(ii) If $\bar{F}(t)$ has increasing hazard rate, then

- (a) $\eta_\alpha(E;t) \leq \eta_\alpha(X;t)$ for $0 < \alpha < 1$.
- (b) $\eta_\alpha(E;t) \geq \eta_\alpha(X;t)$ for $\alpha > 1$.

Proof:

Rewriting equation (3.4.7) , we have

$$\begin{aligned}\eta_\alpha(E;t) &= \frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left(\frac{\bar{F}_E(x)}{\bar{F}_E(t)}\right)^\alpha dx\right)\\ &= \frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)}dx\right).\end{aligned} \tag{3.4.8}$$

(i) If $\bar{F}(t)$ has decreasing hazard rate, we have $r(x) \geq r(t) \;\; \forall\, x \leq t$, i.e.,

$$\frac{r(x)}{r(t)} \geq 1 \;\; \forall\, x \leq t,$$

then we have

$$\int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \geq \;\; \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\,. \tag{3.4.9}$$

(a) For $0 < \alpha < 1$

$$\begin{aligned}
& \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \geq \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \;\; & \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \geq \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \;\; & \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right) \geq \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right).
\end{aligned} \tag{3.4.10}$$

Now using (3.4.8) and (3.2.4) , from (3.4.10) we have

$$\eta_\alpha(E;t) \;\; \geq \;\; \eta_\alpha(X;t)\,.$$

(b) For $\alpha > 1$

$$\begin{aligned}
& \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \leq \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \;\; & \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \leq \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \;\; & \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right) \leq \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right).
\end{aligned} \tag{3.4.11}$$

Now using (3.4.8) and (3.2.4) , from (3.4.11) we have

$$\eta_\alpha(E;t) \;\; \leq \;\; \eta_\alpha(X;t)\,.$$

(ii) If $\bar{F}(t)$ has increasing hazard rate, we have $r(x) \leq r(t) \;\; \forall\, x \leq t$, i.e.,

$$\frac{r(x)}{r(t)} \leq 1 \;\; \forall\, x \leq t,$$

then we have

$$\int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \leq \;\; \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\,. \tag{3.4.12}$$

(a) For $0 < \alpha < 1$

$$\begin{aligned}
& \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \leq \; \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \quad & \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \leq \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \quad & \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right) \;\; \leq \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right).
\end{aligned} \tag{3.4.13}$$

Now using (3.4.8) and (3.2.4) , from (3.4.13) we have

$$\eta_\alpha(E;t) \;\; \leq \;\; \eta_\alpha(X;t)\,.$$

(b) For $\alpha > 1$

$$\begin{aligned}
& \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \geq \; \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \quad & \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \;\; \geq \frac{1}{(\alpha-1)} + \frac{1}{(1-\alpha)} \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx \\
\Rightarrow \quad & \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \left[\frac{r(x)}{r(t)}\right]^\alpha \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right) \;\; \geq \frac{1}{(\alpha-1)} \left(1 - \int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)} dx\right).
\end{aligned} \tag{3.4.14}$$

Now using (3.4.8) and (3.2.4) , from (3.4.14) we have

$$\eta_\alpha(E;t) \;\; \geq \;\; \eta_\alpha(X;t)\,.$$

□

3.5 Empirical cumulative Tsallis entropy

Let $X_1, X_2, ...X_n$ be non-negative, absolutely continuous, independent and identically distributed random variables with distribution function $F(x)$. According to equation (3.2.1), we define the empirical cumulative Tsallis entropy as

$$\eta_\alpha(\hat{F_n}) = \frac{1}{\alpha-1}\left(1 - \int_0^\infty (\hat{F_n}(x))^\alpha dx\right), \quad \alpha > 0,\, \alpha \neq 1, \tag{3.5.1}$$

where

$$\hat{F_n}(x) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \leq x\}}, \quad x \in R,$$

is the empirical distribution of the sample and $X_{(1)} < X_{(2)} < ..... < X_{(n)}$ are the order statistic. Equation (3.5.1) can be expressed as

$$\eta_\alpha(\hat{F_n}) = \frac{1}{\alpha-1}\left(1 - \sum_{j=1}^{n-1}\int_{X_{(j)}}^{X_{(j+1)}} (\hat{F_n}(x))^\alpha dx\right). \tag{3.5.2}$$

Recalling that

$$\hat{F_n}(x) = \begin{cases} 0\;, & x < X_{(1)}, \\ \dfrac{j}{n}\;, & X_{(j)} \leq x < X_{(j+1)}, \;\; j = 1, 2, ..., n-1 \\ 1\;, & x \geq X_{(n)} \end{cases}$$

from equation (3.5.2), we get

$$\eta_\alpha(\hat{F_n}) = \frac{1}{\alpha-1}\left(1 - \sum_{j=1}^{n-1} U_{(j+1)}\left(\frac{j}{n}\right)^\alpha\right), \quad \alpha > 0,\, \alpha \neq 1, \tag{3.5.3}$$

where

$$U_{(i)} = X_{(i)} - X_{(i-1)}, \quad i = 1, 2, ..., n$$

are the sample spacings [cf. Di Crescenzo and Longobardi (2009), (2012)] .

In the following example we study the empirical cumulative Tsallis entropy for exponentially distributed random samples.

Example 3.5.1:
Let $X_1, X_2, ...X_n$ be a random sample of exponentially distributed random variables with parameter λ. By Pyke (1965), the sample spacings are independent, with $U_{(j+1)}$ exponentially distributed with parameter $\lambda(n-j)$. Hence from equation (3.5.3) we obtain the mean and variance of the empirical cumulative Tsallis entropy as follows

$$E\Big(\eta_\alpha(\hat{F_n})\Big) = \frac{1}{(\alpha-1)}\left[1 - \sum_{j=1}^{n-1} \frac{1}{\lambda(n-j)}\left(\frac{j}{n}\right)^{\alpha}\right], \quad \alpha > 0,\, \alpha \neq 1,$$

and

$$Var\Big(\eta_\alpha(\hat{F_n})\Big) = \frac{1}{(\alpha-1)^2} \sum_{j=1}^{n-1} \frac{1}{\lambda^2(n-j)^2}\left(\frac{j}{n}\right)^{2\alpha}, \quad \alpha > 0,\, \alpha \neq 1,$$

Table 3.1: Mean of empirical cumulative Tsallis entropy for different values of n and α

n	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.99$	$\alpha = 1.01$	$\alpha = 1.25$	$\alpha = 1.5$
10	-8.63	-13.43	-27.64	-706.51	707.69	28.83	14.65
50	-59.81	-90.24	-181.31	-4549.44	4550.71	182.59	91.55
100	-125.56	-188.86	-378.55	-9480.62	9481.90	379.84	190.18
500	-656.74	-985.64	-1972.13	-49320.07	49321.36	1973.43	986.97
1000	-1322.48	-1984.26	-3969.36	-99250.80	99252.09	3970.66	1985.59

Table 3.2: Variance of empirical cumulative Tsallis entropy for different values of n and α

n	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.99$	$\alpha = 1.01$	$\alpha = 1.25$	$\alpha = 1.5$
10	2.46	5.03	18.45	10673.12	10606.53	15.77	3.66
50	2.80	6.14	23.97	14664.52	14638.81	22.95	5.62
100	2.86	6.33	24.98	15420.17	15404.57	24.35	6.02
500	2.91	6.52	25.97	16179.85	16175.45	25.79	6.43
1000	2.92	6.55	26.13	16300.87	16298.39	26.03	6.49

Based on the empirical cumulative Tsallis entropy for random samples from exponential distribution with mean 1, we tabulated the values for mean and variance in Table 3.1 and Table 3.2 respectively. It may be observed from tabulated data that the mean of empirical cumulative Tsallis entropy i.e., $E\Big(\eta_\alpha(\hat{F_n})\Big)$ is decreasing for different values of n, whereas the variance of empirical cumulative Tsallis entropy i.e., $Var\Big(\eta_\alpha(\hat{F_n})\Big)$ is increasing for different values of n.

3.6 Conclusion

The dynamic generalized information measure based on cumulative distribution function is more stable than based on density function. We proposed the Dynamic Cumulative Residual Tsallis Entropy which is found to be monotonic in nature. Based on the proposed DCRTE, we characterized some well known life time distributions such as exponential, Weibull, Pareto and the finite range distributions which play a vital role in reliability modeling. Hear we proposed weighted dynamic cumulative residual Tsallis entropy and examined its application in relation to weighted and equilibrium models. Finally, we introduced empirical cumulative Tsallis entropy for empirical samples. It is observed that the mean of empirical cumulative Tsallis entropy decreases and variance increases irrespective of sample size.

Chapter 4

Bivariate Dynamic cumulative residual Tsallis entropy

4.1 Introduction

In the literature several generalization of Shannon entropy are available like Renyi entropy(1961), Varma entropy(1966), Tsallis entropy(1988) etc. In this chapter, we focus on non-extensive entropy.

Tsallis (1988) defined the generalized non-expansive entropy of order α as

$$S_\alpha(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^\infty (f(x))^\alpha dx\right), \ \alpha > 0, \, \alpha \neq 1. \tag{4.1.1}$$

In chapter III, we have proposed dynamic cumulative residual Tsallis entropy (DCRTE) of order α as follows:

$$\eta_\alpha(X;t) = \frac{1}{\alpha - 1}\left(1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx\right), \, \alpha > 0 \, \alpha \neq 1, \tag{4.1.2}$$

and studied its properties and applications.

The multivariate life distributions are used for studying the reliability characteristics of multi-component system with each component having a lifetime depending on the next component. In the univariate case, the reliability characteristics can be extended to higher dimensions. Although, a lot of work have been done on information measures in the univariate case, but very limited works have been done in higher dimensions. For more details, we refer [Rajesh and Nair (2000), Nadarajah and Zografos (2005), Ebrahimi et al. (2007), Sathar et al. (2009), Rajesh et al. (2009), (2014a), (2014b)].

The main objective of the chapter is to extend DCRTE defined in (4.1.2) to bivariate setup and study its properties and connect it to some well known reliability models. In section 2, we propose a bivariate dynamic cumulative residual Tsallis entropy (BDCRTE) of order α and characterize some well known bivariate models using the BDCRTE. In section 3, we define new classes of life distributions based on BDCRTE and study their properties.

4.2 Bivariate Dynamic Cumulative Residual Tsallis Entropy (BDCRTE)

In this section, we extend the definition of DCRTE to the bivariate setup known as the bivariate cumulative residual Tsallis entropy (BDCRTE) and we also give some characterization results of well known bivariate distributions in term of BDCRTE.

Definition 4.2.1:
Let $X = (X_1, X_2)$ be a bivariate random vector admitting an absolutely continuous probability density function $f(x_1, x_2)$, cumulative density function $F(x_1, x_2)$ and survival function $\bar{F}(x_1, x_2)$ with respect to Lebesgue measure in the positive octant $R_2^+ = \{(t_1, t_2)|t_i > 0, i = 1, 2\}$ of the two-dimensional Euclidean space R_2. We define the bivariate DCRTE as

$$\eta_\alpha(X;t_1,\,t_2) = \frac{1}{\alpha-1}\left(1\;-\int\limits_{t_1}^{\infty}\int\limits_{t_2}^{\infty}\left(\frac{\bar F(x_1,x_2)}{\bar F(t_1,t_2)}\right)^{\alpha} dx_2dx_1\right),\ \alpha>0,\ \alpha\neq 1. \tag{4.2.1}$$

Ebrahimi (2007) has proved that the bivariate residual entropy is not invariant under non singular transformations. We can show that the bivariate DCRTE defined in equation (4.2.1) is not invariant under non singular transformations.

If $Y_j = \phi_j(X_j),\ j = 1, 2$ are one to one transformations, then

$$\eta_\alpha(Y_j\,;\phi_1(t_1),\,\phi_2(t_2)) = \frac{1}{\alpha-1}\left(1\;-\int\limits_{t_1}^{\infty}\int\limits_{t_2}^{\infty}\left(\frac{\bar F(x_1,x_2)}{\bar F(t_1,t_2)}\right)^{\alpha} J\,dx_2dx_1\right),$$

where $J = |\frac{\partial}{\partial x_1}\,\phi_1(x_1)\times\frac{\partial}{\partial x_2}\,\phi_2(x_2)|$ is the absolute value of the Jacobian of transformation. In particular, if we take $\phi_j(X_j) = a_jX_j + b_j$, then we get

$$\eta_\alpha(Y_j\,;\phi_1(t_1),\,\phi_2(t_2)) = \frac{(1-a_1\,a_2)}{(\alpha-1)}\;+\;a_1\,a_2\,\eta_\alpha(X;t_1,\,t_2).$$

Now we take into account the behavior of the Dynamic cumulative residual Tsallis entropy for the conditional distributions. Let us consider the random variables $Y_j = (X_j|X_i > t_i,\ i,\ j = 1,\ 2;\ i\neq j)$, where $Y_j,\ j = 1,\ 2$ corresponds to the conditional distributions of X_j given that X_i has survived up to time $t_i,\ i = 1, 2$ and have the survival functions $\frac{\bar F(x_1,\,t_2)}{\bar F(t_1,\,t_2)}$ for $x_1\geq t_1$ & $\frac{\bar F(t_1,\,x_2)}{\bar F(t_1,\,t_2)}$ for $x_2\geq t_2$, respectively. The DCRTE for the random variables $Y_j,\ j = 1, 2$ are defined as follows:

$$\eta_{1\alpha}(X;t_1,\,t_2) = \frac{1}{\alpha-1}\left(1\;-\int\limits_{t_1}^{\infty}\left(\frac{\bar F(x_1,t_2)}{\bar F(t_1,t_2)}\right)^{\alpha} dx_1\right),\ \alpha>0,\ \alpha\neq 1, \tag{4.2.2}$$

and

$$\eta_{2\alpha}(X;t_1,\,t_2) = \frac{1}{\alpha-1}\left(1 - \int_{t_2}^{\infty}\left(\frac{\bar{F}(t_1,x_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha} dx_2\right),\ \alpha>0,\ \alpha\neq 1, \quad (4.2.3)$$

respectively.

For a bivariate random vector $X = (X_1,\, X_2)$, Johnson and Kotz (1975) defined the bivariate hazard rate as

$$r(t_1,\,t_2) = (\,r_1(t_1,\,t_2),\,r_2(t_1,\,t_2)\,),$$

where

$$r_i(t_1,\,t_2) = -\frac{\partial}{\partial t_i}\log \bar{F}(t_1,t_2),\ \ i=1,2. \quad (4.2.4)$$

For a bivariate random vector $X = (X_1,\, X_2)$, Zahedi (1985) defined the bivariate mean residual life function (MRLF) as

$$m(t_1,\,t_2) = (\,m_1(t_1,\,t_2),\,m_2(t_1,\,t_2)\,),$$

where

$$m_i(t_1,\,t_2) = \frac{1}{\bar{F}(t_1,\,t_2)}\int_{t_i}^{\infty}\bar{F}(x_i,\,t)\,dx,\ \ i=1,2. \quad (4.2.5)$$

The following theorem shows that the Bivariate dynamic cumulative residual entropy uniquely determines the survival function $\bar{F}(t_1,t_2)$.

Theorem 4.2.1:

Let $X = (X_1,\, X_2)$ be a non-negative random vector admitting continuous distribution function with respect to Lebsegue measure. Let $\eta_{i\alpha}(X;t_1,\,t_2) < \infty\,;\ i = 1,\,2,\ t = (t_1,\,t_2) \geq 0;\ \forall\,\alpha > 0\,,\alpha\neq 1$. Then for each α, $\eta_{i\alpha}(X;t_1,\,t_2)$ (where $\frac{\partial}{\partial t_i}\eta_{i\alpha}(X;\,t_1,\,t_2) \neq 0, \forall\, i = 1,\,2$) uniquely determines the survival function $\bar{F}(t_1,\,t_2)$.

Proof:

From the equation (4.2.2) we have,

$$(\alpha-1)\,\eta_{1\alpha}(X;\,t_1,\,t_2) = 1 - \frac{\int\limits_{t_1}^{\infty}\left(\bar{F}(x_1,\,t_2)\right)^{\alpha}\,dx_1}{\left(\bar{F}(t_1,\,t_2)\right)^{\alpha}}. \tag{4.2.6}$$

Differentiating (4.2.6) with respect to t_1 and simplifying, we obtain

$$(\alpha-1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\,t_1,\,t_2) = 1+\alpha\,r_1(X;t_1,t_2)\,[(\alpha-1)\eta_{1\alpha}(X;t_1,t_2)-1]. \tag{4.2.7}$$

Similarly for $i=2$, we also get

$$(\alpha-1)\,\frac{\partial}{\partial t_2}\eta_{2\alpha}(X;\,t_1,\,t_2) = 1+\alpha\,r_2(X;t_1,t_2)\,[(\alpha-1)\eta_{2\alpha}(X;t_1,t_2)-1]. \tag{4.2.8}$$

Let $\bar{F}_X(t_1,\,t_2)$ and $\bar{F}_Y(t_1,\,t_2)$ be two survival functions having Bivariate dynamic entropies $\eta_{i\alpha}(X;\,t_1,\,t_2)$ and $\eta_{i\alpha}(Y;\,t_1,\,t_2)$ with hazard rates $r_i(X;\,t_1,\,t_2)$ and $r_i(Y;\,t_1,\,t_2)$, $i=1,2$ respectively.

Consider the following relationship between entropies of random vectors X and Y :

$$\eta_{i\alpha}(X;\,t_1,\,t_2) \;=\; \eta_{i\alpha}(Y;\,t_1,\,t_2)\,,\;\; i=1,2. \tag{4.2.9}$$

Taking $i=1$ and differentiating (4.2.9) with respect to t_1, we get

$$\begin{aligned} \frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\,t_1,\,t_2) \;&=\; \frac{\partial}{\partial t_1}\eta_{1\alpha}(Y;\,t_1,\,t_2), \\ \Rightarrow \qquad (\alpha-1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\,t_1,\,t_2) \;&=\; (\alpha-1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(Y;\,t_1,\,t_2). \end{aligned} \tag{4.2.10}$$

Using (4.2.7), the equation (4.2.10) becomes

$$1+\alpha\,r_1(X;\,t_1,t_2)\,[(\alpha-1)\eta_{1\alpha}(X;t_1,t_2)-1] = 1+\alpha\,r_1(Y;\,t_1,t_2)\,[(\alpha-1)\eta_{1\alpha}(Y;t_1,t_2)-1]. \tag{4.2.11}$$

Since $\eta_{1\alpha}(X;\,t_1,\,t_2) \;=\; \eta_{1\alpha}(Y;\,t_1,\,t_2)$, therefore the equation (4.2.11) reduces to

$$r_1(X;\,t_1,\,t_2) \;=\; r_1(Y;\,t_1,\,t_2).$$

Similarly for $i = 2$, we get

$$r_2(X;\, t_1,\, t_2) \;=\; r_2(Y;\, t_1,\, t_2).$$

Thus, we have

$$\bar{F}_X(t_1,\, t_2) \;=\; \bar{F}_Y(t_1,\, t_2).$$

Hence, $\eta_{i\alpha}(X;\, t_1,\, t_2)$ uniquely determines the survival function $\bar{F}(t_1,\, t_2)$. □

The following theorem characterize some well known bivariate distribution using relationship between BDCRTE and bivariate mean residual life function $m(X; t_1, t_2)$.

Theorem 4.2.2:

For the random vector $X = (X_1,\, X_2)$ admitting continuous distribution function with respect to Lebsegue measure, a relationship of the form

$$(\alpha - 1)\, \eta_{i\alpha}(X;\, t_1,\, t_2) = 1 - K m_i\left(X; t_1,\, t_2\right), \quad i = 1,\, 2,\;\; \alpha > 0\,, \alpha \neq 1, \quad (4.2.12)$$

where $m_i\left(X; t_1,\, t_2\right), i = 1, 2$ are the components of the bivariate mean residual life function and K is a constant independent of t_i, holds for all $t_i \geq 0$, if and only if X follows any one of the three distributions:

(i) the bivariate Pareto distribution with joint survival function

$$\bar{F}(t_1,\, t_2) = (1 + a_1 t_1 + a_2 t_2 + b t_1 t_2)^{-c};\;\; a_1, a_2, c, t_1, t_2 > 0; \quad 0 < b < (c+1) a_1 a_2, \qquad (4.2.13)$$

(ii) the Gumbel's bivariate exponential distribution with joint survival function

$$\bar{F}(t_1,\, t_2) = \exp\left(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2\right);\;\; \lambda_1, \lambda_2, t_1, t_2 > 0; \quad 0 < \theta < \lambda_1 \lambda_2, \qquad (4.2.14)$$

and

(iii) the bivariate finite range distribution with joint survival function

$$\bar{F}(t_1,\, t_2) = (1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)^d\,; \quad p_1, p_2, d > 0; \;\; 0 < t_1 < \frac{1}{p_1};$$
$$0 < t_2 < \frac{1 - p_1 t_1}{p_2 - q t_1}, \qquad (4.2.15)$$

according as $K\alpha < 1,\ K\alpha = 1$ and $K\alpha > 1$, respectively.

Proof:

Differentiating equation (4.2.12) with respect to t_1 by taking $i = 1$, we obtain

$$(\alpha - 1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\, t_1,\, t_2) \;=\; -K\frac{\partial}{\partial t_1}m_1(X; t_1,\, t_2).$$

Using the equation (4.2.7) , we get

$$1 - K\alpha\, r_1(X; t_1, t_2)\, m_1(X; t_1, t_2) = \; -K\frac{\partial}{\partial t_1}m_1(X; t_1,\, t_2). \qquad (4.2.16)$$

Using the relation $r_1(X; t_1, t_2)\, m_1(X; t_1, t_2) = 1 + \frac{\partial}{\partial t_1}m_1(X; t_1,\, t_2)$, the equation (4.2.16) reduces to

$$\begin{aligned}
\frac{1}{K} - \alpha\left(1 + \frac{\partial}{\partial t_1}m_1(X; t_1,\, t_2)\right) \;&=\; -\frac{\partial}{\partial t_1}m_1(X; t_1,\, t_2)\\
(\alpha - 1)\frac{\partial}{\partial t_1}m_1(X; t_1,\, t_2) \;&=\; \frac{1}{K} - \alpha\\
\frac{\partial}{\partial t_1}m_1(X; t_1,\, t_2) \;&=\; \frac{\frac{1}{K} - \alpha}{\alpha - 1} \;=\; C.
\end{aligned}$$

Integrating on both side with respect to t_1, we get

$$m_1(X; t_1,\, t_2) = \; C t_1 + D_1(t_2), \qquad (4.2.17)$$

where D_1 is independent of t_1. Similarly for $i = 2$, we have

$$m_2(X; t_1,\, t_2) = \; C t_2 + D_2(t_1). \qquad (4.2.18)$$

Hence

$$m_i(X; t_1,\, t_2) = \; C t_i + D_i(t_j)\,, \quad i \neq j,\; i, j = 1,\, 2,$$

where $C = \dfrac{\frac{1}{K} - \alpha}{\alpha - 1}$ and $D_i(t_j)$ is a function of t_j only. Based on the characterization theorem given by Sankaran and Nair (2000), we can easily prove that X follows bivariate Pareto distribution with survival function (4.2.13) when $C > 0$, Gumbel's exponential distribution with survival function (4.2.14) when $C = 0$ and bivariate finite range distribution with survival function (4.2.15) when $C < 0$.

Converse:

(i) When X follows bivariate Pareto distribution with survival function (4.2.13), then using the equation (4.2.2), we get

$$\begin{aligned}(\alpha-1)\,\eta_{1\alpha}(X;\,t_1,\,t_2) &= 1 - \frac{\int\limits_{t_1}^{\infty}(1+a_1x_1+a_2t_2+bx_1t_2)^{-c\alpha}\,dx_1}{(1+a_1t_1+a_2t_2+bt_1t_2)^{-c\alpha}} \\ &= 1 - \left[\frac{(c-1)}{(c\alpha-1)}\,\frac{(1+a_1t_1+a_2t_2+bt_1t_2)}{(c-1)(a_1+bt_2)}\right].\end{aligned}$$

Similar result holds for $i = 2$. Hence

$$(\alpha-1)\,\eta_{i\alpha}(X;\,t_1,\,t_2) = 1 - Km_i\,(X;t_1,\,t_2)\,,$$

where $K = \dfrac{(c-1)}{(c\alpha-1)}$, such that $C = \dfrac{\frac{1}{K}-\alpha}{\alpha-1} > 0$.

(ii) When X follows Gumbel's exponential distribution with survival function (4.2.14), then using the equation (4.2.2), we get

$$\begin{aligned}(\alpha-1)\,\eta_{1\alpha}(X;\,t_1,\,t_2) &= 1 - \frac{\int\limits_{t_1}^{\infty}\left(e^{-\lambda_1x_1-\lambda_2t_2-\theta x_1t_2}\right)^{\alpha}\,dx_1}{\left(e^{-\lambda_1t_1-\lambda_2t_2-\theta t_1t_2}\right)^{\alpha}} \\ &= 1 - \frac{1}{\alpha(\lambda_1+\theta t_2)}.\end{aligned}$$

Similar result holds for $i = 2$. Hence

$$(\alpha-1)\,\eta_{i\alpha}(X;\,t_1,\,t_2) = 1 - Km_i\,(X;t_1,\,t_2)\,,$$

where $K = \dfrac{1}{\alpha}$, such that $C = \dfrac{\frac{1}{K} - \alpha}{\alpha - 1} = 0$.

(iii) When X follows bivariate finite range distribution with survival function (4.2.15), then using the equation (4.2.2), we get

$$\begin{aligned}(\alpha - 1)\,\eta_{1\alpha}(X;\, t_1,\, t_2) &= 1 - \frac{\int\limits_{t_1}^{\infty} (1 - p_1 x_1 - p_2 t_2 + q x_1 t_2)^{d\alpha}\, dx_1}{(1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)^{d\alpha}} \\ &= 1 - \left[\frac{(d+1)}{(d\alpha + 1)}\, \frac{(1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)}{(d+1)(p_1 + q t_2)}\right].\end{aligned}$$

Similar result holds for $i = 2$. Hence

$$(\alpha - 1)\,\eta_{i\alpha}(X;\, t_1,\, t_2) = 1 - K m_i\,(X; t_1,\, t_2)\,,$$

where $K = \dfrac{(d+1)}{(d\alpha + 1)}$, such that $C = \dfrac{\frac{1}{K} - \alpha}{\alpha - 1} < 0$.

□

Now we provide characterization result in terms of relationship between bivariate DCRTE and bivariate hazard rate function.

Theorem 4.2.3:

For the random vector $X = (X_1,\, X_2)$ admitting an absolutely continuous function with respect to Lebsegue measure, the relationship of the form

$$(\alpha - 1)\,\frac{\partial}{\partial t_i}\eta_{i\alpha}(X;\, t_1,\, t_2) = c\, r_i\,(X; t_1,\, t_2)\,, \quad i = 1,\, 2,\;\; \alpha > 0\,, \alpha \neq 1, \qquad (4.2.19)$$

hold for all $t_1, t_2 \geq 0$, then X follows the Gumbel's bivariate exponential distribution with joint survival function

$$\bar{F}(t_1,\, t_2) = \exp\left(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2\right);\; \lambda_1, \lambda_2, t_1, t_2 > 0;\;\; 0 < \theta < \lambda_1\lambda_2\,, \text{when } c = 0. \qquad (4.2.20)$$

Proof:

When equation (4.2.19) hold for $i = 1$, then using the equation (4.2.6) and equation (4.2.7), we get

$$1 - \alpha\, r_1(X; t_1, t_2) \frac{\int\limits_{t_1}^{\infty} \left(\bar{F}(x_1,\, t_2)\right)^{\alpha}\, dx_1}{\left(\bar{F}(t_1,\, t_2)\right)^{\alpha}} = c\, r_1(X; t_1, t_2),$$

or equivalently

$$\frac{1}{r_1(X; t_1, t_2)} = c + \alpha\, \frac{\int\limits_{t_1}^{\infty} \left(\bar{F}(x_1,\, t_2)\right)^{\alpha}\, dx_1}{\left(\bar{F}(t_1,\, t_2)\right)^{\alpha}}. \tag{4.2.21}$$

Differentiating equation (4.2.21) with respect to t_1 and simplifying, we obtain

$$\frac{-1}{\left(r_1(X; t_1, t_2)\right)^2} \frac{\partial}{\partial t_1} \{r_1(X; t_1, t_2)\} = -\,\alpha c\, r_1(X; t_1, t_2).$$

$$\frac{\partial}{\partial t_1} \{\log\left(r_1(X; t_1, t_2)\right)\} = \alpha c\, r_1^2(X; t_1, t_2). \tag{4.2.22}$$

For simplification, we assume that $\log\left(r_1(X; t_1, t_2)\right) = y_1(t_1, t_2)$ i.e., $r_1(X; t_1, t_2) = e^{y_1(t_1, t_2)}$, then the equation (4.2.22) reduces to

$$\frac{\partial}{\partial t_1} \{(y_1(t_1, t_2))\} = \alpha c\, e^{2 y_1(t_1, t_2)}.$$

Integrating on both side with respect to t_1, we get

$$r_1(X; t_1, t_2) = \frac{1}{\sqrt{K_1(t_2) -\, 2\alpha c\, t_1}}\, .$$

Similarly for $i = 2$, we get

$$r_2(X; t_1, t_2) = \frac{1}{\sqrt{K_2(t_1) -\, 2\alpha c\, t_2}}\, .$$

Hence

$$r_i(X; t_1, t_2) = \frac{1}{\sqrt{K_i(t_j) -\, 2\alpha c\, t_i}}\,, \quad i \neq j\,, \quad i, j = 1,\, 2, \tag{4.2.23}$$

where $K_i(t_j) > 0$ is constant and independent of t_i.

When $c = 0$, then from the equation (4.2.23), we get $r_i(X; t_1, t_2) = \frac{1}{\sqrt{K_i(t_j)}}$ or equivalently

$$-\frac{\partial}{\partial t_i}\{\log\ \bar{F}(t_1,t_2)\} = \frac{1}{\sqrt{K_i(t_j)}}.$$

Integrating both side with respect to t_i, we get

$$-\log\ \bar{F}(t_1,t_2) = \frac{t_i}{\sqrt{K_i(t_j)}} + Q_i(t_j)$$

$$\bar{F}(t_1,t_2) = e^{-\left[\frac{t_i}{\sqrt{K_i(t_j)}}+Q_i(t_j)\right]}, \quad i \neq j\,, \quad i = 1,\, 2. \tag{4.2.24}$$

Applying for $i = 1, 2$ and equating (4.2.24) we get

$$\frac{t_1}{\sqrt{K_1(t_2)}} + Q_1(t_2) = \frac{t_2}{\sqrt{K_2(t_1)}} + Q_2(t_1) \tag{4.2.25}$$

As $t_1 \to 0$, equation (4.2.25) becomes

$$Q_1(t_2) = \frac{t_2}{\sqrt{K_2(0)}} + Q_2(0)$$

As $t_2 \to 0$, equation (4.2.25) becomes

$$Q_2(t_1) = \frac{t_1}{\sqrt{K_1(0)}} + Q_1(0)$$

Putting the value of $Q_1(t_2)$ and $Q_2(t_1)$ in the equation (4.2.25), we get

$$\frac{t_1}{\sqrt{K_1(t_2)}} + \frac{t_2}{\sqrt{K_2(0)}} + Q_2(0) = \frac{t_2}{\sqrt{K_2(t_1)}} + \frac{t_1}{\sqrt{K_1(0)}} + Q_1(0) \tag{4.2.26}$$

Since $Q_1(0) = Q_2(0) = \bar{F}(0,0)$, equation (4.2.25) become

$$\frac{1}{t_2\sqrt{K_1(t_2)}} - \frac{1}{t_2\sqrt{K_1(0)}} = \frac{1}{t_1\sqrt{K_2(t_1)}} - \frac{1}{t_1\sqrt{K_2(0)}} = \theta(say), \tag{4.2.27}$$

which implies

$$\frac{1}{\sqrt{K_1(t_2)}} = \lambda_1 + \theta t_2$$

Similarly, we get

$$\frac{1}{\sqrt{K_2(t_1)}} = \lambda_2 + \theta t_1,$$

where $\frac{1}{\sqrt{K_1(0)}} = \lambda_1$ and $\frac{1}{\sqrt{K_2(0)}} = \lambda_2$. Substituting these value in the equation (4.2.25), after simplification we get

$$\bar{F}(t_1,\, t_2) = \exp\left(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2\right).$$

The converse part is staightforward. □

4.3 New class of life distributions

In this section, we define new class of life distributions based on proposed bivariate dynamic cumulative residual Tsallis entropy (BDCRTE).

Definition 4.3.1:
The distribution function $F(t_1, t_2)$ is said to be increasing bivariate dynamic cumulative residual Tsallis entropy (IBDCRTE), if $\eta_{i\alpha}(X; t_1, t_2)$ is an increasing function of t_1, t_2. Similarly the distribution function $F(t_1, t_2)$ is said to be decreasing bivariate dynamic cumulative residual Tsallis entropy (DBDCRTE), if $\eta_{i\alpha}(X; t_1, t_2)$ is an decreasing function of t_1, t_2.

The following theorem gives the necessary and sufficient conditions for BDCRTE to be increasing(decreasing) BDCRTE.

Theorem 4.3.1:
The bivariate distribution function $F(t_1, t_2)$ is increasing (decreasing) BDCRTE if and only if for all $t_1, t_2 \geq 0$.

$$\eta_{i\alpha}(X; t_1, t_2) \;\geq (\leq)\; \frac{1}{(\alpha - 1)} \left(1 \;-\; \frac{1}{\alpha\, r_i(t_1, t_2)}\right), \quad i = 1, 2 \;\forall\; \alpha > 0\,, \alpha \neq 1.$$

Proof:
Differentiating (4.2.2) with respect to t_1 and simplifying, we get

$$\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\, t_1,\, t_2) = \frac{1}{\alpha - 1}\,\left(1 + \alpha\, r_1(t_1, t_2)\left[(\alpha - 1)\eta_{1\alpha}(X; t_1, t_2) - 1\right]\right).$$

The BDCRTE of order α, $\eta_{1\alpha}(X;t_1,t_2)$ is increasing (decreasing) function of t_1, if $\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\,t_1,\,t_2)\ \geq (\leq)\ 0$. Therefore

$$\frac{1}{\alpha-1}\ (1+\alpha\, r_1(t_1,t_2)\,[(\alpha-1)\eta_{1\alpha}(X;t_1,t_2)-1])\ \geq (\leq)\ 0.$$

Hence

$$\eta_{1\alpha}(X;t_1,t_2)\ \geq (\leq)\ \frac{1}{(\alpha-1)}\ \left(1\ -\frac{1}{\alpha\, r_1(t_1,t_2)}\right).$$

Similar result holds for $i=2$. □

The following theorem provides lower bound of BDCRTE based on bivariate mean residual life function.

Theorem 4.3.2:

Let $X=(X_1,X_2)$ be a non-negative random vector admitting absolute continuous distribution function with respect to Lebesgue measure and $m_i(t_1,t_2), i=1,2$ are the components of the bivariate mean residual life function, then

$$\eta_{i\alpha}(X;t_1,t_2)\ \geq\ \frac{1}{(\alpha-1)}\ (1\ -\ m_i(t_1,t_2)\ ),\ \ i=1,2\ ,\forall\ \alpha>0\,,\alpha\neq 1.$$

Proof:

We know that

$$\left(\bar{F}(t_1,t_2)\right)^{\alpha}\ \leq (\geq)\ \bar{F}(t_1,t_2),\ \ \forall\ t_1,t_2>0,\ \alpha>1\,(0<\alpha<1)$$

$$\Rightarrow\quad \int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha} dx_1\ \leq (\geq) \int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right) dx_1\ ,\ \alpha>1\,(0<\alpha<1).$$

Case 1: When $\alpha>1$

$$1\ -\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha} dx_1\ \geq\ 1\ -\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right) dx_1$$

$$\frac{1}{(\alpha-1)}\left(1-\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dx_1\right)\geq\frac{1}{(\alpha-1)}\left(1-\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)dx_1\right)$$

$$\Rightarrow \qquad \eta_{1\alpha}(X;t_1,t_2)\geq\frac{1}{(\alpha-1)}\left(1-m_1(t_1,t_2)\right).$$

Case 2: When $0<\alpha<1$

$$\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dx_1-1\geq\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)dx_1-1$$

$$\Rightarrow \quad \frac{1}{(1-\alpha)}\left(\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dx_1-1\right)\geq\frac{1}{(1-\alpha)}\left(\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)dx_1-1\right)$$

$$\Rightarrow \quad \frac{1}{(\alpha-1)}\left(1-\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dx_1\right)\geq\frac{1}{(\alpha-1)}\left(1-\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)dx_1\right)$$

$$\Rightarrow \qquad \eta_{1\alpha}(X;t_1,t_2)\geq\frac{1}{(\alpha-1)}\left(1-m_1(t_1,t_2)\right).$$

Thus

$$\eta_{1\alpha}(X;t_1,t_2)\geq\frac{1}{(\alpha-1)}\left(1-m_1(t_1,t_2)\right),\forall\ \alpha>0,\ \alpha\neq1.$$

Similar result holds for $i=2$. Therefore, we have

$$\eta_{i\alpha}(X;t_1,t_2)\geq\frac{1}{(\alpha-1)}\left(1-m_i(t_1,t_2)\right),\ \ i=1,2\ ,\forall\ \alpha>0,\ \alpha\neq1.$$

□

In the following theorem, we give the bivariate hazard rate ordering based on BDCRTE.

Theorem 4.3.3:

Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two non-negative random vector with survival functions $\bar{F}(t_1, t_2)$ and $\bar{G}(t_1, t_2)$, and hazard rates $r_F(t_1, t_2)$ and $r_G(t_1, t_2)$ respectively. If $X \geq^{hr} Y$, that is $r_F(t_1, t_2) \leq r_G(t_1, t_2)$, then

(i) $\eta_{i\alpha}(X; t_1, t_2) \leq \eta_{i\alpha}(Y; t_1, t_2)$ for $\alpha > 1$, $i = 1, 2$.

(ii) $\eta_{i\alpha}(X; t_1, t_2) \geq \eta_{i\alpha}(Y; t_1, t_2)$ for $0 < \alpha < 1$, $i = 1, 2$.

Proof:

We know that $r_F(t_1, t_2) \leq r_G(t_1, t_2)$ which implies $\bar{F}(t_1, t_2) \geq \bar{G}(t_1, t_2)$. Therefore, we have for all $\alpha > 0$

$$\int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 \geq \int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1.$$

(i) For $\alpha > 1$

$$1 - \int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 \leq 1 - \int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1$$

$$\Rightarrow \quad \frac{1}{(\alpha - 1)} \left(1 - \int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 \right) \leq \frac{1}{(\alpha - 1)} \left(1 - \int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1 \right)$$

$$\Rightarrow \quad \eta_{1\alpha}(X; t_1, t_2) \leq \eta_{1\alpha}(Y; t_1, t_2).$$

Similar result holds for $i = 2$. Hence

$$\eta_{i\alpha}(X; t_1, t_2) \leq \eta_{i\alpha}(Y; t_1, t_2).$$

(ii) For $0 < \alpha < 1$

$$\int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 - 1 \geq \int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1 - 1$$

$$\frac{1}{(1-\alpha)}\left(\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dx_1-1\right) \geq \frac{1}{(1-\alpha)}\left(\int_{t_1}^{\infty}\left(\frac{\bar{G}(x_1,t_2)}{\bar{G}(t_1,t_2)}\right)^{\alpha}dx_1-1\right)$$

$$\Rightarrow \quad \frac{1}{(\alpha-1)}\left(1-\int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dx_1\right) \geq \frac{1}{(\alpha-1)}\left(1-\int_{t_1}^{\infty}\left(\frac{\bar{G}(x_1,t_2)}{\bar{G}(t_1,t_2)}\right)^{\alpha}dx_1\right)$$

$$\Rightarrow \quad \eta_{1\alpha}(X;t_1,t_2) \geq \eta_{1\alpha}(Y;t_1,t_2).$$

Similar result holds for $i=2$. Hence

$$\eta_{i\alpha}(X;t_1,t_2) \geq \eta_{i\alpha}(Y;t_1,t_2).$$

□

Gupta and Sankaran (1998) proposed the bivariate equilibrium distribution. Let $X=(X_1,X_2)$ be a bivariate positive random vector admitting an absolute continuous survival function $\bar{F}(x_1,x_2)$. Then its bivariate equilibrium distribution is the distribution of a random vector $Y=(Y_1,Y_2)$ such that the density function and survival function of $(Y_i|\,Y_j>t_j)$, $i,j=1,2.i\neq j$ are of the form:

$$\begin{aligned} g_i(t_i|Y_j>t_j) &= \frac{P(X_i>t_i|X_j>t_j)}{E(X_i|X_j>t_j)} \\ &= \frac{\bar{F}(t_1,t_2)}{\bar{F}_j(t_j)\,E(X_i|X_j>t_j)},\ \ i\neq j;\ i,j=1,2 \end{aligned} \tag{4.3.1}$$

and

$$\begin{aligned} \bar{G}_i(t_i|Y_j>t_j) &= \int_{t_i}^{\infty} g_i(u|Y_j>t_j)du \\ &= \frac{\bar{F}(t_1,t_2)\,m_i(t_1,t_2)}{\bar{F}_j(t_j)\,E(X_i|X_j>t_j)},\ \ i\neq j;\ i,j=1,2, \end{aligned} \tag{4.3.2}$$

respectively, for $t_1,t_2\geq 0$.

We define the residual Tsallis entropy for the bivariate random vector $X = (X_1, X_2)$ as follows:

$$H_\alpha(X; t_1, t_2) \;=\; (\, H_{1\alpha}(X; t_1, t_2)\,,\; H_{2\alpha}(X; t_1, t_2)\,)\,,$$

where

$$H_{i\alpha}(X; t_1, t_2) \;=\; \frac{1}{(\alpha - 1)} \left(1 - \int_{t_i}^{\infty} \left(\frac{f_i(x_i | X_j > t_j)}{\bar{F}_i(t_i | X_j > t_j)} \right)^{\alpha} dx_i \right), \; i, j = 1, 2, \; i \neq j\,.$$

In the following theorem we establish a relation between BDCRTE and residual Tsallis entropy corresponding to the bivariate equilibrium random vector $Y = (Y_1, Y_2)$.

Theorem 4.3.4:
Let $X = (X_1, X_2)$ be a non-negative random vector and $Y = (Y_1, Y_2)$ be the equilibrium random vector associate with X, then

$$H_{i\alpha}(Y; t_1, t_2) \;=\; \frac{\eta_{i\alpha}(X; t_1, t_2)}{m_i^{\alpha}(t_1, t_2)} + \frac{1 - \, m_i^{-\alpha}(t_1, t_2)}{(\alpha - 1)}; \;\; i = 1, 2, \;\forall\, \alpha > 0\,,\, \alpha \neq 1,$$

where $H_{i\alpha}(Y; t_1, t_2)$ denote the bivariate residual Tsallis entropy corresponding to Y and $m_i(t_1, t_2), i = 1, 2$ are the components of the bivariate mean residual life function.

Proof:
The bivariate residual Tsallis entropy corresponding to Y is defined as

$$H_{i\alpha}(Y; t_1, t_2) \;=\; \frac{1}{(\alpha - 1)} \left(1 - \int_{t_i}^{\infty} \left(\frac{g_i(y_i | Y_j > t_j)}{\bar{G}_i(t_i | Y_j > t_j)} \right)^{\alpha} dy_i \right); \; i, j = 1, 2, \; i \neq j\,. \tag{4.3.3}$$

When the equation (4.3.3) holds for $i = 1$, we have

$$H_{1\alpha}(Y; t_1, t_2) \;=\; \frac{1}{(\alpha - 1)} \left(1 - \int_{t_1}^{\infty} \left(\frac{g_1(y_1 | Y_2 > t_2)}{\bar{G}_1(t_1 | Y_2 > t_2)} \right)^{\alpha} dy_1 \right).$$

Applying the results of the equations (4.3.1) and (4.3.2), we get

$$\begin{aligned}
H_{1\alpha}(Y;t_1,t_2) &= \frac{1}{(\alpha-1)}\left(1-\frac{1}{m_1^{\alpha}(t_1,t_2)}\int\limits_{t_1}^{\infty}\left(\frac{\bar{F}(y_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dy_1\right)\\
&= \frac{1}{(\alpha-1)}\left(1-\frac{1}{m_1^{\alpha}(t_1,t_2)}\int\limits_{t_1}^{\infty}\left(\frac{\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha}dx_1\right)\\
&= \frac{\eta_{1\alpha}(X;t_1,t_2)}{m_1^{\alpha}(t_1,t_2)}+\frac{1-m_1^{-\alpha}(t_1,t_2)}{(\alpha-1)}.
\end{aligned}$$

Similar result holds for $i=2$. Therefore, we have

$$H_{i\alpha}(Y;t_1,t_2) = \frac{\eta_{i\alpha}(X;t_1,t_2)}{m_i^{\alpha}(t_1,t_2)}+\frac{1-m_i^{-\alpha}(t_1,t_2)}{(\alpha-1)},\ \ i=1,2,\ \forall\,\alpha>0,\ \alpha\neq 1.$$

□

Cox (1972) introduced the concept of proportional hazards model (PHM). Let X and X_θ be two continuous random variables with survival functions $\bar{F}_X(x)$ and $\bar{F}_{X_\theta}(x)$, respectively. The relation between survival functions of random life times is given by

$$\bar{F}_{X_\theta}(x) = \left[\bar{F}_X(x)\right]^{\theta},\ \ x\in R\ ,\theta>0,$$

where θ is a positive constant.

The following theorem is based on the proportional hazards model (PHM).

Theorem 4.3.5:

Let X and X_θ be two non-negative continuous random variables with survival functions $\bar{F}_X(x)$ and $\bar{F}_{X_\theta}(x)$ and dynamic cumulative residual Tsallis entropies $\eta_\alpha(X;t)$ and $\eta_\alpha(X_\theta;t)$, respectively. If $F_X(x)$ is the PH model of $F_{X_\theta}(x)$ then

(a) $\eta_\alpha(X_\theta;t)\ \geq\ \eta_\alpha(X;t)$ for $\theta\geq 1$, $\alpha>1$ and $0<\theta<1$, $0<\alpha<1$.

(b) $\eta_\alpha(X_\theta;t)\ \leq\ \eta_\alpha(X;t)$ for $\theta\geq 1$, $0<\alpha<1$ and $0<\theta<1$, $\alpha>1$.

Proof:

We know that $\bar{F}_{X_\theta}(x) = \left[\bar{F}_X(x)\right]^\theta,\ \theta > 0.$

(a) Case 1: When $\theta \geq 1,\ \alpha > 1$ then $\left(\bar{F}_X(x)\right)^{\theta\alpha} \leq \left(\bar{F}_X(x)\right)^\alpha$, this implies

$$\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} \leq \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^\alpha$$

$$\Rightarrow \quad \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} dx \leq \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^\alpha dx$$

$$\Rightarrow \quad 1 - \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} dx \geq 1 - \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^\alpha dx$$

$$\Rightarrow \quad \frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} dx\right) \geq \frac{1}{(\alpha-1)}\left(1 - \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^\alpha dx\right)$$

$$\Rightarrow \quad \eta_\alpha(X_\theta; t) \geq \eta_\alpha(X; t).$$

Case 2: When $0 < \theta < 1,\ 0 < \alpha < 1$ then $\left(\bar{F}_X(x)\right)^{\theta\alpha} \geq \left(\bar{F}_X(x)\right)^\alpha$, this implies

$$\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} \geq \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^\alpha$$

$$\Rightarrow \quad \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} dx \geq \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^\alpha dx$$

$$\Rightarrow \quad \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} dx - 1 \geq \int_t^\infty \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^\alpha dx - 1 \quad ,$$

$$\frac{1}{(1-\alpha)}\left(\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha}dx-1\right) \;\geq\; \frac{1}{(1-\alpha)}\left(\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}dx-1\right)$$

$$\Rightarrow \quad \frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha}dx\right) \;\geq\; \frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}dx\right)$$

$$\Rightarrow \quad \eta_\alpha(X_\theta;t) \;\geq\; \eta_\alpha(X;t).$$

(b) Case 1: When $\theta \geq 1,\ 0<\alpha<1$ then $\left(\bar{F}_X(x)\right)^{\theta\alpha} \;\leq\; \left(\bar{F}_X(x)\right)^{\alpha}$, this implies

$$\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} \;\leq\; \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}$$

$$\Rightarrow \quad \int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha}dx-1 \;\leq\; \int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}dx-1$$

$$\Rightarrow \quad \frac{1}{(1-\alpha)}\left(\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha}dx-1\right) \;\leq\; \frac{1}{(1-\alpha)}\left(\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}dx-1\right)$$

$$\Rightarrow \quad \frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha}dx\right) \;\leq\; \frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}dx\right)$$

$$\Rightarrow \quad \eta_\alpha(X_\theta;t) \;\leq\; \eta_\alpha(X;t).$$

Case 2: When $0<\theta<1,\ \alpha \geq 1$ then $\left(\bar{F}_X(x)\right)^{\theta\alpha} \;\geq\; \left(\bar{F}_X(x)\right)^{\alpha}$, this implies

$$\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha} \;\geq\; \left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}$$

$$\Rightarrow \quad 1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha}dx \;\leq\; 1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}dx$$

$$\Rightarrow \quad \frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\theta\alpha}dx\right) \;\leq\; \frac{1}{(\alpha-1)}\left(1-\int_t^{\infty}\left(\frac{\bar{F}_X(x)}{\bar{F}_X(t)}\right)^{\alpha}dx\right)$$

$$\Rightarrow \quad \eta_\alpha(X_\theta;t) \;\leq\; \eta_\alpha(X;t).$$

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4.4 Conclusion

The definition of dynamic cumulative residual Tsallis entropy (DCRTE) have been extended into bivariate setup consequently proposed the Bivariate Dynamic Cumulative Residual Tsallis Entropy. The monotonic behaviour in the context of bivariate random vector has also been studied. Some well known bivariate life time distributions are characterized. Additionally, we have defined some new classes of life distributions based on BDCRTE.

Chapter 5

Conclusions

If X is an absolutely continuous random variable and the distribution function $F_X(x)$ is log-concave then it is proved that the conditional Varma entropy $H_\alpha^\beta(X|A)$ is partially decreasing (increasing) in interval $A = (a, b)$. Further it is proved that if random variables X_1 and X_2 be independent and identically distributed copies of X and have log-concave probability density function then the conditional Varma's entropy of $U = |X_1 - X_2|$ given $B = \{a \leq X_1, X_2 \leq b\}$ is partially increasing (decreasing) function on B if $\alpha + \beta < 2(\alpha + \beta > 2)$.

The dynamic generalized information measure based on cumulative distribution function is more stable than based on density function. We proposed the Dynamic Cumulative Residual Tsallis Entropy which is found to be monotonic in nature. Based on the proposed DCRTE, we characterized some well known life time distributions such as exponential, Weibull, Pareto and the finite range distributions which play a vital role in reliability modeling. Hear we proposed weighted dynamic cumulative residual Tsallis entropy and examined its application in relation to weighted and equilibrium models. Finally, we introduced empirical cumulative Tsallis entropy for empirical samples. It is observed that the mean of empirical cumulative Tsallis entropy decreases and variance increases irrespective of sample size.

The definition of dynamic cumulative residual Tsallis entropy (DCRTE) have been extended to bivariate setup consequently proposed the Bivariate Dynamic Cumulative Residual Tsallis Entropy. The monotonic behaviour in the context of bivariate random vector has also been studied. Some well known bivariate life time distributions are characterized. Additionally, we have defined some a new class of life distributions based on BDCRTE.

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