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INFORMATION AND SIMILARITY MEASURES OF INTUITIONISTIC FUZZY AND SOFT SETS

BY

TANUJ KUMAR

UNDER THE SUPERVISION OF

RAKESH KUMAR BAJAJ

JAYPEE UNIVERSITY OF INFORMATION TECHNOLOGY

WAKNAGHAT

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DECLARATION

I hereby declare that the work reported in the Ph.D. thesis entitled, "**Information and Similarity Measures of Intuitionistic Fuzzy and Soft Sets**" submitted to **Jaypee University of Information Technology, Waknaghat, India,** is an authentic record of my work carried out under the supervision of **Dr. Rakesh Kumar Bajaj.** I have not submitted this work elsewhere for any other degree or diploma.

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विद्या तत्व ज्योतिसम

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CERTIFICATE

This is to certify that the thesis entitled, "**Information and Similarity Measures of Intuitionistic Fuzzy and Soft Sets**" which is being submitted by **Tanuj Kumar** in fulfillment for the award of degree of **Doctor of Philosophy** in **Mathematics** by the **Jaypee University of Information Technology,** is the record of candidate's own work carried out by him under my supervision. This work has not been submitted partially or wholly to any other University or Institute for the award of this or any other degree or diploma.

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Dedicated to

My Parents

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(TANUJ KUMAR)

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Abstract

Fuzzy sets, introduced by L.A. Zadeh (1965), provide a flexible framework for handling the uncertain situations, containing ambiguity and vagueness. Fuzzy sets find applications in several fields, such as reliability, marketing, image processing, pattern recognition, artificial intelligence, etc. However, fuzzy sets do not handle the situation, where the vague/incomplete/uncertain information involves some degree of hesitation. Atanassov (1986) introduced the concept of Intuitionistic Fuzzy Set (IFS) as a generalization of fuzzy set, which is found to be more useful in capturing the vague, incomplete or uncertain information that involves some degree of hesitation and applicable in various fields of research.

The objective of this thesis entitled, "Information and Similarity Measures of Intuitionistic Fuzzy and Soft Sets" is to study new intuitionistic fuzzy information measures, similarity measures, fuzzy linear regression model, intuitionistic fuzzy reliability and complex intuitionistic fuzzy soft sets with their entropies.

We present fundamental background of fuzzy set, intuitionistic fuzzy set, soft set and complex intuitionistic fuzzy set with their definitions and various properties. In addition to this, we have also presented application of these theories in statistical regression analysis, decision making and reliability evaluation of a system along with a brief literature survey in chapter 1.

In chapter 2, a new R-norm intuitionistic fuzzy entropy and R-norm intuitionistic fuzzy directed divergence measure have been proposed with their proof of validity. Further, empirical study on the proposed information measures has also been done which explains monotonic nature of the information measures with respect to R as well as the λ involved. Computational applications of these information measures in the field of pattern recognition and image thresholding has been proposed with discussion.

In chapter 3, a fuzzy linear regression model with some restrictions in the form of prior information has been considered. The estimators of regression coefficients have been obtained with the help of fuzzy entropy for the restricted/ un-

restricted fuzzy linear regression model by assigning some weights in the distance function. Some numerical examples have also been provided in order to illustrate the proposed model along with the obtained weighted estimators. Further, in order to compare the performance of unrestricted estimator and restricted estimator, a simulation study has been conducted by using two fundamental criteria of dominance-mean squared error matrix and absolute bias.

In chapter 4, we have proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form. The proposed similarity measures have been analogously extended to obtain new entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets along with their proofs of validity. A new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measure in which the weights have been calculated using the proposed entropies. Further, numerical example for illustrating the proposed methodology has also been provided by taking interval-valued intuitionistic fuzzy sets.

In chapter 5, we compute the reliability of k-out-of-n : G-system (particularly, series and parallel system) with independent and non-identically distributed components, where the reliability of the components are unknown. The reliability of each component has been estimated using statistical confidence interval approach. Then we converted these statistical confidence interval into triangular intuitionistic fuzzy numbers. Based on these triangular intuitionistic fuzzy numbers, the reliability of the k-out-of-n : G-system has been calculated. Further, in order to implement the proposed methodology and to analyze the results of k-out-of-n : G-system, a numerical example has been provided.

In chapter 6, we introduce the concept of complex intuitionistic fuzzy soft sets which is parametric in nature. However, the theory of complex fuzzy sets and complex intuitionistic fuzzy sets are independent of the parametrization tools. Some real life problems, for example, multi-criteria decision making problems, involve the parametrization tools. In order to get their new entropies, some important properties and operations on the complex intuitionistic fuzzy soft sets have also been discussed. On the basis of some well-known distance measures, some new distance measures for the complex intuitionistic fuzzy soft sets have also been obtained. Further, we have established correspondence between the proposed entropies and the distance measures of complex intuitionistic fuzzy soft sets.

In chapter 7, we present the conclusions.

Publications Based on Present Work

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2. Bajaj R., Kumar T. and Gupta N. "R-norm Intuitionistic Fuzzy Information Measures and its Computational Applications," *Springer: Communications in Computer and Information Science*, **305 (8)**, pp. 372-380, 2012. [**Springer, ISI Index, DBLP, Scopus**]

3. Kumar T. and Bajaj R. "Reliability Analysis of the k-out-of-$n : G$-System Using Triangular Intuitionistic Fuzzy Numbers," *Inter. Journal of Mathematical, Computational Science & Engineering*, **8(2)**, pp. 64-70, 2014.

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Chapter 1

Introduction

1.1 Background and Motivation

Fuzzy sets, introduced by Zadeh (1965), provide a flexible framework for handling non-statistical imprecision or vague concepts. It has been designed to represent uncertainty and vagueness mathematically to provide formalized tools for dealing with the imprecision intrinsic to many real world problems. Fuzzy set theory has found wide applications in many areas of science and technology, e.g., clustering, image processing, decision making etc. because of its capability to describe the uncertain situations, containing ambiguity and vagueness.

However, fuzzy sets do not handle the situation, where the uncertain information involves some degree of hesitation which mainly arises from the imprecise and/or imperfect nature of the information. Atanassov (1986) introduced the concept of Intuitionistic Fuzzy Set (IFS) as a generalization of fuzzy set, which is found to be more useful in capturing the vague, incomplete or uncertain information that involves some degree of hesitation and applicable in various fields of research. A prominent characteristic of IFS is that it assigns to each element a membership degree and a non-membership degree with certain amount of hesitation degree. Therefore, due to the feasibility and effectiveness of IFSs in various engineering applications, intuitionistic fuzzy sets techniques have been more popular than fuzzy sets techniques in recent years.

1.2 Basic Concepts of Fuzzy Sets

The concept of fuzzy set was first proposed by Lofti A. Zadeh (1965) as a generalization of a crisp set. A crisp set is characterized by a characteristic function while a fuzzy set is characterized by a membership function where an object belongs to a fuzzy set with a continuum grade of membership ranging between zero and one.

Definition 1 (Fuzzy Set): Let X be the universal set. A fuzzy set A in X is characterized by its membership function

$$\mu_A : X \to [0,\, 1]$$

and denoted by

$$A = \{x,\, \mu_A(x) : x \in X\},$$

where $\mu_A(x)$ denotes the degree of membership of an element x in fuzzy set A.

Definition 2: (Support of a Fuzzy Set) The support of a fuzzy set A is the set of all points x in X such that $\mu_A(x) > 0$, i.e., Supp(A)= $\{x \in X \mid \mu_A(x) > 0\}$.

Definition 3: (α-cut) The α-cut or α-level set of A is a crisp set defined by $A_\alpha = \{x|\mu_A(x) \geq \alpha\}$, $\alpha \in [0,\, 1]$.
Similarly, the strong α-cut is defined as $A'_\alpha = \{x|\mu_A(x) > \alpha\}$, $\alpha \in [0,\, 1]$.

Definition 4 (Convexity): A fuzzy set A is convex if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1),\, \mu_A(x_2)\}, x_1,\, x_2 \in X,\, \lambda \in [0,\, 1].$$

Alternatively, a fuzzy set is convex if all α-cuts are convex.

Mathematically, a *fuzzy number* is a convex and normalized fuzzy set whose membership function is at least segmentally continuous having bounded support and has the functional value $\mu_A(x) = 1$ at precisely one element which is called modal value of the fuzzy number. Among the various shapes of fuzzy number in the literature of fuzzy set theory, triangular fuzzy number (TFN) is one of the most popular type of fuzzy number.

A *triangular fuzzy number* $\tilde{A}$ is defined by the membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & \text{where } x \in [a,\, b] \\ \frac{(c-x)}{(c-b)}, & \text{where } x \in (b,\, c] \\ 0 \quad , & \text{otherwise,} \end{cases}$$

where b is known as the model value for which $\mu_{\tilde{A}}(b) = 1$ and $b - a > 0$ and $c - b > 0$ are the left and right spread of $\tilde{A}$, respectively.

Definition 5 (t-norm): A binary operation $* : [0,\, 1] \times [0,\, 1] \to [0,\, 1]$ is continuous t-norm if $*$ satisfies the following properties:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a, \quad \forall a \in [0,\, 1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a,\, b,\, c,\, d \in [0,\, 1]$.

Some examples of continuous t-norm are $a*b = ab$, $a*b = \min\{a,\, b\}$, $a*b = \max\{a + b - 1,\, 0\}$.

Definition 6 (s-norm): A binary operation $\diamond : [0,\, 1] \times [0,\, 1] \to [0,\, 1]$ is continuous s-norm if $\diamond$ satisfies the following properties:

(i) $\diamond$ is commutative and associative;

(ii) $\diamond$ is continuous;

(iii) $a \diamond 1 = a, \quad \forall a \in [0,\, 1]$;

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$, $b \leq d$ and $a,\, b,\, c,\, d \in [0,\, 1]$.

Some examples of continuous s-norm are $a \diamond b = a + b - ab$, $a \diamond b = \max\{a,\, b\}$, $a \diamond b = \min\{a + b,\, 1\}$.

1.3 Intuitionistic Fuzzy Sets

Out of several generalizations of fuzzy set for various objectives, the notion of intuitionistic fuzzy set introduced by Atanassov [(1986), (1989)] has become more popular and highly useful in dealing with vague and imprecise information. The idea of intuitionistic fuzzy set provides a flexible mathematical framework to handle the vagueness having hesitancy originating from imperfect or imprecise information. For example, in decision making problems, particularly in the case of medial diagnosis, sales analysis, new product marketing, financial services, etc. there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object. Thus, a prominent characteristic of IFS is that it assigns to each element in the universe a membership degree and a non-membership degree with certain amount of hesitation degree. Since intuitionistic fuzzy sets allow two degrees of freedom into a set description, and fuzzy sets only allow one, this generalization gives us an additional possibility to represent the lack of information that leads when we try to describe many real problems. Thus, it becomes more convenient to model the situations where human answers are present as 'yes', 'no' or 'does not apply'. A good example of these kind of situations is voting, since human voters can be divided into three groups: vote for, vote against or abstain. Intuitionistic fuzzy sets are characterized by the membership and non-membership functions expressing the degree of membership (belongingness) and the degree of non-membership (non-belongingness) of elements of the universe to the IFS with some degree of hesitation.

Definition 7 (Intuitionistic Fuzzy Set): Atanassov's Intuitionistic Fuzzy Set over the universal set X, is given by

$$\tilde{A} = \{\langle x,\, \mu_{\tilde{A}}(x),\, \nu_{\tilde{A}}(x)\rangle : x \in X\},$$

where $\mu_{\tilde{A}} : X \to [0, 1]$ and $\nu_{\tilde{A}} : X \to [0, 1]$ with the condition $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$, $\forall x \in X$. The numbers $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ denotes the degree of membership and non-membership of an element x to a set $\tilde{A}$ respectively. For each element $x \in X$, the amount $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$ is called the degree of indeterminacy (hesitancy). It is the degree of uncertainty whether x belongs to $\tilde{A}$ or not. We denote $\mathcal{IFS}(X)$ the set of all the IFSs on X.

Definition 8 (**Basic Operations and Relations on Intuitionistic Fuzzy Sets**): Let $\tilde{A}$ and $\tilde{B}$ are two IFSs belonging to $\mathcal{IFS}(X)$, then the following operations have been defined as

- **Union:** $\tilde{A} \cup \tilde{B} = \{\langle x, \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}\rangle | x \in X\}$;
- **Intersection:** $\tilde{A} \cap \tilde{B} = \{\langle x, \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}\rangle | x \in X\}$;
- **Complement:** $\tilde{A}^c = \{\langle x, \nu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)\rangle | x \in X\}$;
- **Inclusion:** $\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) \leq \nu_{\tilde{B}}(x), \nu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x), \forall x \in X$;
- **Equality:** $\tilde{A} = \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x), \nu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \forall x \in X$.

Definition 9 (Vague Set): A vague set $V = \{\langle x, [\mu_{\tilde{V}}(x), 1 - \nu_{\tilde{v}}(x)]\rangle : x \in X\}$, on the universal set X is characterized by a true membership function $\mu_{\tilde{V}} : X \to [0, 1]$ and a false membership (non-membership) function $\nu_{\tilde{V}} : X \to [0, 1]$. The values $\mu_V(x)$ and $\nu_V(x)$ represents the degree of truth membership and degree of false membership of x and always satisfies the condition $0 \leq \mu_{\tilde{V}}(x) + \nu_{\tilde{V}}(x) \leq 1$, for all $x \in X$. The value $1 - \mu_{\tilde{V}}(x) - \nu_{\tilde{V}}(x)$ represents the degree of hesitation of $x \in X$.

The value $\mu_{\tilde{V}}(x)$ is considered as the lower bound of the grade of membership of x derived from the evidence for x and $\nu_{\tilde{V}}(x)$ is the lower bound of the grade of membership of the negation of x derived from the evidence against x. Thus, the grade of membership of x in the vague set $\tilde{A}$ is bounded by a sub-interval $[\mu(x), 1 - \nu(x)]$ of $[0, 1]$. For example, if the membership value of x in vague set $\tilde{V}$ on the universal set X is $[0.5, 0.7]$, then $\mu_{\tilde{V}}(x) = 0.5$ and $1 - \nu_{\tilde{V}}(x) = 0.7$ or

$\nu_{\tilde{V}}(x) = 0.3$. This means that x belongs to vague set $\tilde{V}$ with accept evidence is 0.5, decline evidence is 0.3.

We can see that the difference between vague set and intuitionistic fuzzy set is due to the definition of membership intervals. We have $[\mu_{\tilde{V}}(x), 1 - \nu_{\tilde{V}}(x)]$ for x in $\tilde{V}$ but $\langle \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle$ for x in $\tilde{A}$. Here the semantics of $\mu_{\tilde{A}}$ is same as with $\mu_{\tilde{V}}$ and $\nu_{\tilde{A}}$ is the same as with $\nu_{\tilde{V}}$. However, the boundary $(1 - \nu_{\tilde{V}}(x))$ is able to indicate the possible existence of a data value, as already mentioned by Bustince and Burillo (1996b). This subtle difference gives rise to a simpler but meaningful graphical view of data sets. We now depict a vague set in figure 1.1 and an IFS in figure 1.2, respectively. It can be seen that the shaded part, formed by the boundary in a given vague set in figure 1.1, represents the possible existence of data. Thus, this "hesitation region" corresponds to the intuition of representing the vague data.

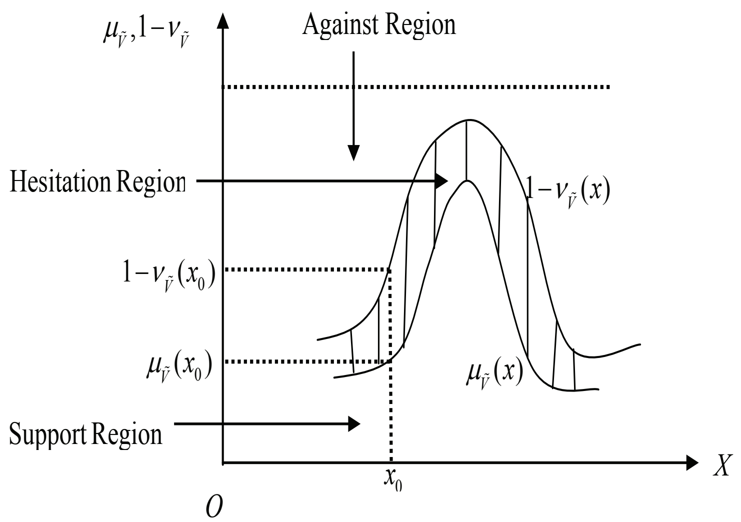

Figure 1.1: *Vague Set*

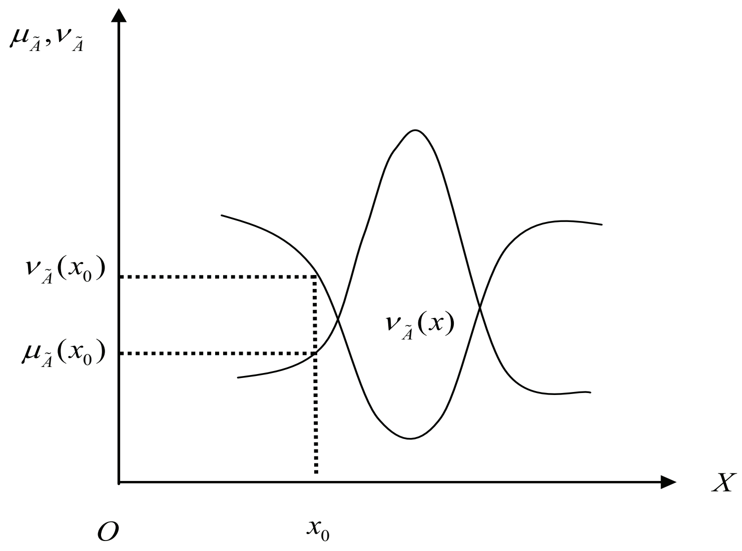

Figure 1.2: *Intuitionistic Fuzzy Set*

Definition 10 (α-Cut of Vague Set or IFS): The α-cut of a membership function, is a crisp set which consists of elements of $\tilde{A}$ having at least degree α. It is denoted by $\tilde{A}^{\mu}(\alpha)$ and is defined mathematically as

$$\tilde{A}^{\mu}(\alpha) = \{x : \mu_{\tilde{A}}(x) \geq \alpha,\, x \in X\},\, \alpha \in [0,\, 1],$$

while for the non-membership function, it is defined as

$$\tilde{A}^{\nu}(\alpha) = \{x : 1 - \nu_{\tilde{A}}(x) \geq \alpha,\, x \in X\},\, \alpha \in [0,\, 1].$$

Definition 11 (Intuitionistic Fuzzy Number): An intuitionistic fuzzy subset $\tilde{A} = \{\langle x,\, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle :\, x \in X\}$ of the real line $\mathbb{R}$ is called an intuitionistic fuzzy number if the following axioms hold:

(i) $\tilde{A}$ is normal, i.e., there exist at least two points $x_1,\, x_2 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_1) = 1$ and $\nu_{\tilde{A}}(x_2) = 0$;

(ii) The membership function $\mu_{\tilde{A}}$ is fuzzy-convex, i.e.,

$$\mu_{\tilde{A}}(\lambda \cdot x_1 + (1 - \lambda) \cdot x_2) \geq \min\{\mu_{\tilde{A}}(x_1),\, \mu_{\tilde{A}}(x_2)\}\ \ \forall x_1,\, x_2 \in X,\ \lambda \in [0,\, 1];$$

(iii) The non-membership function $\nu_{\tilde{A}}$ is fuzzy-concave, i.e.,

$$\nu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda) \cdot x_2) \leq \max\{\nu_{\tilde{A}}(x_1),\, \nu_{\tilde{A}}(x_2)\}\ \forall x_1,\, x_2 \in X,\ \lambda \in [0,\, 1];$$

Definition 12 (Triangular Intuitionistic Fuzzy Number): A Triangular Intuitionistic Fuzzy Number (TIFN) $\tilde{A}$ is an intuitionistic fuzzy set in $\mathbb{R}$ with following membership function $\mu_{\tilde{A}}(x)$ and non-membership function $\nu_{\tilde{A}}(x)$:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{for } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & \text{for } a_2 \leq x \leq a_3 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x}{a_2-a_1'}, & \text{for } a_1' \leq x \leq a_2 \\ \frac{x-a_2}{a_3'-a_2}, & \text{for } a_2 \leq x \leq a_3' \\ 1, & \text{otherwise} \end{cases}$$

where $a_1' < a_1 < a_2 < a_3 < a_3'$ and $\mu_{\tilde{A}}(x),\, \nu_{\tilde{A}}(x) \leq 0.5$ for $\mu_{\tilde{A}}(x) = \nu_{\tilde{A}}(x)$, $\forall x \in \mathbb{R}$, and TIFN is denoted by $\tilde{A}_{TIFN} = (a_1,\, a_2,\, a_3,\, a_1',\, a_2',\, a_3')$.

Definition 13 (α-Cut of Intuitionistic Fuzzy Number): The α-cut representation of an IFN $\tilde{A} = \langle (a,\, b,\, c); \mu,\, \nu \rangle$ defined on $\mathbb{R}$, is given by the following pair of intervals and denoted by

$$\left(\tilde{A}(\alpha_\mu) = \left[A^l(\alpha_\mu),\, A^r(\alpha_\mu)\right];\, \tilde{A}(\alpha_\nu) = \left[A^l(\alpha_\nu),\, A^r(\alpha_\nu)\right]\right),$$

where $A^l(\alpha_\mu)$, $A^l(\alpha_\nu)$ are the increasing functions and $A^r(\alpha_\mu)$, $A^r(\alpha_\nu)$ are decreasing functions of α_μ and α_ν, respectively.

The intervals of confidence defined by the α-cut of TIFN $\tilde{A}$ are given by

$$\tilde{A}(\alpha_\mu) = \left[a + \frac{\alpha_\mu}{\mu}(b-a),\, c - \frac{\alpha_\mu}{\mu}(c-b)\right],\ \forall \alpha_\mu \in [0,\, \mu],$$

and

$$\tilde{A}(\alpha_\nu) = \left[a + \frac{\alpha_\nu}{(1-\nu)}(b-a),\, c - \frac{\alpha_\nu}{(1-\nu)}(c-b)\right],\ \forall \alpha_\nu \in [0,\, 1-\nu].$$

1.4 Intuitionistic Fuzzy Information Measures

Szmidt and Kacprzyk (2001) extended the axioms of De Luca and Termini (1972) and proposed the following definition for an entropy measure of intuitionistic fuzzy set $\tilde{A} \in \mathcal{IFS}(X)$:

- **(IFS1) :** $H(\tilde{A}) = 0$ iff $\tilde{A}$ is a crisp set, i.e., $\mu_{\tilde{A}}(x_i) = 0$ & $\nu_{\tilde{A}}(x_i) = 1$ or $\mu_{\tilde{A}}(x_i) = 1$ & $\nu_{\tilde{A}}(x_i) = 0,\ \forall x_i \in X$.
- **(IFS2) :** $H(\tilde{A}) = 1$ iff $\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i), \forall x_i \in X$.
- **(IFS3) :** $H(\tilde{A}) \leq H(\tilde{B})$ iff $\tilde{A}$ is less fuzzy than $\tilde{B}$, i.e., $\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i)$ & $\nu_{\tilde{A}}(x_i) \geq \nu_{\tilde{B}}(x_i)$, for $\mu_{\tilde{B}}(x_i) \leq \nu_{\tilde{B}}(x_i)$ or $\mu_{\tilde{A}}(x_i) \geq \mu_{\tilde{B}}(x_i)$ & $\nu_{\tilde{A}}(x_i) \leq \nu_{\tilde{B}}(x_i)$, for $\mu_{\tilde{B}}(x_i) \geq \nu_{\tilde{B}}(x_i)$, $\forall x_i \in X$.
- **(IFS4) :** $H(\tilde{A}) = H(\tilde{A}^c)$, where $\tilde{A}^c$ is complement of $\tilde{A}$.

Vlachos and Sergiadis (2007) proposed the following intuitionistic fuzzy entropy of $\tilde{A}$ and cross-entropy between two IFSs $\tilde{A}$ & $\tilde{B}$:

$$I(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \left[\mu_{\tilde{A}}(x_i) \ln \frac{\mu_{\tilde{A}}(x_i)}{\frac{1}{2}\mu_{\tilde{A}}(x_i) + \frac{1}{2}\nu_{\tilde{B}}(x_i)} + \nu_{\tilde{A}}(x_i) \ln \frac{\nu_{\tilde{A}}(x_i)}{\frac{1}{2}\mu_{\tilde{A}}(x_i) + \frac{1}{2}\nu_{\tilde{B}}(x_i)} \right]$$

and

$$H_{LT}(\tilde{A}) = -\frac{1}{n \ln 2} \sum_{i=1}^{n} \left[\mu_{\tilde{A}}(x_i) \ln \left(\frac{\mu_{\tilde{A}}(x_i)}{\mu_{\tilde{A}}(x_i) + \nu_{\tilde{A}}(x_i)} \right) + \nu_{\tilde{A}}(x_i) \ln \left(\frac{\nu_{\tilde{A}}(x_i)}{\mu_{\tilde{A}}(x_i) + \nu_{\tilde{A}}(x_i)} \right) - \pi_{\tilde{A}}(x_i) \ln 2 \right],$$

respectively.

Hung & Yang (2006) introduced the following axiomatic definition of IFS entropy in a probabilistic setting as a real valued functional $H : \mathcal{IFS}(X) \rightarrow \mathbb{R}^+$:

- **IE1(Sharpness):** $H(\tilde{A}) = 0$ iff $\tilde{A}$ is a crisp set.
- **IE2(Maximality):** $H(\tilde{A})$ assumes a unique maximum if
$$\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = \tfrac{1}{3}, \forall x_i \in X.$$
- **IE3(Resolution):** $H(\tilde{A}) \leq H(\tilde{B})$ if $\tilde{A}$ is crisper than $\tilde{B}$, i.e., if $\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i)$ & $\nu_{\tilde{A}}(x_i) \leq \nu_{\tilde{B}}(x_i)$, for $\max\{\mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i)\} \leq \frac{1}{3}$ and $\mu_{\tilde{A}}(x_i) \geq \mu_{\tilde{B}}(x_i)$ & $\nu_{\tilde{A}}(x_i) \geq \nu_{\tilde{B}}(x_i)$, for $\min\{\mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i)\} \geq \frac{1}{3}, \forall x_i \in X$.

- **IE4(Symmetry):** $H(\tilde{A}^c) = H(\tilde{A})$.

Under the above axioms, Hung and Yang (2006) proposed two families of fuzzy entropies of an IFS $\tilde{A}$ given by

$$H_{hc}^{\alpha}(\tilde{A}) = \begin{cases} \frac{1}{(\alpha-1)n}\sum_{i=1}^{n}\left[1-\left(\mu_{\tilde{A}}^{\alpha}(x_i)+\nu_{\tilde{A}}^{\alpha}(x_i)+\pi_{\tilde{A}}^{\alpha}(x_i)\right)\right]; \quad \alpha \neq 1(\alpha > 0) \\ \\ -\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}(x_i)\log\mu_{\tilde{A}}(x_i)+\nu_{\tilde{A}}(x_i)\log\nu_{\tilde{A}}(x_i)+\pi_{\tilde{A}}(x_i)\log\pi_{\tilde{A}}(x_i)\right); \alpha = 1, \end{cases}$$

and

$$H_r^{\beta}(\tilde{A}) = \frac{1}{1-\beta}\sum_{i=1}^{n}\log\left(\mu_{\tilde{A}}^{\beta}(x_i)+\nu_{\tilde{A}}^{\beta}(x_i)+\pi_{\tilde{A}}^{\beta}(x_i)\right); \ 0 < \beta < 1.$$

1.5 Interval-valued Intuitionistic Fuzzy Sets

In many real-world decision problems the values of the membership function and the non-membership function in an IFS are difficult to be expressed as exact numbers. Instead, the ranges of their values can usually be specified. In such cases, Atanassov and Gargov (1989) generalized the concept of IFS to interval-valued intuitionistic fuzzy set (IVIFS), and define some basic operational laws of IVIFSs. In this section, we present the basics of interval-valued intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy numbers which are well known in literature.

Definition 14 (Interval-Valued Intuitionistic Fuzzy Number): Let X be a universe of discourse and $\mathrm{int}\,(0,\,1)$ be the set of all closed subintervals of the interval $[0,\,1]$. An interval-valued intuitionistic fuzzy set (IVIFS) $\tilde{A}_*$ in X is an object having the form:

$$\tilde{A}_* = \left\{\left\langle x,\, m_{\tilde{A}_*}^{\mu}(x),\, n_{\tilde{A}_*}^{\nu}(x)\right\rangle | \, x \in X\right\}$$

where $m_{\tilde{A}_*}^{\mu} : X \to \mathrm{int}\,(0,\,1)$, $n_{\tilde{A}_*}^{\nu} : X \to \mathrm{int}\,(0,\,1)$, with the condition

$$0 \leq sup\left(m_{\tilde{A}_*}^{\mu}(x)\right) + sup\left(n_{\tilde{A}_*}^{\nu}(x)\right) \leq 1,\ \forall x \in X.$$

Here, the intervals $m^{\mu}_{\tilde{A}_*}(x) = [\mu^L_{\tilde{A}_*}(x), \mu^U_{\tilde{A}_*}(x)]$ and $n^{\nu}_{\tilde{A}_*} = [\nu^L_{\tilde{A}_*}(x), \nu^U_{\tilde{A}_*}(x)]$ denote the degree of the membership and the non-membership of an element x belonging to IVIFS $\tilde{A}_*$, respectively. For each IVIFS $\tilde{A}_*$ in X, the amount $\pi_{\tilde{A}_*}(x) = \left[1 - \mu^U_{\tilde{A}_*}(x) - \nu^U_{\tilde{A}_*}(x), 1 - \nu^L_{\tilde{A}_*}(x) - \nu^L_{\tilde{A}_*}(x)\right]$, is called the interval-valued intuitionistic index of x in $\tilde{A}_*$, which is a hesitancy degree of x to $\tilde{A}_*$. It is the degree of uncertainty whether an element x belongs to $\tilde{A}_*$ or not. We denote $\mathcal{IVIFS}(X)$ the set of all the IVIFSs on X.

Definition 15 (Basic Operations and Relations on IVIFSs): For all $x \in X$ and $\tilde{A}_*, \tilde{B}_* \in \mathcal{IVIFS}(X)$, the following relations and operations have been defined as follows:

- **Union:** $\tilde{A}_* \cup \tilde{B}_* = \left\{\left\langle x, m^{\mu}_{\tilde{A}_* \cup \tilde{B}_*}(x), n^{\nu}_{\tilde{A}_* \cup \tilde{B}_*}(x) | x \in X\right\rangle | x \in X\right\}$, where

$$m^{\mu}_{\tilde{A}_* \cup \tilde{B}_*}(x) = \left[\min\left\{\inf\left(m^{\mu}_{\tilde{A}_*}(x)\right), \inf\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\}, \max\left\{\sup\left(m^{\mu}_{\tilde{A}_*}(x)\right), \sup\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\}\right],$$
$$n^{\nu}_{\tilde{A}_* \cup \tilde{B}_*}(x) = \left[\max\left\{\inf\left(n^{\nu}_{\tilde{A}_*}(x)\right), \inf\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\}, \min\left\{\sup\left(n^{\nu}_{\tilde{A}_*}(x)\right), \sup\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\}\right].$$

- **Intersection:** $\tilde{A}_* \cap \tilde{B}_* = \left\{\left\langle x, m^{\mu}_{\tilde{A}_* \cap \tilde{B}_*}(x), n^{\nu}_{\tilde{A}_* \cap \tilde{B}_*}(x)\right\rangle | x \in X\right\}$, where

$$m^{\mu}_{\tilde{A}_* \cap \tilde{B}_*}(x) = \left[\max\left\{\inf\left(m^{\mu}_{\tilde{A}_*}(x)\right), \inf\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\}, \min\left\{\sup\left(m^{\mu}_{\tilde{A}_*}(x)\right), \sup\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\}\right],$$
$$n^{\nu}_{\tilde{A}_* \cap \tilde{B}_*}(x) = \left[\min\left\{\inf\left(n^{\nu}_{\tilde{A}_*}(x)\right), \inf\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\}, \max\left\{\sup\left(n^{\nu}_{\tilde{A}_*}(x)\right), \sup\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\}\right].$$

- **Addition:** $\tilde{A}_* + \tilde{B}_* = \left\{\left\langle x, m^{\mu}_{\tilde{A}_* + \tilde{B}_*}(x), n^{\nu}_{\tilde{A}_* + \tilde{B}_*}(x)\right\rangle | x \in X\right\}$, where

$$\begin{aligned} m^{\mu}_{\tilde{A}_* + \tilde{B}_*}(x) &= \Big[\inf\left(m^{\mu}_{\tilde{A}_*}(x)\right) + \inf\left(m^{\mu}_{\tilde{B}_*}(x)\right) - \inf\left(m^{\mu}_{\tilde{A}_*}(x)\right) \cdot \inf\left(m^{\mu}_{\tilde{B}_*}(x)\right), \\ &\quad \sup\left(m^{\mu}_{\tilde{A}_*}(x)\right) + \sup\left(m^{\mu}_{\tilde{B}_*}(x)\right) - \sup\left(m^{\mu}_{\tilde{A}_*}(x)\right) \cdot \sup\left(m^{\mu}_{\tilde{B}_*}(x)\right)\Big], \\ n^{\nu}_{\tilde{A}_* + \tilde{B}_*}(x) &= \left[\inf\left(n^{\nu}_{\tilde{A}_*}(x)\right) + \inf\left(n^{\nu}_{\tilde{B}_*}(x)\right), \sup\left(n^{\nu}_{\tilde{A}_*}(x)\right) + \sup\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right]. \end{aligned}$$

- **Product:** $\tilde{A}_* \cdot \tilde{B}_* = \left\{\left\langle x, m^{\mu}_{\tilde{A}_* \cdot \tilde{B}_*}(x), n^{\nu}_{\tilde{A}_* \cdot \tilde{B}_*}(x)\right\rangle | x \in X\right\}$, where

$$\begin{aligned} m^{\mu}_{\tilde{A}_* \cdot \tilde{B}_*}(x) &= \left[\inf\left(m^{\mu}_{\tilde{A}_*}(x)\right) + \inf\left(m^{\mu}_{\tilde{B}_*}(x)\right), \sup\left(m^{\mu}_{\tilde{A}_*}(x)\right) + \sup\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right], \\ n^{\nu}_{\tilde{A}_* \cdot \tilde{B}_*}(x) &= \Big[\inf\left(n^{\nu}_{\tilde{A}_*}(x)\right) + \inf\left(n^{\nu}_{\tilde{B}_*}(x)\right) - \inf\left(n^{\nu}_{\tilde{A}_*}(x)\right) \cdot \inf\left(n^{\nu}_{\tilde{B}_*}(x)\right), \\ &\quad \sup\left(n^{\nu}_{\tilde{A}_*}(x)\right) + \sup\left(n^{\nu}_{\tilde{B}_*}(x)\right) - \sup\left(n^{\nu}_{\tilde{A}_*}(x)\right) \cdot \sup\left(n^{\nu}_{\tilde{B}_*}(x)\right)\Big]. \end{aligned}$$

- **Complement:** $\tilde{A}^c_* = \left\{ \left\langle x, [\nu^L_{\tilde{A}_*}(x), \nu^R_{\tilde{A}_*}(x)], [\mu^L_{\tilde{A}_*}(x), \mu^R_{\tilde{A}_*}(x)] \right\rangle \right\}$;

- **Subset:** $\tilde{A}_* \subseteq \tilde{B}_* \Leftrightarrow \mu^L_{\tilde{A}_*}(x) \leq \mu^L_{\tilde{B}_*}(x)$, $\mu^U_{\tilde{A}_*}(x) \leq \mu^U_{\tilde{B}_*}$, $\nu^L_{\tilde{A}_*}(x) \geq \nu^L_{\tilde{B}_*}(x)$ and $\nu^U_{\tilde{A}_*}(x) \geq \nu^U_{\tilde{B}_*}$.

- **Domination:** $\tilde{A}_* \preceq \tilde{B}_* \Leftrightarrow \mu^L_{\tilde{A}_*}(x) \leq \mu^L_{\tilde{B}_*}(x)$, $\mu^U_{\tilde{A}_*}(x) \leq \mu^U_{\tilde{B}_*}$, $\nu^L_{\tilde{A}_*}(x) \leq \nu^L_{\tilde{B}_*}(x)$ and $\nu^U_{\tilde{A}_*}(x) \leq \nu^U_{\tilde{B}_*}$.

- **Equality:** $\tilde{A}_* = \tilde{B}_* \Leftrightarrow \mu^L_{\tilde{A}_*}(x) = \mu^L_{\tilde{B}_*}(x)$, $\mu^U_{\tilde{A}_*}(x) = \mu^U_{\tilde{B}_*}$, $\nu^L_{\tilde{A}_*}(x) = \nu^L_{\tilde{B}_*}(x)$ and $\nu^U_{\tilde{A}_*}(x) = \nu^U_{\tilde{B}_*}$.

1.6 Complex Intuitionistic Fuzzy Sets

Ramot *et al.* [(2002), (2003)] introduced a new innovative concept of complex fuzzy set (CFS), where the membership function μ instead of being a real valued function with the range of [0, 1] is replaced by a complex-valued function of the form $r_A(x) \cdot e^{i\Omega_A(x)}$, $(i = \sqrt{-1})$, where $r_A(x)$ is a real valued function such that $r_A(x) \in [0, 1]$ and $\Omega_A(x)$ is a periodic function. The key feature of complex fuzzy sets is the presence of phase and its membership. Several examples are given in Ramot *et al.* (2003), which demonstrate the utility of these complex fuzzy sets. They also defined several important operations such as complement, union, intersection and discussed fuzzy relations for such complex fuzzy sets. On the other hand, Jun *et al.* (2012) used the complex fuzzy set to represent the information with uncertainty and periodicity, where they introduced a product-sum aggregation operator based prediction (PSAOP) method to find the solution of the multiple periodic factor prediction (MPFP) problems. Further, Chen *et al.* (2011) proposed a neurofuzzy system architecture to implement the complex fuzzy rule as a practical application of the concept of complex fuzzy logic.

Definition 16 (Complex Fuzzy Set): A complex fuzzy set A, defined on universal set X, is characterized by the membership function $\mu_A(x)$, which assign to each element $x \in X$ a complex-valued grade of membership in A.

The a complex fuzzy set A may be represented as the set of ordered pairs

$$A = \{\langle x,\, \mu_A(x)\rangle : x \in X\},$$

where $\mu_A(x) : X \to \{a | a \in \mathcal{C},\, |a| \leq 1\}$.

It may be noted that the membership function $\mu_A(x)$ receives the values lying within the unit circle in the complex plane and are of the form

$$\mu_A(x) = r_A(x) \cdot e^{i\Omega_A(x)}, (i = \sqrt{-1}),$$

where r_A is a real valued function such that $r_A(x) \in [0,\, 1]$ and Ω_A is a periodic function whose periodic law and principal period are respectively, 2π and $0 \leq \omega_A(x) \leq 2\pi$, i.e., $\Omega_A(x) = \omega_A(x) + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, where $\omega_A(x)$ is the principal argument.

Definition 17 (Operations on Complex Fuzzy Set): Let A and B be two complex fuzzy sets on X, where $\mu_A(x) = r_A(x) \cdot e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{i\omega_B(x)}$ are their membership functions, respectively. Then their set theoretic operations have been defined as follows:

- **Complement:** $\mu_{\bar{A}}(x) = (1 - r_A(x)) \cdot e^{i\omega_{\bar{A}}(x)}$;
- **Union:** $A \bigcup B = \left\{\langle x,\, r_A(x) \diamond r_B(x) \cdot e^{\omega_{A\cup B}(x)}\rangle | x \in X\right\}$;
- **Intersection:** $A \bigcap B = \left\{\langle x,\, r_A(x) * r_B(x) \cdot e^{\omega_{A\cup B}(x)}\rangle | x \in X\right\}$,

where the $\diamond$ and $*$ are s-norm and t-norm operators, respectively.

Ramot *et al.* (2002) obtained several possible methods for calculating the membership phase of complex fuzzy complement, $\omega_{\bar{C}}(x)$. For example, $\omega_{\bar{C}}(x)$ may be defined as $\omega_{\bar{C}}(x) = \omega_C(x)$ or $\omega_{\bar{C}}(x)$ may be defined by the relation $\omega_{\bar{C}}(x) = 2\pi - \omega_C(x)$, which is described by Zhang *et al.* (2009) to define the complement for the phase component, also the rotation of $\omega_C(x)$ by π radians, may be a good method to calculate the complement for a phase term as $\omega_{\bar{C}}(x) = \pi + \omega_C(x)$.

Definition 18 (Phase Union and Intersection): Let A and B be two complex fuzzy sets in X. Then complex fuzzy union and intersection are specified by the functions u and v as follows:

$$u : \{a|a \in \mathbb{C},\ |a| \leq 1\} \times \{b|b \in \mathbb{C},\ |b| \leq 1\} \rightarrow \{c|c \in \mathbb{C},\ |c| \leq 1\},$$

$$v : \{a|a \in \mathbb{C},\ |a| \leq 1\} \times \{b|b \in \mathbb{C},\ |b| \leq 1\} \rightarrow \{c|c \in \mathbb{C},\ |c| \leq 1\},$$

where u satisfies at least the following axiomatic requirements:

- (boundary conditions): $u(a,\ 0) = a$;
- (monotonicity): $|b| \leq |d| \Rightarrow |u(a,\ b) \leq |u(a,\ d)|$;
- (commutativity): $u(a,\ b) = u(b,\ a)$;
- (associativity): $u(a,\ u(b,\ d)) = u(u(a,\ b),\ d)$.

In some cases, it may be desirable that u also satisfies the following axiomatic requirements:

- (continuity): u is a continuous function;
- (superidempotency): $|u(a,\ a)| > |a|$;
- (strict monotonicity): $|a| \leq |c|$ and $|b| \leq |d| \Rightarrow |u(a,\ b)| \leq |u(c,\ d)|$.

and v must satisfies the following axiomatic requirements:

- (boundary conditions): if $|b| = 1$, then $v(a,\ b) = |a|$;
- (monotonicity): $|b| \leq |d| \Rightarrow |v(a,\ b) \leq |v(a,\ d)|$;
- (commutativity): $v(a,\ b) = v(b,\ a)$;
- (associativity): $v(a,\ v(b,\ d)) = v(v(a,\ b),\ d)$.

In some cases, it may be required that v also satisfies the following axiomatic requirements:

- (continuity): v is a continuous function;
- (superidempotency): $|v(a,\, a)| < |a|$;
- (strict monotonicity): $|a| \leq |c|$ and $|b| \leq |d| \Rightarrow |v(a,\, b)| \leq |v(c,\, d)|$.

The following are several possibilities for calculation of $\omega_{A\cup B}(x)$ and $\omega_{A\cap B}(x)$ which, if combined with an appropriate function for determining $r_{A\cup B}(x)$ and $r_{A\cap B}(x)$, satisfies the above axiomatic requirements.

(i) Sum: $\omega_A(x) + \omega_B(x)$;

(ii) Max: $\max(\omega_A(x),\, \omega_B(x))$;

(iii) Min: $\min(\omega_A(x),\, \omega_B(x))$;

(iv) "Winner Take All": $\begin{cases} \omega_A(x), & \text{if } r_A(x) > r_B(x) \\ \omega_B(x), & \text{if } r_A(x) \leq r_B(x). \end{cases}$

Further, Alkouri and Salleh [(2012), (2013)] gave the generalization of complex fuzzy set to the complex intuitionistic fuzzy set by adding the non-membership term to the definition of CFS. The range of values are extended to the unit circle in complex plane for both membership and non-membership functions instead of [0, 1] as in the conventional intuitionistic fuzzy sets.

Definition 19 (Complex Intuitionistic Fuzzy Set): A complex intuitionistic fuzzy set $\tilde{A}$, defined on a universal set X, is characterized by the membership and non-membership functions $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$, respectively that assign to each element $x \in X$ a complex-valued grade of membership & non-membership in $\tilde{A}$.

The complex intuitionistic fuzzy set $\tilde{A}$ may be represented as

$$\tilde{A} = \{\langle x,\, \mu_{\tilde{A}}(x),\, \nu_{\tilde{A}}(x)\rangle : x \in X\},$$

where $\mu_{\tilde{A}}(x) : X \to \{a | a \in \mathbb{C},\, |a| \leq 1\}$ and $\nu_{\tilde{A}}(x) : X \to \{a^{'} | a^{'} \in \mathbb{C},\, |a^{'}| \leq 1\}$.

It may be noted that the values of $\mu_{\tilde{A}}(x)$, $\nu_{\tilde{A}}(x)$ and their sum are lying within the unit circle in the complex plane and are of the form $\mu_{\tilde{A}}(x) = r_{\tilde{A}}(x) \cdot$

$e^{i\omega^r_{\tilde{A}}(x)}$ and $\nu_{\tilde{A}}(x) = k_{\tilde{A}}(x) \cdot e^{i\omega^k_{\tilde{A}}(x)}$, where $r_{\tilde{A}}(x)$, $k_{\tilde{A}}(x)$ are real valued functions such that $r_{\tilde{A}}(x)$, $k_{\tilde{A}}(x) \in [0,\, 1]$ and satisfies the condition $0 \leq r_{\tilde{A}}(x) + k_{\tilde{A}}(x) \leq 1$. The phase terms $\omega^r_{\tilde{A}}(x)$ and $\omega^s_{\tilde{A}}(x)$ belong to $(0,\, 2\pi]$.

1.7 Intuitionistic Fuzzy Soft Sets

Molodtsov (1999) pointed out that the important existing theories viz. probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory etc., which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. The inadequacy of the parametrization tools of these theories make them very limited and difficult. In order to overcome the above stated difficulties, Molodtsov (1999) introduced the concept of Soft Sets for dealing with uncertainties in parameterized form. Later on Maji *et al.* [(2001), (2004a), (2004b)] extended Soft Sets to Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets (IFSSs). Pei and Miao (2005) and Chen *et al.* (2005) have studied and extended the work of Maji *et al.* [(2002), (2003)]. Also, Majumdar and Samanta (2010) have further generalized the concept of fuzzy soft sets.

Definition 20 (Soft Sets): Let X be the universal set, E be the set of parameters under consideration and $\mathcal{P}(X)$ denotes the power set of X. A soft set may be represented by the set of ordered pairs as

$$\langle F,\, E\rangle = \{\langle \varepsilon,\, F(\varepsilon)\rangle | \varepsilon \in E,\, F(\varepsilon) \in \mathcal{P}(X)\},$$

where F is a mapping given by

$$F : E \to \mathcal{P}(X).$$

In other words, the soft set is a parameterized family of subsets of the universe X. For each $\varepsilon \in E$, $F(\varepsilon)$ may be considered as a set of ε-elements or as a set of ε-approximate elements of the soft set $\langle F,\, E\rangle$.

Definition 21 (Fuzzy Soft Sets): Let X be the universal set, E be the set of parameters under consideration and $\mathcal{FS}(X)$ denotes the set of all fuzzy subset of

X. A fuzzy soft set may be represented by the set of ordered pairs as

$$\langle F,\, E\rangle = \{\langle \varepsilon,\, F(\varepsilon)\rangle | \varepsilon \in E,\, F(\varepsilon) \in \mathcal{FS}(X)\},$$

where F is a mapping given by $F : E \to \mathcal{FS}(X)$ such that $F(\varepsilon) = \phi$, i.e., $\mu_{F(\varepsilon)}(x) = 0$, $\forall x \in X$, if $\varepsilon \notin E$.

Definition 22 (Intuitionistic Fuzzy Soft Sets): Let X be the universal set, E be the set of parameters under consideration and $\mathcal{IFS}(X)$ denotes the set of all intuitionistic fuzzy subset of X. An intuitionistic fuzzy soft set may be represented by the set of ordered pairs as

$$\langle \tilde{F},\, E\rangle = \{\langle \varepsilon,\, \tilde{F}(\varepsilon)\rangle | \varepsilon \in E,\, \tilde{F}(\varepsilon) \in \mathcal{IFS}(X)\},$$

where $\tilde{F}$ is a mapping given by $\tilde{F} : E \to \mathcal{IFS}(X)$ such that $\tilde{F}(\varepsilon) = \phi$, i.e., $\mu_{\tilde{F}(\varepsilon)}(x) = 0$ and $\nu_{\tilde{F}(\varepsilon)}(x) = 1$, $\forall x \in X$, if $\varepsilon \notin E$.

Definition 23 (Operations on IFSSs): Suppose that $\langle \tilde{F},\, E\rangle$ and $\langle \tilde{G},\, E\rangle$ are two intuitionistic fuzzy soft sets over a universal set X. Then in view of the above definition, the following operations have been defined as:

- **Union:** $\langle \tilde{F},\, E\rangle \cup \langle \tilde{G},\, E\rangle = \langle \tilde{H},\, E\rangle$,
  where $\tilde{H}(\varepsilon) = \Big\{\Big\langle x,\, \mu_{\tilde{F}(\varepsilon)}(x) \diamond \mu_{\tilde{G}(\varepsilon)}(x),\, \nu_{\tilde{F}(\varepsilon)}(x) * \nu_{\tilde{G}(\varepsilon)}(x)\Big\rangle | x \in X,\, \forall \varepsilon \in E\Big\}$.

- **Intersection:**$\langle \tilde{F},\, E\rangle \cap \langle \tilde{G},\, E\rangle = \langle \tilde{H},\, E\rangle$,
  where $\tilde{H}(\varepsilon) = \Big\{\Big\langle x,\, \mu_{\tilde{F}(\varepsilon)}(x) * \mu_{\tilde{G}(\varepsilon)}(x),\, \nu_{\tilde{F}(\varepsilon)}(x) \diamond \nu_{\tilde{G}(\varepsilon)}(x)\Big\rangle | x \in X,\, \forall \varepsilon \in E\Big\}$.
  Here $\diamond$ and $*$ are s-norm and t-norm operators respectively.

- **Complement:** $(\tilde{F},\, E)^c = (\tilde{F}^c,\, \neg E)$,
  where mapping $\tilde{F}^c : \neg E \to \mathcal{IFS}(X)$ is given by

$$\begin{aligned} \tilde{F}^c(\neg\varepsilon) &= \Big\{\Big\langle x,\, \nu_{\tilde{F}(\neg\neg\varepsilon)}(x),\, \mu_{\tilde{F}(\neg\neg\varepsilon)}(x)\Big\rangle | x \in X\Big\} \\ &= \Big\{\Big\langle x,\, \nu_{\tilde{F}(\varepsilon)}(x),\, \mu_{\tilde{F}(\varepsilon)}(x)\Big\rangle | x \in X\Big\},\ \forall \neg\varepsilon \in \neg E. \end{aligned}$$

- **Subset:** $\langle \tilde{F},\, E\rangle \subseteq \langle \tilde{G},\, E\rangle$, if and only if $\mu_{\tilde{F}(\varepsilon)}(x) \le \mu_{\tilde{G}(\varepsilon)}(x)$ and $\nu_{\tilde{F}(\varepsilon)}(x) \ge \nu_{\tilde{G}(\varepsilon)}(x), \forall x \in X$, and $\varepsilon \in E$.

- **Equality:** $\langle \tilde{F}, E\rangle = \langle \tilde{G}, E\rangle$, if and only if $\mu_{\tilde{F}(\varepsilon)}(x) = \mu_{\tilde{G}(\varepsilon)}(x)$ and $\nu_{\tilde{F}(\varepsilon)}(x) = \nu_{\tilde{G}(\varepsilon)}(x)$, $\forall x \in X$, and $\varepsilon \in E$,

1.8 Literature Survey

The concept of fuzzy entropy has been widely used in different areas, e.g., machine learning, pattern recognition, image processing, decision making, finance and medical diagnosis etc. A large number of generalizations of fuzzy entropy are available in the literature, among them some famous generalizations are given by De Luca and Termini (1972), Kaufman [(1975), (1980)], Yager (1979), Kosko [(1986), (1990)], Pal and Pal (1989). Further, Bhandari and Pal (1993) made a survey on entropy of fuzzy sets and gave some new measures of fuzzy information. Vlachos and Sergiadis (2007) extended the De Luca and Termini's (1972) non-probabilistic entropy for fuzzy sets in the study of the intuitionistic fuzzy information measure.

Tanaka *et al.* [(1980), (1982)] initiated the research in the area of linear regression analysis in a fuzzy environment, where a fuzzy linear system is used as a regression model. Tanaka and Warada (1988), Tanaka *et al.* (1989), Tanaka and Ishibuchi (1991) considered more general models in fuzzy regression. The comparison among various fuzzy regression models and the difference between the approaches of fuzzy regression analysis and conventional regression analysis have been presented by Redden and Woodall (1994). Chang and Lee (1994) & Redden and Woodall (1994) pointed out some weaknesses of the approaches proposed by Tanaka *et al.* (1989).

Further, Cai *et al.* (1993) developed the fuzzy system reliability based on the basis of fuzzy state and probability assumptions. Next, Cai *et al.* (1995) also discussed the system reliability for coherent system based on the fuzzy-state and probability assumptions. Cai *et al.* (1993) presented a fuzzy set-based approach to failure rate and reliability analysis. Cheng and Mon (1993) used interval of confidence in order to analyze fuzzy system reliability. Chen (1994) presented a new method for fuzzy system reliability analysis using fuzzy number arithmetic operations in which the reliability of each component is considered as fuzzy number and used simplified fuzzy arithmetic operations rather than complicated interval fuzzy arithmetic operations of fuzzy numbers. Mahapatra and Roy (2009) presented a method to analyze the fuzzy reliability of the series and parallel system using triangular intuitionistic fuzzy numbers (TIFNs) arithmetic operations. Yao *et al.* (2008) applied a statistical methodology in fuzzy system reliability analysis.

Szmidt and Kacprzyk (2001) showed that intuitionistic fuzzy sets are useful in situations when description of a problem by a linguistic variable given in terms of a membership function only seems too rough. De *et al.* (2001) gave an intuitionistic fuzzy sets approach in medical diagnosis. Burillo and Bustince (1996a) introduced the notions of entropy of IFSs and interval-valued fuzzy sets (IVFS) to measure the degree of intuitionism of an IFS and IVFS, respectively. Further, Szmidt and Kacprzyk (2005) defined a similarity measure using distance measure of IFSs and applied these measures in group decision making problems and medical diagnostic reasoning. Xu (2007a) defined some similarity measures for IVIFSs and applied these similarity measures in pattern recognitions. Hung and Yang (2004) presented a similarity measure of IFSs based on Hausdorff metric and applied it to pattern recognition problems. In the study of fuzzy sets, Wang (1997) defined two similarity measures and Pappis and Karacapilidis (1993) defined three kinds of similarity measures. Hung and Yang (2008) extend these similarity measures from the fuzzy sets to IFSs.

Xu and Yager (2006) developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and gave an application of the IFHG operator to multi-criteria decision-making problems with intuitionistic fuzzy information. Xu (2007a) developed some arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Xu (2007b) defined the concept of interval-valued intuitionistic fuzzy number (IVIFN), and gave some basic operational laws of IVIFNs. He gave an interval-valued intuitionistic fuzzy weighted averaging operator and an interval-valued intuitionistic fuzzy weighted geometric operator and defines the score function and the accuracy function of IVIFNs. Xu and Chen (2007) developed some arithmetic aggregation operators, such as the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted averaging (IIFOWA) operator and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator, and gave an application of the IIFHA operator to multi-criteria decision making problems with interval-valued intuitionistic fuzzy information by using the score function and accuracy function of interval-valued intuitionistic fuzzy numbers.

Chen *et al.* (2011) proposed a neurofuzzy system architecture to implement the complex fuzzy rule as a practical application of the concept of complex fuzzy logic. Alkouri and Salleh (2012) introduced the concept of Complex Intuitionistic Fuzzy Set (CIFS) to represent the information which is happening repeatedly over a period of time. Further, as an application, Alkouri and Salleh (2013) presented an example of suppler selection model which is based on the distance measure of complex intuitionistic fuzzy sets.

1.9 Outline of Thesis

We presented fundamental background of fuzzy set, intuitionistic fuzzy set, soft set and complex intuitionistic fuzzy set with their definitions and various properties along with a brief literature survey in chapter 1.

In chapter 2, a new R-norm intuitionistic fuzzy entropy and R-norm intuitionistic fuzzy directed divergence measure have been proposed with their proof of validity with their monotonic behavior. Computational applications of these information measures in the field of pattern recognition and image thresholding has been proposed with discussion.

In chapter 3, the estimators of regression coefficients of the proposed fuzzy linear regression model (restricted/unrestriced) have been obtained with the help of fuzzy entropy. Some numerical examples have also been provided in order to illustrate the proposed model. Further, in order to compare the performance of unrestricted estimator and restricted estimator, a simulation study has been conducted by using two fundamental criteria of dominance-mean squared error matrix and absolute bias.

In chapter 4, new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form has been proposed. The proposed similarity measures have been analogously extended to obtain new entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets along with their proofs of validity. A new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measures and entropies.

In chapter 5, we compute the reliability of k-out-of-n : G-system (particularly, series and parallel system) with independent and non-identically distributed components, where the reliability of the components are unknown. The reliability of each component has been estimated using statistical confidence interval approach. Then we converted these statistical confidence interval into triangular intuitionistic fuzzy numbers. Based on these triangular intuitionistic fuzzy numbers, the reliability of the k-out-of-n : G-system has been calculated.

In chapter 6, we introduce the concept of complex intuitionistic fuzzy soft sets which is parametric in nature. In order to get their new entropies, some important properties and operations on the complex intuitionistic fuzzy soft sets have also been discussed. On the basis of some well-known distance measures, some new distance measures for the complex intuitionistic fuzzy soft sets have also been obtained. Further, we have established correspondence between the proposed entropies and the distance measures of complex intuitionistic fuzzy soft sets.

Finally, the conclusions have been presented in chapter 7.

Chapter 2

$\boldsymbol{R}$-norm Intuitionistic Fuzzy Information Measures and their Computational Applications

2.1 Introduction

Intuitionistic Fuzzy Set (IFS), developed by Atanassov (1986) is a controlling tool to deal with vagueness and uncertainty. A prominent characteristic of IFS is that it assigns to each element a membership degree and a non-membership degree with certain amount of hesitation degree. Thus, the IFS constitutes an extension of Zadeh's fuzzy set (1965), which only assigns to each element a membership degree. Szmidt and Kacprzyk (2001) showed that intuitionistic fuzzy sets are useful in situations when description of a problem by a linguistic variable given in terms of a membership function only seems too rough. Due to the flexibility of IFS in handling uncertainty, they are tool for a more human consistent reasoning under imperfectly defined facts and imprecise knowledge [Szmidt and Kacprzyk (2004)]. De *et al.* (2001) gave an intuitionistic fuzzy sets approach in medical diagnosis. Intuitionistic fuzzy set is a tool in modeling real life problems like sale analysis, new product marketing, financial services, negotiation process, psychological investigations etc. since there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object [Szmidt and Kacprzyk, (1997), (2001)]. In the context of pattern recognition, Dengfeng and Chuntian (2002) and Mitchell (2003) applied similarity measures for IFSs to perform classification.

In this chapter, a new R-norm intuitionistic fuzzy entropy and R-norm intuitionistic fuzzy directed divergence measure for intuitionistic fuzzy sets have been proposed with their proof of validity in section 2.2 and section 2.3 respectively. Further, in section 2.4, empirical study on the proposed information measures has also been done which explains the monotonic nature of the information measures with respect to the parameters involved. Applications of the proposed new information measures in the field of pattern recognition and image thresholding have also been discussed and suggested in section 2.5.

2.2 R-norm Information Measure of IFS

Hung & Yang (2006) introduced the following axiomatic definition of IFS entropy in a probabilistic setting as a real valued functional $H : \mathcal{IFS}(X) \to \mathbb{R}^+$:

- **IE1(Sharpness):** $H(\tilde{A}) = 0$ iff $\tilde{A}$ is a crisp set;
- **IE2(Maximality):** $H(\tilde{A})$ assumes maximum value if
$$\mu_{\tilde{A}} = \nu_{\tilde{A}} = \pi_{\tilde{A}} = \frac{1}{3}, \forall x_i \in X \qquad ;$$
- **IE3(Resolution):** $H(\tilde{A}) \leq H(\tilde{B})$ if $\tilde{A}$ is crisper than $\tilde{B}$, i.e., if $\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i)$ & $\nu_{\tilde{A}}(x_i) \leq \nu_{\tilde{B}}(x_i)$, for $\max\{\mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i)\} \leq \frac{1}{3}$ and $\mu_{\tilde{A}}(x_i) \geq \mu_{\tilde{B}}(x_i)$ & $\nu_{\tilde{A}}(x_i) \geq \nu_{\tilde{B}}(x_i)$, for $\min\{\mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i)\} \geq \frac{1}{3}, \forall x_i \in X$;
- **IE4(Symmetry):** $H(\tilde{A}^c) = H(\tilde{A})$.

Let $\triangle_n = \{P = (p_1, p_2, \ldots, p_n), 0 \leq p_i \leq 1; \sum\limits_{i=1}^{n} p_i = 1\}$ be the set of all probability distributions associated with a discrete random variable X. Boekee and Lubbe (1980) defined R-norm information measure of the probability distribution P for $R \in \mathbb{R}^+$ as given by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \; ; R > 0,\, R \neq 1. \tag{2.2.1}$$

The measure (2.2.1) is a real function from $\triangle_n$ to $\mathbb{R}^+$ and is called R-norm information measure. The most important property of this measure is that when $R \to 1$, it approaches to Shannon's entropy (1948) and when $R \to \infty$, $H_R(P) \to (1 - \max p_i)$; $i = 1, 2, \ldots, n$.

For an IFS $\tilde{A}$ in X, we have $0 \leq \mu_{\tilde{A}}(x_i),\, \nu_{\tilde{A}}(x_i),\, \pi_{\tilde{A}}(x_i) \leq 1$ and $\mu_{\tilde{A}}(x_i) + \nu_{\tilde{A}}(x_i) + \pi_{\tilde{A}}(x_i) = 1, \forall x_i \in X$. This implies that $\langle \mu_{\tilde{A}}(x_i),\, \nu_{\tilde{A}}(x_i),\, \pi_{\tilde{A}}(x_i)\rangle$ may be regarded as a probability distribution. Therefore, corresponding to R-norm information measure (2.2.1), we propose the following intuitionistic fuzzy entropy:

$$H_R(\tilde{A}) = \frac{R}{(R-1)}\sum_{i=1}^{n}\frac{1}{n}\left[1 - \left((\mu_{\tilde{A}}^R(x_i) + \nu_{\tilde{A}}^R(x_i) + \pi_{\tilde{A}}^R(x_i))\right)^{\frac{1}{R}}\right],\; R > 0,\, R \neq 1. \tag{2.2.2}$$

It may be observed that when $R \to 1$, the R-norm intuitionistic fuzzy entropy

$$H_R(\tilde{A}) \to -\tfrac{1}{n}\log\left(\sum_{i=1}^{n}\mu(x_i)^{\mu(x_i)} \cdot \sum_{i=1}^{n}\nu(x_i)^{\nu(x_i)} \cdot \sum_{i=1}^{n}\pi(x_i)^{\pi(x_i)}\right).$$

We present following properties for proving the validity of the above proposed measure:

Property 2.2.1: Under the condition of **IE3**, we have

$$\left|\mu_{\tilde{A}}(x_i) - \frac{1}{3}\right| + \left|\nu_{\tilde{A}}(x_i) - \frac{1}{3}\right| + \left|\pi_{\tilde{A}}(x_i) - \frac{1}{3}\right| \geq \left|\mu_{\tilde{B}}(x_i) - \frac{1}{3}\right| + \left|\nu_{\tilde{B}}(x_i) - \frac{1}{3}\right| + \left|\pi_{\tilde{B}}(x_i) - \frac{1}{3}\right| \tag{2.2.3}$$

and

$$\begin{aligned}&\left(\mu_{\tilde{A}}(x_i) - \frac{1}{3}\right)^2 + \left(\nu_{\tilde{A}}(x_i) - \frac{1}{3}\right)^2 + \left(\pi_{\tilde{A}}(x_i) - \frac{1}{3}\right)^2\\ &\qquad \geq \left(\mu_{\tilde{B}}(x_i) - \frac{1}{3}\right)^2 + \left(\nu_{\tilde{B}}(x_i) - \frac{1}{3}\right)^2 + \left(\pi_{\tilde{B}}(x_i) - \frac{1}{3}\right)^2.\end{aligned} \tag{2.2.4}$$

Proof.

If $\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i) \leq \nu_{\tilde{B}}(x_i)$ for $\max\{\mu_{\tilde{B}}(x_i),\, \nu_{\tilde{B}}(x_i)\} \leq \frac{1}{3}$, then $\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i) \leq \frac{1}{3}$, $\nu_{\tilde{A}}(x_i) \leq \nu_{\tilde{B}}(x_i) \leq \frac{1}{3}$ and $\pi_{\tilde{A}}(x_i) \geq \pi_{\tilde{B}}(x_i) \geq \frac{1}{3}$, which implies that

$$\left|\mu_{\tilde{A}}(x_i) - \frac{1}{3}\right| \geq \left|\mu_{\tilde{B}}(x_i) - \frac{1}{3}\right|,\; \left|\nu_{\tilde{A}}(x_i) - \frac{1}{3}\right| \geq \left|\nu_{\tilde{B}}(x_i) - \frac{1}{3}\right| \text{ and } \left|\pi_{\tilde{A}}(x_i) - \frac{1}{3}\right| \geq \left|\pi_{\tilde{B}}(x_i) - \frac{1}{3}\right|.$$

This implies that the equation (2.2.3) and (2.2.4) hold.
Similarly, if $\mu_{\tilde{A}}(x_i) \geq \mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i) \geq \nu_{\tilde{B}}(x_i)$ for $\max\{\mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i)\} \geq \frac{1}{3}$, then equation (2.2.3) and (2.2.4) hold.

Theorem 2.2.1: $H_R(\tilde{A})$ is a valid intuitionistic fuzzy information measure.
Proof.
In order to prove that the measure (2.2.2) is a valid intuitionistic fuzzy information measure, we shall show that four properties (IE1-IE4) are satisfied.
(IE1)(Sharpness): Since $R(\neq 1) > 0$, therefore this is possible only in the following cases:

- Either $\mu_{\tilde{A}}(x_i) = 1$, i.e., $\nu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = 0$ or
- $\nu_{\tilde{A}}(x_i) = 1$, i.e., $\mu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = 0$ or
- $\pi_{\tilde{A}}(x_i) = 1$, i.e., $\nu_{\tilde{A}}(x_i) = \mu_{\tilde{A}}(x_i) = 0$.

In all the cases, $H_R(\tilde{A}) = 0$ implies that $\tilde{A}$ is a crisp set. Conversely, if $\tilde{A}$ be a crisp set, i.e., either $\mu_{\tilde{A}}(x_i) = 1$, or $\nu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = 0$ and either $\nu_{\tilde{A}}(x_i) = 1$ or $\mu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = 0$ and either $\pi_{\tilde{A}}(x_i) = 1$ or $\nu_{\tilde{A}}(x_i) = \mu_{\tilde{A}}(x_i) = 0$.

It implies that $\mu^R_{\tilde{A}}(x_i) + \nu^R_{\tilde{A}}(x_i) + \pi^R_{\tilde{A}}(x_i))^{\frac{1}{R}} = 1$ for $R(\neq 1) > 0$, which gives $H_R(\tilde{A}) = 0$. Hence $H_R(\tilde{A}) = 0$ if and only if $\tilde{A}$ is a crisp set.

(IE2)(Maximality): Since $\mu_{\tilde{A}}(x_i) + \nu_{\tilde{A}}(x_i) + \pi_{\tilde{A}}(x_i) = 1$, therefore to obtain the maximum value of intuitionistic fuzzy entropy, we write

$$g(\mu_{\tilde{A}}, \nu_{\tilde{A}}, \pi_{\tilde{A}}) = \mu_{\tilde{A}}(x_i) + \nu_{\tilde{A}}(x_i) + \pi_{\tilde{A}}(x) - 1$$

and taking the Lagrange's multiplier λ, we consider

$$G(\mu_{\tilde{A}}, \nu_{\tilde{A}}, \pi_{\tilde{A}}) = H_R(\mu_{\tilde{A}}, \nu_{\tilde{A}}, \pi_{\tilde{A}}) + \lambda g(\mu_{\tilde{A}}, \nu_{\tilde{A}}, \pi_{\tilde{A}}). \tag{2.2.5}$$

To find the maximum value of $H_R(\tilde{A})$, we differentiate (2.2.5) partially with respect to $\mu_{\tilde{A}}$, $\nu_{\tilde{A}}$, $\pi_{\tilde{A}}$ and λ and equating them to zero, we get

$$\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = \frac{1}{3}.$$

It may be noted that all the first order partial derivatives vanish if and only $\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = \frac{1}{3}$.
Hence $H_R(\tilde{A})$ has the stationary point $\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i) = \pi_{\tilde{A}}(x_i) = \frac{1}{3}$.

Next, we show that $H_R(\tilde{A})$ is a concave function on the IFS $\tilde{A} \in \mathcal{IFS}(X)$ by calculating its Hessian at the stationary point. The Hessian of $H_R(\tilde{A})$ is given by

$$\hat{H} = \frac{R \cdot 3^{\frac{1}{R}-1}}{n} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

For any $R > 0$, $\hat{H}$ is a negative semi-definite matrix and hence $H_R(\tilde{A})$ is a concave function and has its maximum value at the point $\mu_{\tilde{A}} = \nu_{\tilde{A}} = \pi_{\tilde{A}} = \frac{1}{3}$, $\forall x_i \in X$.
(IE3)(Resolution): Since $H_R(\tilde{A})$ is a concave function on the IFS $\tilde{A} \in \mathcal{IFS}(X)$, therefore if $\max\{\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\} \leq \frac{1}{3}$, then $\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i) \leq \nu_{\tilde{B}}(x_i)$, which implies that

$$\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i) \leq \frac{1}{3},\ \nu_{\tilde{A}}(x_i) \leq \nu_{\tilde{B}}(x_i) \leq \frac{1}{3} \text{ and } \pi_{\tilde{A}}(x_i) \geq \pi_{\tilde{B}}(x_i) \geq \frac{1}{3}.$$

According to the result of property 2.2.1, we conclude that $(\mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i), \pi_{\tilde{B}}(x_i))$ is more around $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ than $(\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i), \pi_{\tilde{A}}(x_i))$.
Hence, $H_R(\tilde{A}) \leq H_R(\tilde{B})$.
Similarly, if $\min\{\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\} \geq \frac{1}{3}$, then $\mu_{\tilde{A}}(x_i) \geq \mu_{\tilde{B}}(x_i)$, and $\nu_{\tilde{A}}(x_i) \geq \nu_{\tilde{B}}(x_i)$.
By property 2.2.1, we again conclude that $H_R(\tilde{A}) \leq H_R(\tilde{B})$.

(IE4)(Symmetry): It may be noted that from the definition of the complement of intuitionistic fuzzy set, it is clear that $H_R(\bar{\tilde{A}}) = H_R(\tilde{A})$.

Hence $H_R(\tilde{A})$ satisfies all the properties of intuitionistic fuzzy entropy. Therefore, $H_R(\tilde{A})$ is a valid measure of intuitionistic fuzzy entropy. $\square$

2.3 R-norm Intuitionistic Fuzzy Directed Divergence Measure

Based on the parametric fuzzy directed divergence measure given by Hooda and Bajaj (2008), we propose the following measure of intuitionistic fuzzy directed divergence of IFS $\tilde{A}$ from IFS $\tilde{B}$:

$$I_R^{\lambda}(\tilde{A}, \tilde{B}) = \lambda M_R(\tilde{A}, \tilde{B}) + (1-\lambda)\, N_R(\tilde{A}, \tilde{B});\ 0 < \lambda < 1,\ R > 0,\ R \neq 1, \tag{2.3.1}$$

where

$$N_R(\tilde{A},\, \tilde{B}) = \frac{R}{R-1}\sum_{i=1}^{n}\left[\left(\begin{array}{l} (\mu_{\tilde{A}}(x_i)+\pi_{\tilde{A}}(x_i))^R(\mu_{\tilde{B}}(x_i)+\pi_{\tilde{B}}(x_i))^{1-R} \\ +(1-(\mu_{\tilde{A}}(x_i)+\pi_{\tilde{A}}(x_i))^R(1-(\mu_{\tilde{B}}(x_i)+\pi_{\tilde{B}}(x_i))^{1-R} \end{array}\right)^{\frac{1}{R}} - 1\right]$$

and

$$M_R(\tilde{A}, \tilde{B}) = \frac{R}{R-1}\sum_{i=1}^{n}\left[\left(\begin{array}{l} (\nu_{\tilde{A}}(x_i)+\pi_{\tilde{A}}(x_i))^R(\nu_{\tilde{B}}(x_i)+\pi_{\tilde{B}}(x_i))^{1-R} \\ +(1-(\nu_{\tilde{A}}(x_i)+\pi_{\tilde{A}}(x_i))^R(1-(\nu_{\tilde{B}}(x_i)+\pi_{\tilde{B}}(x_i))^{1-R} \end{array}\right)^{\frac{1}{R}} - 1\right].$$

Also, the proposed R-norm intuitionistic fuzzy directed divergence measure may be used to define a symmetric intuitionistic fuzzy directed divergence, denoted by $J_R^\lambda(\tilde{A},\, \tilde{B})$ and defined as $J_R^\lambda(\tilde{A},\, \tilde{B}) = I_R^\lambda(\tilde{A},\, \tilde{B}) + I_R^\lambda(\tilde{B},\, \tilde{A})$.

Theorem 2.3.1: Divergence measure $I_R^\lambda(\tilde{A},\, \tilde{B})$ is a valid intuitionistic fuzzy directed divergence measure between $\tilde{A}$ and $\tilde{B}$.

Proof. It may be noted that linear combination of two valid fuzzy directed divergence measures is a valid fuzzy directed divergence measure. Therefore, in order to prove that (2.3.1) is a valid measure of intuitionistic fuzzy directed divergence, it is sufficient to show that $M_R(A,\, B) \geq 0$ with equality if $\mu_A(x_i) = \mu_B(x_i)$, $\forall x_i \in X$, as $M_R(A\,, B)$ and $N_R(A,\, B)$ are defined in similar way.

Let $\sum\limits_{i=1}^{n}\mu_A(x_i) = s$, $\sum\limits_{i=1}^{n}\mu_B(x_i) = t$, then

$$\sum_{i=1}^{n}\left[\left(\frac{\mu_A(x_i)}{s}\right)^R\left(\frac{\mu_B(x_i)}{t}\right)^{1-R} - 1\right] \geq 0$$

$$\sum_{i=1}^{n}\mu_A^R(x_i)(\mu_B(x_i))^{1-R} \geq s^R t^{1-R}. \tag{2.3.2}$$

Similarly, we write

$$\sum_{i=1}^{n}(1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} \geq (n-s)^R(n-t)^{1-R}. \tag{2.3.3}$$

Adding (2.3.2) and (2.3.3), we get

$$\sum_{i=1}^{n} \mu_A^R(x_i)(\mu_B(x_i))^{1-R} + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}$$
$$\geq s^R t^{1-R} + (n-s)^R(n-t)^{1-R}. \qquad (2.3.4)$$

Case 1: When $0 < R < 1$
Let $\mu_A^R(x_i)(\mu_B(x_i))^{1-R} + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} = x_i$,
then $x_i < 1$ and $\frac{1}{R} > 1$, implies that, $x_i - 1 > (x_i)^{1/R} - 1$.
Since $\frac{R}{R-1} < 0$, therefore $\sum_{i=1}^{n} x_i - 1 > (x_i)^{\frac{1}{R}} - 1$.
Thus, we have

$$M_R(A,\, B) = \frac{R}{R-1}\big[s^R t^{1-R} + (n-s)^R(n-t)^{1-R} - n\big]. \qquad (2.3.5)$$

Further, let $\varphi(s) = \frac{R}{R-1}\big[s^R t^{1-R} + (n-s)^R(n-t)^{1-R} - n\big]$,
then $\varphi'(s) = \frac{R}{R-1}\big[R(s/t)^{R-1} - R((n-s)/(n-t))^{R-1}\big]$
and
$\varphi''(s) = R^2\big[(1/t)(s/t)^{R-2} - (1/n-t)((n-s)/(n-t))^{R-2}\big] > 0$.
This shows that $\varphi(s)$ is a convex function of s whose minimum value arises when $(s/t)(= (n-s)/(n-t)) = 1$ and is equal to zero. Hence, $\varphi(s) > 0$ and vanishes only when $s = t$.

Case 2: When $R > 1$, In this case, equation (2.3.4) can be written as

$$\left(\sum_{i=1}^{n} \mu_A^R(x_i)(\mu_B(x_i))^{1-R} + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}\right)^{1/R}$$
$$\geq \left(s^R t^{1-R} + (n-s)^R(n-t)^{1-R}\right)^{1/R}. \qquad (2.3.6)$$

also

$$\sum_{i=1}^{n} \left[\left(\mu_A^R(x_i)(\mu_B(x_i))^{1-R} + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R}\right)^{1/R} - 1\right]$$
$$\geq \left(\sum_{i=1}^{n} \mu_A^R(x_i)(\mu_B(x_i))^{1-R} + (1-\mu_A(x_i))^R(1-\mu_B(x_i))^{1-R} - 1\right)^{1/R}. \qquad (2.3.7)$$

Now equations (2.3.6) and (2.3.7) implies that

$$M_R(A,\, B) \geq \frac{R}{R-1}\left(s^R t^{1-R} + (n-s)^R (n-t)^{1-R} - n\right)^{1/R}.$$

If $\varphi(s) = \frac{1}{R-1}\left[s^R t^{1-R} + (n-s)^R (n-t)^{1-R} - n\right]$,
then

$$\varphi^{'}(s) = \frac{R}{R-1}\left[\left(\frac{s}{t}\right)^{R-1} - \left(\frac{n-s}{n-t}\right)^{R-1}\right] \tag{2.3.8}$$

and

$$\varphi^{''}(s) = R^2\left[\left(\frac{1}{t}\right)\left(\frac{s}{t}\right)^{R-2} - \left(\frac{1}{n-t}\right)\left(\frac{n-s}{n-t}\right)^{R-2}\right] > 0. \tag{2.3.9}$$

Therefore, $\varphi(s)$ is a convex function of s whose minimum value arises when $(s/t)(= (n-s)/(n-t)) = 1$ and is equal to zero. Hence, $\varphi(s) > 0$ and vanishes only when $s = t$, i.e., $\forall R \neq 1(>0)$, $M_R(\tilde{A},\, \tilde{B}) \geq 0$ and vanishes only when $\tilde{A} = \tilde{B}$. Thus $M_R(A,\, B)$ is a valid intuitionistic fuzzy directed divergence measure. Hence, measure (2.3.1) is a valid intuitionistic fuzzy directed divergence measure for intuitionistic fuzzy sets. □

2.4 Monotonicity of *R*-norm Intuitionistic Fuzzy Information Measures

Let $\tilde{A}_1$ and $\tilde{A}_2$ be two intuitionistic fuzzy sets over $X = \{x_1,\, x_2,\, x_3,\, x_4\}$, where

$$\tilde{A}_1 = \{(x_1,\, 0.2,\, 0.5),\, (x_2,\, 0.4,\, 0.4),\, (x_3,\, 0.5,\, 0.2),\, (x_4,\, 0.6,\, 0.3)\},$$
$$\tilde{A}_2 = \{(x_1,\, 0.3,\, 0.4),\, (x_2,\, 0.2,\, 0.6),\, (x_3,\, 0.5,\, 0.3),\, (x_4,\, 0.6,\, 0.2)\},$$

Considering various values of R, and using (2.2.2), we compute and tabulate all the values. On the basis of the tabulated data, we plot the figure 2.1.

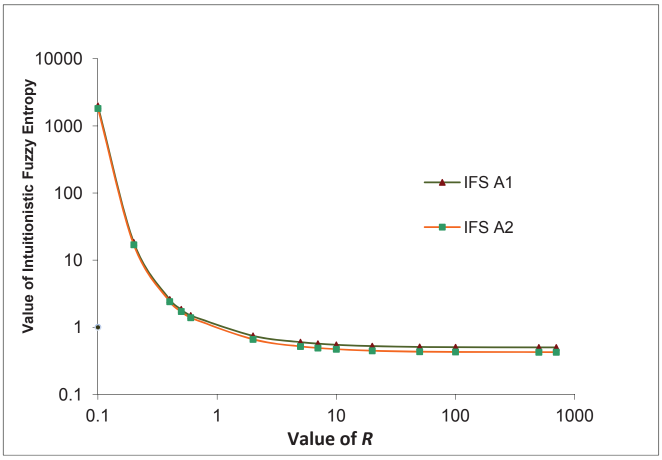

Figure 2.1: *Monotonicity of Intuitionistic Fuzzy Information Measure* H_R

Let $\tilde{A}$ and $\tilde{B}$ be a pair intuitionistic fuzzy sets over $X = \{x_1,\ x_2,\ x_3,\ x_4\}$, where

$$\tilde{A} = \{(x_1,\ 0.2,\ 0.5),\ (x_2,\ 0.4,\ 0.4),\ (x_3,\ 0.3,\ 0.4),\ (x_4,\ 0.5,\ 0.3)\},$$
$$\tilde{B} = \{(x_1,\ 0.3,\ 0.4),\ (x_2,\ 0.2,\ 0.6),\ (x_3,\ 0.5,\ 0.3),\ (x_4,\ 0.6,\ 0.2)\}.$$

Considering various values of R & λ, we compute $I_R^\lambda(\tilde{A},\ \tilde{B})$ by using (2.3.1) and tabulate them. On the basis of the tabulated data, we plot figure 2.2.

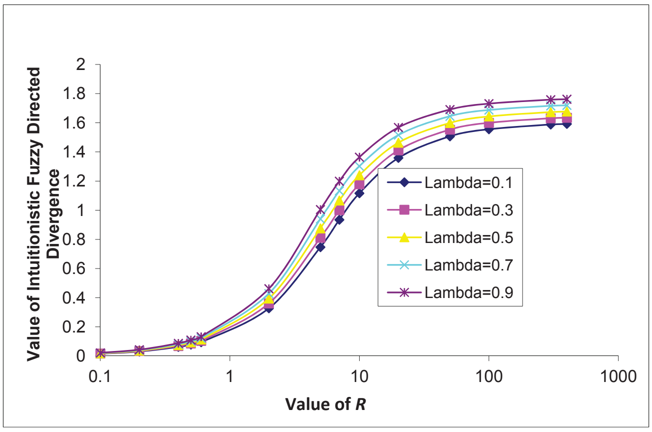

Figure 2.2: *Monotonicity of Intuitionistic Fuzzy Directed Divergence Measure* I_R^λ

Empirically, it may be observed that $H_R(\tilde{A})$ is monotonically decreasing function of R and $I_R^\lambda(\tilde{A}, \tilde{B})$ is monotonically increasing function of R and λ.

2.5 Computational Applications in Pattern Recognition and Image Thresholding

In the field of pattern recognition, divergence measure describes dissimilarity between pairs of probability distribution which is widely used for the process of statistical inference. It may be noted that the divergence measure and similarity measure are dual concepts. The similarity measure may be defined as a decreasing function of divergence measure, especially when the range of divergence measure is [0, 1].

Let f be any monotonic decreasing function. Since $0 \leq J_R^{\lambda}(\tilde{A},\, \tilde{B}) \leq G(R)$; where $G(R)$ is a calculated upper bound of the symmetric divergence measure $J_R^{\lambda}(\tilde{A},\, \tilde{B})$, therefore $f(G(R)) \leq f(J_R^{\lambda}(\tilde{A},\, \tilde{B})) \leq f(0)$, provided that $f(G(R)) < f(0)$. This implies that the similarity measure between IFSs $\tilde{A}$ and $\tilde{B}$ is defined as

$$S_R^{\lambda}(\tilde{A},\, \tilde{B}) = \frac{f(J_R^{\lambda}(\tilde{A},\, \tilde{B}))) - f(G(R))}{f(0) - f(G(R))}, \text{ where } 0 \leq S_R(\tilde{A},\, \tilde{B}) \leq 1. \tag{2.5.1}$$

For example, if we choose $f(x) = \frac{1}{1+x}$, then our similarity measure can be defined as follows:

$$S_R^{\lambda}(\tilde{A},\, \tilde{B}) = \frac{G(R) - J_R^{\lambda}(\tilde{A},\, \tilde{B})}{(1 + J_R^{\lambda}(\tilde{A},\, \tilde{B})) \cdot G(R)}.$$

On the basis of the proposed similarity measure (2.5.1) between two IFSs, the concept of similarity based clustering method (SCM) can be explored and the structure of the considered data set may be studied.

Another application of the theory of intuitionistic fuzzy sets may be found in the field of image thresholding, where an image is considered as a intuitionistic fuzzy set. The membership degree of a pixel to the image is in proportion with their gray level, the non-membership degree of the pixel to the image is in inverse proportion with their gray level having a certain amount of hesitation degree. Let us consider the original image as an IFS $\tilde{A}$, the degraded image as an IFS $\tilde{B}$, and the reconstructed image as an IFS $\tilde{C}$. We wish to transformed the $\tilde{B}$ to denoised version image as $\tilde{C}$ by an algorithm. Pasha *et al.* (2006) introduced a cost function with the help of fuzzy entropy to choose a threshold value for the denoising the degraded image. In order to accomplish the task, they used Euclidian distance and Kaufmann's entropy. Further, Fatemi (2011) used stochastic fuzzy entropy in place of fuzzy entropy and stochastic fuzzy discrimination information for the Euclidean distance. The algorithm first finds the noised pixels then change them with mean of 8 neighbor pixels. The problem is chose a threshold h as unexpected jumping of gray level in the algorithm to find the noised pixels.

Here, it is being suggested that a new cost function which includes the intuitionistic fuzzy theory may be used to find the best threshold by using R-norm intuitionistic fuzzy entropy in place of fuzzy entropy and R-norm intuitionistic fuzzy symmetric directed divergence for the Euclidean distance. The basic and fundamental equation for the algorithm is $C(\tilde{A}) = H_R(\tilde{C}) + J_R^\lambda(\tilde{A},\ \tilde{C})$.

2.6 Conclusions

The validity of the proposed R-norm intuitionistic fuzzy entropy and R-norm intuitionistic fuzzy directed divergence measure has been checked and found correct. Further, after empirical study on the proposed information measures we find that R-norm fuzzy intuitionistic fuzzy entropy is a decreasing function of R, while the R-norm intuitionistic fuzzy directed divergence measure is increasing function of R as well as the λ involved. The computational applications of the proposed intuitionistic fuzzy information measures in the field of pattern recognition and image processing have been discussed and suggested.

Chapter 3

Fuzzy Weighted Linear Regression Model under Linear Restrictions

3.1 Introduction

In statistical analysis, regression is a used to explore the relationship between k input variables $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_k$ (also known as *independent variables* or *explanatory variables*) and the output variable $\mathbf{y}$ (also called *dependent variable* or *response variable*) from n sets of observations. In linear regression, the method of least-squares is applied to find the regression coefficients $\beta_j, j = 0, 1, \ldots, k$ which describe the contribution of the corresponding independent variable $\mathbf{x}_j$ in explaining the dependent variable $\mathbf{y}$. The aim of regression analysis is to estimate the parameters on the basis of available/observed empirical data. Traditional studies on regression assume the observations to have crisp values. In the crisp linear regression model, the parameters (regression coefficients are crisp) appear in a linear form, i.e.,

$$\mathbf{y} = \beta_0 + \beta_1\mathbf{x}_1 + \beta_2\mathbf{x}_2 + \cdots + \beta_k\mathbf{x}_k + \textit{random error}. \tag{3.1.1}$$

Once the coefficients β_0, β_1, β_2, $\ldots$, β_k are determined from the observed samples, the responses are estimated from any given sets of $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_k$ values.

The regression models are used frequently in various areas of researches wherever we deal with the relations among several variables in a system. In literature, linear regression has been studied rigorously and used extensively. One of the important reasons to study these models which depend linearly on their unknown parameters is that they are easier to fit than the models which are non-linearly related to their parameters. In addition, the statistical properties of the resulting estimators are also easier to determine. Linear regression has wide applications in biological, behavioral and social sciences to describe possible relationships between the variables. It ranks as one of the most important tools used in these disciplines.

Vague or fuzzy data find application in several fields, such as psychometry, reliability, marketing, quality control, ballistics, ergonomics, image recognition, artificial intelligence, etc. A typical problem where vague data arise is that of assigning numbers to subjective perceptions or to linguistic variables (such as "enough", "good", "sufficiently", etc.). In fact, there are many cases where observations cannot be known or quantified exactly, and, thus, we can only provide an approximate description of them, or intervals to enclose them. For instance, "*in measuring the influence of character size on the reading ability from a video display terminal [...] the reading ability of the subject, which is essentially the experimental output, depends on his/her eyesight, age, the environment, individual responses, and even how tired is the individual. Some of these factors cannot be described accurately and [...] the best description of these kinds of output is that they are fuzzy outputs*" [Chang *et al.* (1994)]. Also, if a system under consideration is not governed by random variables and/or crisp observations but is governed by possibility variables and/or imprecise observations, it is more beneficial to use fuzzy regression analysis for such a system.

Tanaka *et al.* [(1980), (1982)] initiated the research in the area of linear regression analysis in a fuzzy environment, where a fuzzy linear system is used as a regression model. They consider a regression model in which the relation of the variables are subject to fuzziness, i.e., the model with crisp input and fuzzy parameters. In general, fuzzy regression can be classified into two categories:

- when the relations of the variables are subject to fuzziness,
- when the variables themselves are fuzzy.

There exist several conceptual and methodological approaches to fuzzy regression with respect to the characterization mentioned above. Tanaka and Warada (1988), Tanaka *et al.* (1989), Tanaka and Ishibuchi (1991) considered more general models in fuzzy regression. In the approaches of Tanaka *et al.* they considered the L-R fuzzy data and minimized the index of fuzziness of the fuzzy linear regression model. As described by Tanaka and Warada (1988), "*A fuzzy number is a fuzzy subset of the real line whose highest membership values are clustered around a given real number called the mean value; the membership function is monotonic on both sides of this mean value*". Hence, fuzzy number can be decomposed into position and fuzziness, where the position is represented by the element with the highest membership value and the fuzziness of a fuzzy number is represented by the membership function. The comparison among various fuzzy regression models and the difference between the approaches of fuzzy regression analysis and conventional regression analysis have been presented by Redden and Woodall (1994). Chang and Lee (1994) & Redden and Woodall (1994) pointed out some weaknesses of the approaches proposed by Tanaka *et al.* (1989). A fuzzy linear regression model based on Tanaka's approach by considering the fuzzy linear programming problem has also been introduced by Peters (1994).

In fuzzy set theory, the entropy is a measure of degree of fuzziness which expresses the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. The following are the four properties introduced in De Luca and Termini (1972), which are widely accepted as a criterion for defining any new fuzzy entropy measure $H(\cdot)$ of the fuzzy set $\tilde{A}$:

- P1 (*Sharpness*): $H(\tilde{A})$ is minimum iff $\tilde{A}$ is a crisp set, i.e., $\mu_{\tilde{A}}(x) = 0$ or 1; $\forall x \in X$;

- P2 (*Maximality*): $H(\tilde{A})$ is maximum iff $\mu_{\tilde{A}}(x) = 0.5$; $\forall x \in X$;

- P3 (*Resolution*): $H(\tilde{A}^*) \leq H(\tilde{A})$, where $\tilde{A}^*$ is sharpened version of $\tilde{A}$;

- P4 (*Symmetry*): $H(\tilde{A}) = H(\tilde{A}^c)$, where $\tilde{A}^c$ is the complement of $\tilde{A}$, i.e., $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$.

Dubosis and Prade (1980) interpreted the measure of fuzziness $H(\tilde{A})$ as quantity of information which is being lost in going from a crisp number to a fuzzy number. It may be noted that the entropy of an element with a given membership function $\mu_{\tilde{A}}(x)$ is increasing if $\mu_{\tilde{A}}(x)$ is in $[0,\ 0.5]$ and decreasing if $\mu_{\tilde{A}}(x)$ is in $[0.5,\ 1]$. We accept the definition of fuzzy number given by Tanaka and Warada (1988), where the mean value is also called apex.

Let $X = (x_1, x_2, \ldots, x_n)$ is a discrete random variable with probability distribution $P = (p_1, p_2, \ldots, p_n)$ in an experiment, then according to Shannon (1948), the information contained in this experiment is given by:

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i.$$

Based on this famous Shannon's entropy, De Luca and Termini (1972) indicated the following measure of fuzzy entropy:

$$H(\tilde{A}) = -K \int_{x \in X} [\mu_{\tilde{A}}(x) \log \mu_{\tilde{A}}(x) + (1 - \mu_{\tilde{A}}(x)) \log(1 - \mu_{\tilde{A}}(x))] dx. \qquad (3.1.2)$$

However, we have other fuzzy entropies but (3.1.2) can be regarded as one of the most fundamental measure of ambiguity of a fuzzy set. In addition, Yager (1979) also defined entropy of a fuzzy set based on the distance from the set to its complement set. Similarly, Kosko [(1986), (1990)] introduced another kind of fuzzy entropy by considering the distance from a set to its nearest non-fuzzy set and the distance from the set to its farthest non-fuzzy set. Another kind of fuzzy entropy with an exponential function was introduced by Pal and Pal (1989).

Later on, they introduced the concept of higher r^{th} order entropy of a fuzzy set in their paper Pal and Pal (1992). Further, Bhandari and Pal (1993) made a survey on entropy of fuzzy sets and gave some new measures of fuzzy information.

The concept of fuzzy regression model was first introduced in Tanaka et.al. (1980) and its general form in triangular fuzzy setup is given by

$$\tilde{\mathbf{y}} = \tilde{\beta}_0 + \tilde{\beta}_1\tilde{\mathbf{x}}_1 + \cdots + \tilde{\beta}_k\tilde{\mathbf{x}}_k + random\ error, \tag{3.1.3}$$

where the value of the output variable $\tilde{\mathbf{y}}$ defined by (3.1.3) is a fuzzy number; $\tilde{\beta}_0,\, \tilde{\beta}_1,\, \ldots,\, \tilde{\beta}_k$ is a vector of fuzzy parameters where $\beta_j = (a_j,\, b_j,\, c_j)$ is a fuzzy number for $j = 0,\, 1,\, \ldots,\, k$ and $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2,\, \ldots,\, \tilde{\mathbf{x}}_k$ are triangular fuzzy (explanatory) variables.

Among the various shapes of fuzzy number in the literature of fuzzy set theory, triangular fuzzy number (TFN) is one of the most popular type of fuzzy number. A general triangular fuzzy number is represented by three points $\tilde{A} = (a,\, b,\, c)$; where a is the left vertex, b is the apex and c is the right vertex. This representation is interpreted by a membership function which can be defined as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & \text{where } x \in [a,\, b] \\ \frac{(c-x)}{(c-b)}, & \text{where } x \in (b,\, c] \\ 0 \quad, & \text{otherwise.} \end{cases} \tag{3.1.4}$$

The entropy calculated from equation (3.1.2) for the triangular fuzzy number given by (3.1.4) can be expressed as follows:

$$\begin{aligned} H(\tilde{A}) = -K \Bigg[& \int\limits_{x\in[a,\, b]} [\mu_{\tilde{A}}(x)\log\mu_{\tilde{A}}(x) + (1-\mu_{\tilde{A}}(x))\log(1-\mu_{\tilde{A}}(x))]dx \\ & + \int\limits_{x\in[b,\, c]} [\mu_{\tilde{A}}(x)\log\mu_{\tilde{A}}(x) + (1-\mu_{\tilde{A}}(x))\log(1-\mu_{\tilde{A}}(x))]dx \Bigg] \\ = {} & H_L(\tilde{A}) + H_R(\tilde{A}), \end{aligned} \tag{3.1.5}$$

where $H_L(\tilde{A}) = K(b-a)/2$ and $H_R(\tilde{A}) = K(c-b)/2$. It follows that $H(\tilde{A}) = K(c-a)/2$, which does not depend on b. It may be observed that in case of symmetrical triangular fuzzy number (TFN), the left and the right entropies are identical. On the other hand, in case of non-symmetric TFN, the left entropy is a function of $(b-a)$ and the right entropy is a function of $(c-b)$. As described by Pedrycz (1994), the position, the left entropy, and the right entropy uniquely determine the triangular fuzzy number.

Sometimes experimenter's past experiences may be available as prior information about unknown regression coefficients to estimate more efficient estimators. Here, we assume that such prior information is provided in the form of exact linear restrictions on regression coefficients. In this chapter, we first find the unrestricted estimators of regression coefficients with the help of fuzzy entropy. Next, we introduce the restricted linear regression model with fuzzy entropy. Further, the restricted estimators of the regression coefficients are obtained by incorporating the prior information in the form of linear restrictions. In order to illustrate the proposed model along with the obtained weighted estimators, few numerical examples have also been provided. A simulation study has been conducted to compare the performance of unrestricted estimator and restricted estimator using two basic criteria of dominance: mean squared error matrix and absolute bias.

3.2 Restricted FLR Model with Fuzzy Entropy

Without loss of generality, suppose that all observations $(\tilde{\mathbf{y}}_i, \tilde{\mathbf{x}}_{i1}, \tilde{\mathbf{x}}_{i2}, \ldots, \tilde{\mathbf{x}}_{ik})$, $i = 1, \ldots, n$ in the regression analysis are triangular fuzzy numbers. The idea of regression using entropy is to construct three conventional regression equations (one for apex, one for left entropy, and one for right entropy) for the response variable $\tilde{\mathbf{y}}$ using the corresponding attributes of the k fuzzy explanatory variables $\tilde{\mathbf{x}}_j$. In order to be specific, we denote $\mathbf{y^a}, \mathbf{x}_1^{\mathbf{a}}, \mathbf{x}_2^{\mathbf{a}}, \ldots, \mathbf{x}_k^{\mathbf{a}}$ as the apexes of $\tilde{\mathbf{y}}, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_k$, respectively; $\mathbf{e}_{\mathbf{y}}^{\mathbf{l}}, \mathbf{e}_{\mathbf{x}_1}^{\mathbf{l}}, \mathbf{e}_{\mathbf{x}_2}^{\mathbf{l}}, \ldots, \mathbf{e}_{\mathbf{x}_k}^{\mathbf{l}}$ as the left entropy of $\tilde{\mathbf{y}}, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_k$, respectively; and $\mathbf{e}_{\mathbf{y}}^{\mathbf{r}}, \mathbf{e}_{\mathbf{x}_1}^{\mathbf{r}}, \mathbf{e}_{\mathbf{x}_2}^{\mathbf{r}}, \ldots, \mathbf{e}_{\mathbf{x}_k}^{\mathbf{r}}$ as the right entropy of $\tilde{\mathbf{y}}, \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_k$, respectively. Therefore, the three fundamental regression equations in a non-recursive(non-adaptive) setup may be written as:

$$\mathbf{y^a} = A_0^a + \sum_{i=1}^{k} (A_i^a \mathbf{x}_i^\mathbf{a} + B_i^a \mathbf{e}_{\mathbf{x}_i}^\mathbf{l} + C_i^a \mathbf{e}_{\mathbf{x}_i}^\mathbf{r}) + \boldsymbol{\varepsilon}_{\mathbf{y^a}}; \tag{3.2.1}$$

$$\mathbf{e}_\mathbf{y}^\mathbf{l} = A_0^l + \sum_{i=1}^{k} (A_i^l \mathbf{x}_i^\mathbf{a} + B_i^l \mathbf{e}_{\mathbf{x}_i}^\mathbf{l} + C_i^\mathbf{l} \mathbf{e}_{\mathbf{x}_i}^\mathbf{r}) + \boldsymbol{\varepsilon}_{\mathbf{e}_\mathbf{y}^\mathbf{l}}; \tag{3.2.2}$$

$$\mathbf{e}_\mathbf{y}^\mathbf{r} = A_0^r + \sum_{i=1}^{k} (A_i^r \mathbf{x}_\mathbf{i}^\mathbf{a} + B_i^r \mathbf{e}_{\mathbf{x}_i}^\mathbf{l} + C_i^r \mathbf{e}_{\mathbf{x}_i}^\mathbf{r}) + \boldsymbol{\varepsilon}_{\mathbf{e}_\mathbf{Y}^\mathbf{r}}; \tag{3.2.3}$$

where $\boldsymbol{\varepsilon}_{\mathbf{y^a}}$, $\boldsymbol{\varepsilon}_{\mathbf{e}_\mathbf{y}^\mathbf{l}}$ and $\boldsymbol{\varepsilon}_{\mathbf{e}_\mathbf{Y}^\mathbf{r}}$ are the error vectors of dimension $n \times 1$. The compact form of the above mentioned non-recursive or non-adaptive equations is given by:

$$\begin{aligned} \mathbf{y^a} &= \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{\mathbf{y^a}}, \\ \mathbf{e}_\mathbf{y}^\mathbf{l} &= \mathbf{X}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}_{\mathbf{e}_\mathbf{y}^\mathbf{l}}, \\ \mathbf{e}_\mathbf{y}^\mathbf{r} &= \mathbf{X}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_{\mathbf{e}_\mathbf{Y}^\mathbf{r}}, \end{aligned} \tag{3.2.4}$$

where

$$\begin{aligned} \mathbf{X} &= \left(\mathbf{1} \vdots \mathbf{x}_1^\mathbf{a}, \mathbf{x}_2^\mathbf{a}, \ldots, \mathbf{x}_k^\mathbf{a} \vdots \mathbf{e}_{\mathbf{x}_1}^\mathbf{l}, \mathbf{e}_{\mathbf{x}_2}^\mathbf{l}, \ldots, \mathbf{e}_{\mathbf{x}_k}^\mathbf{l} \vdots \mathbf{e}_{\mathbf{x}_1}^\mathbf{r}, \mathbf{e}_{\mathbf{x}_2}^\mathbf{r}, \ldots, \mathbf{e}_{\mathbf{x}_k}^\mathbf{r} \right)_{n \times (3k+1)}, \\ \boldsymbol{\beta} &= \left(A_0^a \vdots A_1^a, A_2^a, \ldots, A_k^a \vdots B_1^a, B_2^a, \ldots, B_k^a \vdots C_1^a, C_2^a, \ldots, C_k^a \right)^T_{(3k+1) \times 1}, \\ \boldsymbol{\alpha} &= \left(A_0^l \vdots A_1^l, A_2^l, \ldots, A_k^l \vdots B_1^l, B_2^l, \ldots, B_k^l \vdots C_1^l, C_2^l, \ldots, C_k^l \right)^T_{(3k+1) \times 1}, \\ \boldsymbol{\gamma} &= \left(A_0^r \vdots A_1^r, A_2^r, \ldots, A_k^r \vdots B_1^r, B_2^r, \ldots, B_k^r \vdots C_1^r, C_2^r, \ldots, C_k^r \right)^T_{(3k+1) \times 1}. \end{aligned}$$

In many real life situations, where the measurements are carried out (e.g., car speed or astronomical distance), it is natural to think that the spread (vagueness) in the measure of a phenomenon is proportional to its intensity. D'Urso and Gastaldi (2000) have done several simulations and observed that even if we consider an adaptive or recursive regression model along with non-adaptive or non-recursive regression model, they yield identical solutions when there is only one independent variable. But, if there are more than one independent variable, then the estimated values of the left entropies and right entropies obtained through the recursive fuzzy regression model will have a less variance as compared to the non-recursive fuzzy regression model. With this consideration, we rewrite the proposed fuzzy regression model in a recursive/adaptive setup where dynamic of the entropies is dependent on the magnitude of the estimated apexes as follows:

$$\begin{aligned}
\mathbf{y^a} &= \mathbf{y^{a^*}} + \boldsymbol{\varepsilon}_{\mathbf{y^a}}; \quad \text{where} \quad \mathbf{y^{a^*}} = \mathbf{X}\boldsymbol{\beta}, \\
\mathbf{e_y^l} &= \mathbf{e_y^{l^*}} + \boldsymbol{\varepsilon}^*_{\mathbf{e_y^l}}; \quad \text{where} \quad \mathbf{e_y^{l^*}} = \mathbf{X}\boldsymbol{\beta}b + \mathbf{1}d, \\
\mathbf{e_y^r} &= \mathbf{e_y^{r^*}} + \boldsymbol{\varepsilon}^*_{\mathbf{e_y^r}}; \quad \text{where} \quad \mathbf{e_y^{r^*}} = \mathbf{X}\boldsymbol{\beta}f + \mathbf{1}g,
\end{aligned} \tag{3.2.5}$$

where $\mathbf{X}$ is the $n \times (3k+1)$-matrix containing the values of the input variables (data matrix), $\boldsymbol{\beta}$ is a column $3k+1$-vector containing the regression parameters for the apexes of the first model (referred to as core regression model), $\mathbf{y^a}$ and $\mathbf{y^{a*}}$ are the vector of the observed apexes and the vector of the interpolated apexes, respectively, both having dimension $n \times 1$, $\mathbf{e_y^l}$ and $\mathbf{e_y^l}^*$ are the vector of the observed left entropies and the vector of the interpolated left entropies, respectively, both having dimension $n \times 1$, $\mathbf{e_y^r}$ and $\mathbf{e_y^r}^*$ are the vector of the observed right entropies and the vector of the interpolated right entropies, respectively, both having dimension $n \times 1$, and $\mathbf{1}$ is a $(n \times 1)$-vector of all 1's, b and d are regression parameters for the second regression model (referred to as left entropy regression model), f and g are regression parameters for the third regression model (referred to as right entropy regression model). The error term in the regression equation of apexes will remain the same while the error terms in the regression equations of entropies may be different. The error vectors in left and right entropies are $(n \times 1)$ dimensional vectors denoted by $\boldsymbol{\varepsilon}^*_{\mathbf{e_y^l}}$ and $\boldsymbol{\varepsilon}^*_{\mathbf{e_y^r}}$ respectively.

If some prior information about unknown regression coefficients is available on the basis of past experiences, then it may be used to estimate more efficient estimators. We assume that such prior information is in the form of exact linear restrictions on regression coefficients. In the present model, we associate such restrictions in the equations for the estimation of regression coefficients in the linear regression model with fuzzy entropy. Therefore, we make the model capable to take into account possible linear relations between the size of the entropies and the magnitude of the estimated apexes. Moreover, we assume that the regression coefficients $\boldsymbol{\beta}$ are subjected to the $j(j < 3k+1)$ exact linear restrictions, which are given by

$$\mathbf{h} = \mathbf{H}\boldsymbol{\beta}, \tag{3.2.6}$$

where $\mathbf{h}$ and $\mathbf{H}$ are known and the matrix $\mathbf{H}$ is of full row rank.

3.3 Estimation of Regression Coefficients

In many applications, it is possible that the values of the variables are on completely different scales of measurement. Also, the possible larger variations in the values will have larger inter-sample differences, so they will dominate in the calculation of Euclidean distances. Therefore, some form of standardization is necessary to balance out the individual contributions. Consider the Euclidean distance between two fuzzy numbers $y_i = (y_i^a,\, e_{y_i}^l,\, e_{y_i}^r)$ and $y_i^* = (y_i^{a^*},\, e_{y_i}^{l^*},\, e_{y_i}^{r^*})$ along with weights w_1, w_2 and w_3 as follows:

$$\delta_i \equiv \delta(y_i,\, y_i^*) = \sqrt{w_1(y_i^a - y_i^{a^*})^2 + w_2(e_{y_i}^l - e_{y_i}^{l^*})^2 + w_3(e_{y_i}^r - e_{y_i}^{r^*})^2}. \tag{3.3.1}$$

It may be observed that we compute the usual squared differences between the values of variables on their original scales, as in the usual Euclidean distance, but then multiply these squared differences by their corresponding weights.

Next, similar to common linear regression (based on crisp data), the regression parameters are estimated by minimizing the following sum of square errors (we use a compact matrix notation):

$$
\begin{aligned}
&\varphi(\boldsymbol{\beta}, b, d, f, g)\\
&= \sum_{i=1}^{n} \delta_i^2\\
&= \sum_{i=1}^{n} w_1(y_i^a - y_i^{a*})^2 + \sum_{i=1}^{n} w_2(e_{y_i}^l - e_{y_i}^{l*})^2 + \sum_{i=1}^{n} w_3(e_{y_i}^r - e_{y_i}^{r*})^2\\
&= w_1(\mathbf{y^a} - \mathbf{y^{a*}})'(\mathbf{y^a} - \mathbf{y^{a*}}) + w_2(\mathbf{e_y^l} - \mathbf{e_y^{l}}^*)'(\mathbf{e_y^l} - \mathbf{e_y^{l}}^*)\\
&\quad + w_3(\mathbf{e_y^r} - \mathbf{e_y^{r^*}})'(\mathbf{e_y^r} - \mathbf{e_y^{r^*}})\\
&= w_1\left(\mathbf{y^a}'\mathbf{y^a} - 2\mathbf{y^a}'\mathbf{y^{a*}} + \mathbf{y^{a*}}'\mathbf{y^{a*}}\right) + w_2\left(\mathbf{e_y^l}'\mathbf{e_y^l} - 2\mathbf{e_y^l}'\mathbf{e_y^l}^* + \mathbf{e_y^l}^{*\prime}\mathbf{e_y^l}^*\right)\\
&\quad + w_3\left(\mathbf{e_y^r}'\mathbf{e_y^r} - 2\mathbf{e_y^r}'\mathbf{e_y^r}^* + \mathbf{e_y^r}^{*\prime}\mathbf{e_y^r}^*\right)\\
&= w_1\left(\mathbf{y^a}'\mathbf{y^a} - 2\mathbf{y^a}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}\right) + w_2\left(\mathbf{e_y^l}'\mathbf{e_y^l} - 2\mathbf{e_y^l}'(\mathbf{X}\boldsymbol{\beta}b + \mathbf{1}d)\right)\\
&\quad + w_2\left((\mathbf{X}\boldsymbol{\beta}b + \mathbf{1}d)'(\mathbf{X}\boldsymbol{\beta}b + \mathbf{1}d)\right) + w_3\left(\mathbf{e_y^r}'\mathbf{e_y^r} - 2\mathbf{e_y^r}'(\mathbf{X}\boldsymbol{\beta}f + \mathbf{1}g)\right)\\
&\quad + w_3\left((\mathbf{X}\boldsymbol{\beta}f + \mathbf{1}g)'(\mathbf{X}\boldsymbol{\beta}f + \mathbf{1}g)\right)\\
&= w_1\left(\mathbf{y^a}'\mathbf{y^a} - 2\mathbf{y^a}'\mathbf{X}\boldsymbol{\beta}\right) + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}(w_1 + w_2b^2 + w_3f^2)\\
&\quad + w_2\left(\mathbf{e_y^l}'\mathbf{e_y^l} - 2\mathbf{e_y^l}'\mathbf{X}\boldsymbol{\beta}b - 2\mathbf{e_y^l}'\mathbf{1}d\right) + 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{1}(w_2bd + w_3fg)\\
&\quad + w_3\left(\mathbf{e_y^r}'\mathbf{e_y^r} - 2\mathbf{e_y^r}'\mathbf{X}\boldsymbol{\beta}f - 2\mathbf{e_y^r}'\mathbf{1}g\right) + n(w_2d^2 + w_3g^2). \qquad (3.3.2)
\end{aligned}
$$

Differentiating $\varphi(\boldsymbol{\beta}, b, d, f, g)$ i.e., (3.3.2) partially with respect to $\boldsymbol{\beta}$ and equating it to zero, we get

$$
\begin{aligned}
&\frac{\partial\varphi(\boldsymbol{\beta}, b, d, f, g)}{\partial\boldsymbol{\beta}} = 0\\
&\Rightarrow -w_1\mathbf{X}'\mathbf{y^a} + \mathbf{X}'\mathbf{X}\boldsymbol{\beta}(w_1 + w_2b^2 + w_3f^2) - w_2\mathbf{X}'\mathbf{e_y^l}b + \mathbf{X}'\,\mathbf{1}(w_2bd + w_3fg) - w_3\mathbf{X}'\mathbf{e_y^r}f = \mathbf{0}\\
&\Rightarrow \boldsymbol{\beta} = \frac{1}{(w_1 + w_2b^2 + w_3f^2)}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\left[w_1\mathbf{y^a} + w_2\mathbf{e_y^l}b + w_3\mathbf{e_y^r}f - \mathbf{1}(w_2bd + w_3fg)\right]. \quad (3.3.3)
\end{aligned}
$$

Similarly, differentiating (3.3.2) partially with respect to b, d, f and g, we get

$$b = \left(\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}\right)^{-1}\left[\mathbf{e_y^l}'\mathbf{X}\boldsymbol{\beta} - \boldsymbol{\beta}'\mathbf{X}'\mathbf{1}d\right]; \qquad (3.3.4)$$

$$d = \frac{1}{n}\left[\mathbf{e_y^l}'\mathbf{1} - \boldsymbol{\beta}'\mathbf{X}'\mathbf{1}b\right]; \qquad (3.3.5)$$

$$f = \left(\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}\right)^{-1}\left[\mathbf{e_y^r}'\mathbf{X}\boldsymbol{\beta} - \boldsymbol{\beta}'\mathbf{X}'\mathbf{1}g\right]; \qquad (3.3.6)$$

$$g = \frac{1}{n}\left[\mathbf{e_y^r}'\mathbf{1} - \boldsymbol{\beta}'\mathbf{X}'\mathbf{1}f\right]; \qquad (3.3.7)$$

respectively.

The equations (3.3.3-3.3.7) are recursive solutions for the problem of least square estimation with fuzzy data. Therefore, we rewrite the system of equations explicitly in a recursive way as follows:

$$\boldsymbol{\beta}_{i+1} = \frac{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'}{(w_1 + w_2 b_i^2 + w_3 f_i^2)}\left[w_1 \mathbf{y}^{\mathbf{a}} + w_2 \mathbf{e}_{\mathbf{y}}^{\mathbf{l}} b_i + w_3 \mathbf{e}_{\mathbf{y}}^{\mathbf{r}} f_i - \mathbf{1}(w_2 b_i d_i + w_3 f_i g_i)\right] \tag{3.3.8}$$

$$b_{\,i+1} = \left(\boldsymbol{\beta}_{i+1}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}_{i+1}\right)^{-1}\left[\mathbf{e}_{\mathbf{y}}^{\mathbf{l}\,\prime}\mathbf{X}\boldsymbol{\beta}_{i+1} - \boldsymbol{\beta}_{i+1}'\mathbf{X}'\mathbf{1}d_i\right]; \tag{3.3.9}$$

$$d_{i+1} = \frac{1}{n}\left[\mathbf{e}_{\mathbf{y}}^{\mathbf{l}\,\prime}\mathbf{1} - \boldsymbol{\beta}_{i+1}'\mathbf{X}'\mathbf{1}b_i\right]; \tag{3.3.10}$$

$$f_{\,i+1} = \left(\boldsymbol{\beta}_{i+1}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}_{i+1}\right)^{-1}\left[\mathbf{e}_{\mathbf{y}}^{\mathbf{r}\,\prime}\mathbf{X}\boldsymbol{\beta}_{i+1} - \boldsymbol{\beta}_{i+1}'\mathbf{X}'\mathbf{1}g_i\right]; \tag{3.3.11}$$

$$g_{i+1} = \frac{1}{n}\left[\mathbf{e}_{\mathbf{y}}^{\mathbf{r}\,\prime}\mathbf{1} - \boldsymbol{\beta}_{i+1}'\mathbf{X}'\mathbf{1}f_i\right]. \tag{3.3.12}$$

In order to initiate the recursive process of obtaining the estimators, we take some initial values for b, d, f, g and $\boldsymbol{\beta}$. After several number of iterations, the values of estimators get corrected to a pre-defined error of tolerance. We denote these values by $\hat{b}$, $\hat{d}$, $\hat{f}$, $\hat{g}$ and $\hat{\boldsymbol{\beta}}$ in order to differentiate them from the eventually obtained restricted estimator $\tilde{\boldsymbol{\beta}}$ in the next commutation.

In a more general setup, if in the linear regression model (3.2.5), we consider k_1 crisp and k_2 fuzzy input variables then the dimensions of $\mathbf{X}$ and $\boldsymbol{\beta}$ will be $n \times (k_1 + 3k_2 + 1)$ and $(k_1 + 3k_2 + 1) \times 1$ respectively. It may further be noted that the core of the solution's structure will remain the same and we will have similar kind of estimators.

Remark: If we take $w_1 = 2$, $w_2 = 1$, $w_3 = 1$, $b = f$ and $d = g$, then our non-symmetric fuzzy regression model reduces to symmetric fuzzy regression model defined as in Bajaj *et al.* (2009).

Next, we assume that the regression coefficients are subjected to the linear restrictions which are given by (3.2.6). It may be noted that the unrestricted estimator obtained above in (3.3.3) does not satisfy the given restrictions (3.2.6). We aim to obtain the restricted estimator which satisfies the given restrictions under the regression model (3.2.5). For this, we propose to minimize the following score function:

$$
\begin{aligned}
S(\lambda,\, \boldsymbol{\beta},\, b,\, d,\, f,\, g) &= \varphi(\boldsymbol{\beta},\, b,\, d,\, f,\, g) - 2\lambda(\mathbf{H}\boldsymbol{\beta} - \mathbf{h}) \\
&= w_1\left(\mathbf{y^{a}}'\mathbf{y^{a}} - 2\mathbf{y^{a}}'\mathbf{X}\boldsymbol{\beta}\right) + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}(w_1 + w_2 b^2 + w_3 f^2) \\
&\quad + w_2\left(\mathbf{e^{l}_{y}}'\mathbf{e^{l}_{y}} - 2\mathbf{e^{l}_{y}}'\mathbf{X}\boldsymbol{\beta}b - 2\mathbf{e^{l}_{y}}'\mathbf{1}d\right) + 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{1}(w_2 bd + w_3 fg) \\
&\quad + w_3\left(\mathbf{e^{r}_{y}}'\mathbf{e^{r}_{y}} - 2\mathbf{e^{r}_{y}}'\mathbf{X}\boldsymbol{\beta}f - 2\mathbf{e^{r}_{y}}'\mathbf{1}g\right) + n(w_2 d^2 + w_3 g^2) \\
&\quad - 2\lambda(\mathbf{H}\boldsymbol{\beta} - \mathbf{h}), \qquad (3.3.13)
\end{aligned}
$$

where 2λ is the vector of Lagrange's Multiplier.

Differentiating $S(\lambda,\, \boldsymbol{\beta},\, b,\, d,\, f,\, g)$ partially with respect to $\boldsymbol{\beta}$ and equating it to zero, we get

$$
\Rightarrow -w_1\mathbf{X}'\mathbf{y^{a}} + \mathbf{X}'\mathbf{X}\boldsymbol{\beta}(w_1 + w_2 b^2 + w_3 f^2) - w_2\mathbf{X}'\mathbf{e^{l}_{y}}b - w_3\mathbf{X}'\mathbf{e^{r}_{y}}f + \mathbf{X}'\mathbf{1}(w_2 bd + w_3 fg) - \mathbf{H}'\lambda = \mathbf{0} \qquad (3.3.14)
$$

Here, we again relabel the computed restricted estimator by $\tilde{\boldsymbol{\beta}}$. Therefore, in view of equations (3.3.3) and (3.3.14), we get

$$
\begin{aligned}
\Rightarrow \tilde{\boldsymbol{\beta}} &= \frac{1}{(w_1 + w_2 b^2 + w_3 f^2)}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\left[w_1\mathbf{y^{a}} + w_2\mathbf{e^{l}_{y}}b + w_3\mathbf{e^{r}_{y}}f - \mathbf{1}(w_2 bd + w_3 fg)\right] \\
&\quad + \frac{1}{(w_1 + w_2 b^2 + w_3 f^2)}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\lambda \\
\Rightarrow \tilde{\boldsymbol{\beta}} &= \hat{\boldsymbol{\beta}} + \frac{1}{(w_1 + w_2 b^2 + w_3 f^2)}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\lambda. \qquad (3.3.15)
\end{aligned}
$$

Similarly, differentiating $S(\lambda,\, \boldsymbol{\beta},\, b,\, d,\, f,\, g)$ partially with respect to λ and equating it to zero, we get

$$
\begin{aligned}
&\Rightarrow \mathbf{H}\tilde{\boldsymbol{\beta}} = \mathbf{h} \\
&\Rightarrow \mathbf{H}\hat{\boldsymbol{\beta}} + \frac{1}{(w_1 + w_2 b^2 + w_3 f^2)}\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\lambda = \mathbf{h} \\
&\Rightarrow \hat{\lambda} = (w_1 + w_2 b^2 + w_3 f^2)\left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\right]^{-1}\left(\mathbf{h} - \mathbf{H}\hat{\boldsymbol{\beta}}\right). \qquad (3.3.16)
\end{aligned}
$$

From equation (3.3.15) and (3.3.16), we have

$$
\Rightarrow \tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\right]^{-1}\left(\mathbf{h} - \mathbf{H}\hat{\boldsymbol{\beta}}\right). \qquad (3.3.17)
$$

Also, differentiating (3.3.13) partially with respect to b, d, f, g and equating all to zero, we get

$$\tilde{b} = \hat{b}, \quad \tilde{d} = \hat{d}, \quad \tilde{f} = \hat{f}, \quad \tilde{g} = \hat{g},$$

respectively. From equation (3.3.17) we see that

$$\Rightarrow \mathbf{H}\tilde{\boldsymbol{\beta}} = \mathbf{H}\hat{\boldsymbol{\beta}} + \left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\right]\left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\right]^{-1}\left(\mathbf{h} - \mathbf{H}\hat{\boldsymbol{\beta}}\right)$$

$$\Rightarrow \mathbf{H}\tilde{\boldsymbol{\beta}} = \mathbf{H}\hat{\boldsymbol{\beta}} + \left(\mathbf{h} - \mathbf{H}\hat{\boldsymbol{\beta}}\right) = \mathbf{h}.$$

Therefore, the estimator $\tilde{\boldsymbol{\beta}}$ satisfies the given restrictions (3.2.6).

3.4 Numerical Examples

We take the following examples to illustrate the theory discussed:

Example 3.4.1: We apply our procedure to estimate the fuzzy output value for a data consisting of the crisp input and fuzzy output (where left entropy and right entropy are equal) and tabulate the data in the following table 3.1:

Table 3.1: Crisp Input-Int. Fuzzy Output Data

Object i	Crisp Input $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$			Fuzzy Output $\mathbf{y} = \left(\mathbf{e}_y^l, \mathbf{y}^a, \mathbf{e}_y^r\right)$			Estimated Fuzzy Output $\mathbf{y}^* = \left(\mathbf{e}_y^{l*}, \mathbf{y}^{a*}, \mathbf{e}_y^{r*}\right)$		
	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{e}_y^l$	$\mathbf{y}^a$	$\mathbf{e}_y^r$	$\mathbf{e}_y^{l*}$	$\mathbf{y}^{a*}$	$\mathbf{e}_y^{r*}$
1	3	5	9	42	96	42	42.3763	93.8615	42.3763
2	14	8	3	47	120	47	50.6310	122.0379	50.6310
3	7	1	4	33	52	33	29.5587	50.1105	29.5587
4	11	7	3	45	106	45	45.3840	104.1280	45.3840
5	7	12	15	79	189	79	71.5124	193.3134	71.5124
6	8	15	10	65	194	65	71.2972	192.5788	71.2972
7	3	9	6	42	107	42	47.5526	108.1166	47.5526
8	12	15	11	78	216	78	76.9011	211.7069	76.9011
9	10	5	8	52	108	52	47.8304	112.4784	47.8304
10	9	7	4	44	103	44	44.9563	102.6679	44.9563

We obtain $\hat{\boldsymbol{\beta}} = (-3.1355,\ 3.4314,\ 7.6158,\ 5.4027)'$, $\hat{b} = 0.2930$, $\hat{d} = 14.8779$, $\hat{f} = 0.2930$, $\hat{g} = 14.8779$ where the number of iterations required is 106.

Example 3.4.2: We apply our procedure to estimate fuzzy output value for a data consisting of crisp input and fuzzy output (where left and right entropy are not equal) and tabulate the data in the following table 3.2:

Table 3.2: Crisp Input-Fuzzy Output Data

Object i	Crisp Input $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$			Fuzzy Output $\mathbf{y} = (\mathbf{e}_y^l, \mathbf{y}^a, \mathbf{e}_y^r)$			Estimated Fuzzy Output $\mathbf{y}^* = (\mathbf{e}_y^{l*}, \mathbf{y}^{a*}, \mathbf{e}_y^{r*})$		
	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{e}_y^l$	$\mathbf{y}^a$	$\mathbf{e}_y^r$	$\mathbf{e}_y^{l*}$	$\mathbf{y}^{a*}$	$\mathbf{e}_y^{r*}$
1	3	5	9	42	96	47	42.8709	96.0280	45.6421
2	14	8	3	47	120	43	50.1534	120.5313	52.2050
3	7	1	4	33	52	50	29.8881	52.3450	33.9421
4	11	7	3	45	106	45	45.7137	102.2286	47.3028
5	7	12	15	79	189	80	72.9202	197.1349	72.7223
6	8	15	10	65	194	60	70.7116	189.7035	70.7319
7	3	9	6	42	107	40	45.3219	104.2749	47.8509
8	12	15	11	78	216	88	77.1474	211.3582	76.5318
9	10	5	8	52	108	50	48.9491	116.4793	51.1197
10	9	7	4	44	103	42	44.3237	100.9164	46.9514

We obtain $\hat{\boldsymbol{\beta}} = (-5.0772,\ 3.6423,\ 7.2026,\ 5.9013)'$, $\hat{b} = 0.2952$, $\hat{d} = 14.5913$, $\hat{f} = 0.2645$, $\hat{g} = 20.3551$ where the number of iterations required is 109.

Example 3.4.3: We apply our procedure to estimate the fuzzy output value for a data consisting of the fuzzy explanatory and fuzzy response variables where left entropy and right entropy are equal. We take weights for computing the distance as $w_1 = 2$, $w_2 = 1$, $w_3 = 1$ and tabulate the data in the following table 3.3:

We obtain $\hat{\boldsymbol{\beta}} = (5.9349,\ 4.9782,\ -3.3014,\ 1.5408,\ 1.5394)'$, $\hat{b} = -0.1311$, $\hat{d} = 17.9292$, $\hat{f} = -0.1311$, $\hat{g} = 17.9292$, where the number of iterations required is 252.

Example 3.4.4: We apply our procedure to estimate the fuzzy output value for a data consisting of the fuzzy explanatory and fuzzy response variables where left entropy and right entropy are equal. We take weights for computing the distance as $w_1 = 2$, $w_2 = 1$, $w_3 = 1$ and tabulate the data in the following table 3.4:

Table 3.3: Crisp and Fuzzy Input-Fuzzy Output Data

Object i	Crisp and Fuzzy Input $\mathbf{X}=\left(\mathbf{x}_1,\mathbf{e}^{l}_{x_1},\mathbf{x}^{a}_1,\mathbf{e}^{r}_{x_1}\right)$				Fuzzy Output $\mathbf{y}=\left(\mathbf{e}^{l}_{y},\mathbf{y}^{a},\mathbf{e}^{r}_{y}\right)$		Estimated Fuzzy Output $\mathbf{y}^{*}=\left(\mathbf{e}^{l*}_{y},\mathbf{y}^{a*},\mathbf{e}^{r*}_{y}\right)$			
	$\mathbf{x}_1$	$\mathbf{e}^{l}_{x_1}$	$\mathbf{x}^{a}_1$	$\mathbf{e}^{r}_{x_1}$	$\mathbf{e}^{l}_{y}$	$\mathbf{y}^{a}$	$\mathbf{e}^{r}_{y}$	$\mathbf{e}^{l*}_{y}$	$\mathbf{y}^{a*}$	$\mathbf{e}^{r*}_{y}$
1	6	2.0	4.2	2.0	11.7	41.8	11.7	12.649	40.291	12.649
2	7	1.0	6.0	1.0	12.7	50.4	12.7	11.604	48.266	11.604
3	8	1.1	5.0	1.1	12.1	49.9	12.1	11.176	51.527	11.176
4	9	1.0	4.0	1.0	12.3	53.9	12.3	10.703	55.140	10.703
5	10	1.5	3.6	1.5	9.8	57.7	9.8	10.146	59.391	10.146
6	11	2.0	3.0	2.0	8.2	60.5	8.2	10.032	60.255	10.032
7	12	1.9	3.5	1.9	8.7	69.1	8.7	8.832	69.412	8.832
8	13	0.9	3.5	0.9	6.7	74.3	6.7	7.767	77.541	7.767
9	14	0.6	4.0	0.6	6.4	84.3	6.4	7.348	80.736	7.348
10	15	1.7	8.0	1.7	7.8	90.6	7.8	6.142	89.941	6.142

Table 3.4: Fuzzy Input-Fuzzy Output Data

Object i	Fuzzy Input $\mathbf{X}=\left(\mathbf{e}^{l}_{x_1},\mathbf{x}^{a}_1,\mathbf{e}^{r}_{x_1}\right)$			Fuzzy Output $\mathbf{y}=\left(\mathbf{e}^{l}_{y},\mathbf{y}^{a},\mathbf{e}^{r}_{y}\right)$			Estimated Fuzzy Output $\mathbf{y}^{*}=\left(\mathbf{e}^{l*}_{y},\mathbf{y}^{a*},\mathbf{e}^{r*}_{y}\right)$		
	$\mathbf{e}^{l}_{x_1}$	$\mathbf{x}^{a}_1$	$\mathbf{e}^{r}_{x_1}$	$\mathbf{e}^{l}_{y}$	$\mathbf{y}^{a}$	$\mathbf{e}^{r}_{y}$	$\mathbf{e}^{l*}_{y}$	$\mathbf{y}^{a*}$	$\mathbf{e}^{r*}_{y}$
1	3	4	5	4	12	4	4.3573	10.1808	4.3573
2	6	7	8	5	7	5	4.3629	10.1948	4.3629
3	3	6	8	3	9	3	3.2566	7.4409	3.3566
4	2	7	9	1	4	1	2.1164	4.6026	2.1164
5	2	5	7	2	6	2	3.2547	7.4362	3.2547
6	3	6	7	4	8	4	3.2189	7.3471	3.2189
7	2	4	9	3	9	3	3.9369	9.1345	3.9369
8	5	8	13	5	10	5	3.3734	7.7316	3.3734
9	7	12	27	3	5	3	2.6156	5.8453	2.6156
10	23	30	45	2	3	2	1.5072	3.0862	1.5072

We obtain $\hat{\boldsymbol{\beta}} = (11.4898,\ 1.4215,\ -1.5106,\ 0.0938)'$, $\hat{b} = 0.4017$, $\hat{d} = 0.2674$, $\hat{f} = 0.4017$, $\hat{g} = 0.2674$, where the number of iterations required is 97.

For example, let us assume that the regression coefficients $\boldsymbol{\beta}$ are subjected to three exact linear restrictions ($j = 3;\ k = 1;\ j < 3k+1$), which are given by (3.2.6), where $\mathbf{h} = [3,\ 5,\ 7]^T$ and $\mathbf{H} = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 4 & 7 & 3 \\ 3 & 6 & 8 & 9 \end{bmatrix}$. On the basis of computation, we get the restricted estimator $\tilde{\boldsymbol{\beta}} = (2.2679, -1.6056, 0.9924, 0.2208)^T$. For the sake of verification, it may be noted that the obtained restricted estimator satisfies the assumed linear restrictions, i.e.,

$$\mathbf{H}\tilde{\boldsymbol{\beta}} = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 4 & 7 & 3 \\ 3 & 6 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2.2679 \\ -1.6056 \\ 0.9924 \\ 0.2208 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \mathbf{h}.$$

3.5 Simulation Study and Results

In this section, we conduct a simulation study in order to compare the performance of unrestricted estimator $\hat{\boldsymbol{\beta}}$ and restricted estimator $\tilde{\boldsymbol{\beta}}$. For this purpose, we adopt two criteria of dominance mean squared error matrix and absolute bias. We have obtained absolute bias and mean squared error matrices of both of the estimators empirically using 5000 repetitions for various set of weights w_1, w_2 and w_3. We adopted many values of $\boldsymbol{\beta}$ to generate the observations. Some important outcomes of simulation study are presented below:

Case1: When $w_1 = 1$, $w_2 = 1$ and $w_3 = 1$:

Absolute Bias (AB) of the estimators: $\text{AB}(\hat{\boldsymbol{\beta}}) = 0.0151$ and $\text{AB}(\tilde{\boldsymbol{\beta}}) = 0.0012$.
Mean Squared Error Matrices (MSEM) of the estimators:

$$\mathrm{MSEM}(\hat{\boldsymbol{\beta}}) = \begin{pmatrix} 0.0442 & -0.0585 & 0.0011 & -0.0078 & -0.0064 \\ -0.0585 & 0.5166 & -0.0806 & 0.0122 & -0.0118 \\ 0.0011 & -0.0806 & 0.0259 & -0.0084 & 0.0059 \\ -0.0078 & 0.0122 & -0.0084 & 0.0084 & -0.0085 \\ -0.0064 & -0.0118 & 0.0059 & -0.0085 & 0.0700 \end{pmatrix},$$

$$\mathrm{MSEM}(\tilde{\boldsymbol{\beta}}) = \begin{pmatrix} 0.0000 & -0.0003 & -0.0001 & -0.0001 & 0.0004 \\ -0.0003 & 0.0025 & 0.0007 & 0.0010 & -0.0033 \\ -0.0001 & 0.0007 & 0.0002 & 0.0003 & -0.0009 \\ -0.0001 & 0.0010 & 0.0003 & 0.0004 & -0.0014 \\ 0.0004 & -0.0033 & -0.0009 & -0.0014 & 0.0045 \end{pmatrix}.$$

Case 2: When $w_1 = 2$, $w_2 = 1$ and $w_3 = 1$:

Absolute Bias (AB) of the estimators: $\mathrm{AB}(\hat{\boldsymbol{\beta}}) = 2.1465$ and $\mathrm{AB}(\tilde{\boldsymbol{\beta}}) = 0.1704$.
Mean Squared Error Matrices (MSEM) of the estimators:

$$\mathrm{MSEM}(\hat{\boldsymbol{\beta}}) = \begin{pmatrix} 1.9155 & -1.2358 & 1.0290 & -1.3777 & 0.8558 \\ -1.2358 & 1.0505 & -0.6992 & 0.8806 & -0.5498 \\ 1.0290 & -0.6992 & 0.5746 & -0.7524 & 0.4715 \\ -1.3777 & 0.8806 & -0.7524 & 1.0029 & -0.6303 \\ 0.8558 & -0.5498 & 0.4715 & -0.6303 & 0.4323 \end{pmatrix},$$

$$\mathrm{MSEM}(\tilde{\boldsymbol{\beta}}) = \begin{pmatrix} 0.0001 & -0.0012 & -0.0003 & -0.0005 & 0.0016 \\ -0.0012 & 0.0108 & 0.0029 & 0.0045 & -0.0145 \\ -0.0003 & 0.0029 & 0.0008 & 0.0012 & -0.0040 \\ -0.0005 & 0.0045 & 0.0012 & 0.0019 & -0.0061 \\ 0.0016 & -0.0145 & -0.0040 & -0.0061 & 0.0196 \end{pmatrix}.$$

Case 3: When $w_1 = 1/3$, $w_2 = 1/3$ and $w_3 = 1/3$:
Absolute Bias (AB) of the estimators: $\mathrm{AB}(\hat{\boldsymbol{\beta}}) = 0.0118$ and $\mathrm{AB}(\tilde{\boldsymbol{\beta}}) = 0.0001$.
Mean Squared Error Matrices (MSEM) of the estimators:

$$\mathrm{MSEM}(\hat{\boldsymbol{\beta}}) = \begin{pmatrix} 0.0457 & -0.0582 & -0.0001 & -0.0075 & -0.0082 \\ -0.0582 & 0.5036 & -0.0768 & 0.0110 & -0.0030 \\ -0.0001 & -0.0768 & 0.0254 & -0.0081 & 0.0048 \\ -0.0075 & 0.0110 & -0.0081 & 0.0081 & -0.0079 \\ -0.0082 & -0.0030 & 0.0048 & -0.0079 & 0.0700 \end{pmatrix},$$

$$\mathrm{MSEM}(\tilde{\boldsymbol{\beta}}) = \begin{pmatrix} 0.0000 & -0.0003 & -0.0001 & -0.0001 & 0.0003 \\ -0.0003 & 0.0023 & 0.0006 & 0.0010 & -0.0031 \\ -0.0001 & 0.0006 & 0.0002 & 0.0003 & -0.0009 \\ -0.0001 & 0.0010 & 0.0003 & 0.0004 & -0.0013 \\ 0.0003 & -0.0031 & -0.0009 & -0.0013 & 0.0042 \end{pmatrix}.$$

It may be noticed that in all the cases,

- AB($\hat{\boldsymbol{\beta}}$) >AB($\tilde{\boldsymbol{\beta}}$);
- all the eigen values of (MSEM($\hat{\boldsymbol{\beta}}$) - MSEM($\tilde{\beta}$)) are non-negative.

Therefore, the restricted estimator $\tilde{\boldsymbol{\beta}}$ is better than unrestricted estimator $\hat{\boldsymbol{\beta}}$ in the sense of absolute bias as well as MSEM. Thus, when some prior information is available in terms of exact linear restrictions on regression coefficients $\boldsymbol{\beta}$, it is advised to use restricted estimator $\tilde{\boldsymbol{\beta}}$ in place of unrestricted estimator $\hat{\boldsymbol{\beta}}$.

3.6 Conclusions

A fuzzy linear regression (FLR) model with and without some linear restrictions in the form of prior information have been studied. The estimators of regression coefficients have also been obtained with the help of fuzzy entropy for the restricted/unrestriced FLR model by assigning some weights in the distance function. Some numerical examples illustrating the outcomes of the studied models have been provided. Further, simulation study over the obtained estimators has been conducted to compare their performance. It has been observed that the restricted estimator is better than unrestricted estimator in the sense of absolute bias as well as mean square error matrix. Thus, whenever some prior information is available in terms of exact linear restrictions on regression coefficients, it is advised to use restricted estimator $\tilde{\boldsymbol{\beta}}$ in place of unrestricted estimator $\hat{\boldsymbol{\beta}}$.

Chapter 4

'NTV' Metric based Entropies of Interval Valued Intuitionistic Fuzzy Sets and their Applications in Decision Making

4.1 Intorduction

In many real-world decision problems the values of the membership function and the non-membership function in an IFS are difficult to be expressed as exact numbers. Instead, the ranges of their values can usually be specified. In such cases, Atanassov and Gargov (1989) generalized the concept of Intuitionistic Fuzzy Set (IFS) to Interval Valued Intuitionistic Fuzzy Set (IVIFS) and studied its various properties. It may be noted that the entropy and similarity measures are two important concepts in the field of fuzzy set theory and are widely investigated by many researchers from different point of view. The similarity measure of IFSs indicates the degree of similarity between two IFSs and plays a significant role in many applications such as pattern recognition, approximate reasoning and decision making.

Vlachos and Sergiadis (2007) extended the De Luca and Termini's (1972) non-probabilistic entropy for fuzzy sets in the study of the intuitionistic fuzzy information measure. Burillo and Bustince (1996a) introduced the notions of entropy of IFSs and interval-valued fuzzy sets (IVFS) to measure the degree of intuitionism of an IFS and IVFS, respectively. Hung and Yang (2006) gave their axiomatic definitions and characterization of entropy of IFSs and IVFSs with the help of probability theory. Dengfeng and Chuntian (2002) proposed some similarity measures on IFSs and applied them in pattern recognition problems. Further, Liang and Shi (2003) pointed out the drawbacks of Li and Cheng (2002) methods and to overcome them, they proposed several new similarity measures and also discussed relationships between these measures. Further, Szmidt and Kacprzyk (2005) defined a similarity measure using distance measure of IFSs and applied these measures in group decision making problems and medical diagnostic reasoning. Xu (2007a) defined some similarity measures for IVIFSs and applied these similarity measures in pattern recognitions. Hung and Yang (2004) presented a similarity measure of IFSs based on Hausdorff metric and applied it to pattern recognition problems. In the study of fuzzy sets, Wang (1997) defined two similarity measures and Pappis and Karacapilidis (1993) defined three kinds of similarity measures. Hung and Yang (2008) extend these similarity measures from the fuzzy sets to IFSs. Further, Xu and Chen (2008) generalized some formulas of similarity measures of IFSs to IVIFSs. Zeng and Guo (2008) proved that some similarity measures and entropies of IVFSs can be deduced by normalized distances of IVFSs based on their axiomatic definitions. Zeng and Li (2006), Zhang *et al.* (2009) showed that similarity measures and entropies of IVFSs can be obtained by the transformation from each other. Zeng *et al.* (2009) put straight forward some entropy formulas of IFSs according to the relationship between entropies and similarity measures of IFSs. Later on, Wei *et al.* (2011) proposed the entropy for the IVIFSs and obtained the similarity measure for the IVIFSs on the basis of proposed entropy.

Xu and Yager (2006) developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and gave an application of the IFHG operator to multi-criteria decision-making problems with intuitionistic fuzzy information. Xu (2007b) developed some arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Xu (2007c) defined the concept of interval-valued intuitionistic fuzzy number (IVIFN), and gave some basic operational laws of IVIFNs. He gave an interval-valued intuitionistic fuzzy weighted averaging operator and an interval-valued intuitionistic fuzzy weighted geometric operator and defines the score function and the accuracy function of IVIFNs. Xu and Chen (2007) developed some arithmetic aggregation operators, such as the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted averaging (IIFOWA) operator and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator, and gave an application of the IIFHA operator to multi-criteria decision making problems with interval-valued intuitionistic fuzzy information by using the score function and accuracy function of interval-valued intuitionistic fuzzy numbers.

In this chapter, we study some basic definitions related to the intuitionistic fuzzy sets and the interval-valued intuitionistic fuzzy sets in section 4.2. New similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form have been proposed in section 4.3. The proposed similarity measures have also been analogously extended to obtain new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets with the proof of their validity in section 4.4. Further, a new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies in section

4.5. Numerical example by taking interval-valued intuitionistic fuzzy sets has been illustrated in section 4.6.

4.2 Preliminaries

In this section, we present some axiomatic definitions of the similarity measure, entropy measure for intuitionistic fuzzy set and interval-valued intuitionistic fuzzy set which are well known in literature.

Similarity Measure on IFSs:

Hung and Yang (2004) proposed that a real-valued function $S : \mathcal{IFS}(X) \times \mathcal{IFS}(X) \to [0,\, 1]$, is called the similarity measure on $\mathcal{IFS}(X)$, if S satisfies the following axiomatic requirements:

$(S1)$ If $\tilde{A}$ is a crisp set, then $S(\tilde{A},\, \tilde{A}^c) = 0$;

$(S2)$ $S(\tilde{A},\, \tilde{B}) = 1 \Leftrightarrow \tilde{A} = \tilde{B}$, i.e., $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ & $\nu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x)$;

$(S3)$ $S(\tilde{A},\, \tilde{B}) = S(\tilde{B},\, \tilde{A})$;

$(S4)$ If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $S(\tilde{A}, \tilde{C}) \leq S(\tilde{A},\, \tilde{B})$ and $S(\tilde{A},\, \tilde{C}) \leq S(\tilde{B},\, \tilde{C})$.

Similarity Measure on IVIFSs:

Xu and Chen (2008) proposed that a real-valued function $S : \mathcal{IVIFS}(X) \times \mathcal{IVIFS}(X) \to [0,\, 1]$, is called the similarity measure on $\mathcal{IVIFS}(X)$, if S satisfies the following axiomatic requirements:

$(S1)$ $0 \leq S(\tilde{A}_*,\, \tilde{B}_*) \leq 1$;

$(S2)$ $S(\tilde{A}_*,\, \tilde{B}_*) = 1 \Leftrightarrow \tilde{A}_* = \tilde{B}_*$;

$(S3)$ $S(\tilde{A}_*,\, \tilde{B}_*) = S(\tilde{B}_*,\, \tilde{A}_*)$;

$(S4)$ If $\tilde{A}_* \subseteq \tilde{B}_* \subseteq \tilde{C}_*$, then $S(\tilde{A}_*,\, \tilde{C}_*) \leq S(\tilde{A}_*,\, \tilde{B}_*)$ and $S(\tilde{A}_*,\, \tilde{C}_*) \leq S(\tilde{B}_*,\, \tilde{C}_*)$.

Apart from similarity measures for IFSs, we have the entropies (information measures) for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. These entropies play an important role in many fields of research such as pattern recognition, approximate reasoning, decision making etc.

Entropy Measure on IFSs:

Szmidt and Kacprzyk (2001) proposed that a real-valued function $E : \mathcal{IFS}(X) \to [0,\ 1]$ is called the entropy measure on $\mathcal{IFS}(X)$, if E satisfies the following properties:

$(E1)$ $E(\tilde{A}) = 0 \Leftrightarrow \tilde{A}$ is crisp set;

$(E2)$ $E(\tilde{A}) = 1 \Leftrightarrow \mu_{\tilde{A}}(x) = \nu_{\tilde{A}}(x),\ \forall x \in X;$

$(E3)$ $E(\tilde{A}) \leq E(\tilde{B})$ if $\tilde{A}$ is less fuzzy than $\tilde{B}$, i.e., $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x)$ for $\mu_{\tilde{B}}(x) \leq \nu_{\tilde{B}}(x)$ or $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \leq \nu_{\tilde{B}}(x)$ for $\mu_{\tilde{B}}(x) \geq \nu_{\tilde{B}}(x)$, $\forall x \in X$;

$(E4)$ $E(\tilde{A}) = E(\tilde{A}^c),$ where $\tilde{A}^c$ is the complement of $\tilde{A}$.

Entropy Measure on IVIFSs:

Liu *et al.* (2005) proposed that a real-valued function $E : \mathcal{IVIFS}(X) \to [0,\ 1]$ is called the entropy measure on $\mathcal{IVIFS}(X)$, if E satisfies the following properties:

$(E1)$ $E(\tilde{A}_*) = 0 \Leftrightarrow \tilde{A}_*$ is crisp set;

$(E2)$ $E(\tilde{A}_*) = 1 \Leftrightarrow \mu^L_{\tilde{A}_*}(x) = \mu^U_{\tilde{A}_*}(x)$ and $\nu^L_{\tilde{A}_*}(x) = \nu^U_{\tilde{A}_*}(x)$, $\forall x \in X$;

$(E3)$ $E(\tilde{A}_*) \leq E(\tilde{B}_*)$ if $\tilde{A}_*$ is less fuzzy than $\tilde{B}_*$, i.e., $\tilde{A}_* \subseteq \tilde{B}_*$, for $\mu^L_{\tilde{B}_*}(x) \leq \nu^L_{\tilde{B}_*}(x)$ and $\mu^U_{\tilde{B}_*}(x) \leq \nu^U_{\tilde{B}_*}(x)$, or $\tilde{B}_* \subseteq \tilde{A}_*$ for $\mu^L_{\tilde{B}_*}(x) \geq \nu^L_{\tilde{B}_*}(x)$ and $\mu^U_{\tilde{B}_*}(x) \geq \nu^U_{\tilde{B}_*}(x), \forall x \in X$;

$(E4)$ $E(\tilde{A}_*) = E(\tilde{A}^c_*)$, where $\tilde{A}^c_*$ is the complement of $\tilde{A}_*$.

4.3 'NTV' Based Similarity Measures for IFSs and IVIFSs

In this section, we propose similarity measures for IFSs and IVIFSs along with their weighted form based on the 'NTV' metric defined by Neito *et al.* (2003) on n-dimensional unit hypercube I^n.

Neito *et al.* (2003) defined 'NTV' metric, $d_{NTV}(p,\, q)$, on I^n as follows:

$$d_{NTV}(p,\, q) = \frac{\sum\limits_{i=1}^{n} |p_i - q_i|}{\sum\limits_{i=1}^{n} \max\{p_i,\, q_i\}},$$

where $p = (p_1,\, p_2,\, \ldots,\, p_n)$ and $q = (q_1,\, q_2,\, \ldots,\, q_n)$ are n-dimensional vectors in I^n.

Let $\tilde{A} = \{\langle x,\, \mu_{\tilde{A}}(x),\, \nu_{\tilde{A}}(x)\rangle\}$ and $\tilde{B} = \{\langle x,\, \mu_{\tilde{B}}(x),\, \nu_{\tilde{B}}(x)\rangle\}$ are two single-element IFSs. Based on the 'NTV' metric, we propose a new similarity measure between $\tilde{A}$ and $\tilde{B}$ as follows:

$$S^1_{NTV}(\tilde{A}, \tilde{B}) = 1 - \frac{|\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)| + |\nu_{\tilde{A}}(x) - \nu_{\tilde{B}}(x)| + |\pi_{\tilde{A}}(x) - \pi_{\tilde{B}}(x)|}{\max\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\} + \max\{\nu_{\tilde{A}}(x),\, \nu_{\tilde{B}}(x)\} + \max\{\pi_{\tilde{A}}(x),\, \pi_{\tilde{B}}(x)\}}. \quad (4.3.1)$$

Also, we know that

$$|\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)| = \max\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\} - \min\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\},$$

$$|\nu_{\tilde{A}}(x) - \nu_{\tilde{B}}(x)| = \max\{\nu_{\tilde{A}}(x),\, \nu_{\tilde{B}}(x)\} - \min\{\nu_{\tilde{A}}(x),\, \nu_{\tilde{B}}(x)\},$$

$$|\pi_{\tilde{A}}(x) - \pi_{\tilde{B}}(x)| = \max\{\pi_{\tilde{A}}(x),\, \pi_{\tilde{B}}(x)\} - \min\{\pi_{\tilde{A}}(x),\, \pi_{\tilde{B}}(x)\}.$$

Hence, the similarity measure (4.3.1) reduces to

$$S^1_{NTV}(\tilde{A}, \tilde{B}) = \frac{\min\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\} + \min\{\nu_{\tilde{A}}(x),\, \nu_{\tilde{B}}(x)\} + \min\{\pi_{\tilde{A}}(x),\, \pi_{\tilde{B}}(x)\}}{\max\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\} + \max\{\nu_{\tilde{A}}(x),\, \nu_{\tilde{B}}(x)\} + \max\{\pi_{\tilde{A}}(x),\, \pi_{\tilde{B}}(x)\}}. \quad (4.3.2)$$

The similarity measure (4.3.2) is defined for single-element IFS. Further, we define similarity measure of two IFSs $\tilde{A}$ and $\tilde{B}$ under the universe of discourse $X = \{x_1,\, x_2,\, \ldots,\, x_n\}$.

Let $\tilde{A} = \{\langle x_i,\, \mu_{\tilde{A}}(x_i),\, \nu_{\tilde{A}}(x_i)\rangle | x_i \in X\}$ and $\tilde{B} = \{\langle x_i,\, \mu_{\tilde{B}}(x_i),\, \nu_{\tilde{B}}(x_i)\rangle | x_i \in X\}$ are two IFSs, then similarity measure between $\tilde{A}$ and $\tilde{B}$ is defined as

$$S_{NTV}(\tilde{A},\, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} + \min\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} + \max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\} + \max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}}\right). \tag{4.3.3}$$

Theorem 4.3.1: $S_{NTV}(\tilde{A},\, \tilde{B})$ is a valid similarity measure.

Proof. In order to prove that similarity measure (4.3.3) is a valid similarity measure, we prove the four properties $(S1)$ to $(S4)$ as listed by Hung and Yang (2006):

$(S1)$ By the definition of equality of two IFSs, it is easy to show that $S_{NTV}(\tilde{A},\, \tilde{B}) = 1$ if and only if $\tilde{A} = \tilde{B}$.

$(S2)$ If $\tilde{A}$ is a crisp set, then either $\mu_{\tilde{A}}(x_i) = 1, \nu_{\tilde{A}}(x_i) = 0,\, \pi_{\tilde{A}}(x_i) = 0$ or $\mu_{\tilde{A}}(x_i) = 0,\, \nu_{\tilde{A}}(x_i) = 1,\, \pi_{\tilde{A}}(x_i) = 0,\, \forall\, x_i \in X$.
Moreover, for $\tilde{A}^c$, either $\mu_{\tilde{A}^c}(x_i) = 0,\, \nu_{\tilde{A}^c}(x_i) = 1,\, \pi_{\tilde{A}^c}(x_i) = 0$ or $\mu_{\tilde{A}^c}(x_i) = 1, \nu_{\tilde{A}^c}(x_i) = 0, \pi_{\tilde{A}^c}(x_i) = 0, \forall\, x_i \in X$;
$\Rightarrow S_{NTV}(\tilde{A},\, \tilde{A}^c) = 0$.

$(S3)$ In view of the proposed similarity measure, it is easy to verify that

$$S_{NTV}(\tilde{A},\, \tilde{B}) = S_{NTV}(\tilde{B},\, \tilde{A}).$$

$(S4)$ Let $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then, we have $\mu_{\tilde{A}}(x_i) \le \mu_{\tilde{B}}(x_i) \le \mu_{\tilde{C}}(x_i), \nu_{\tilde{A}}(x_i) \ge \nu_{\tilde{B}}(x_i) \ge \nu_{\tilde{C}}(x_i)$ and $\pi_{\tilde{A}}(x_i) \le \pi_{\tilde{B}}(x_i) \le \pi_{\tilde{C}}(x_i)$, $\forall x_i \in X$ which implies

$$\begin{aligned}
\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} &= \min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};\\
\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} &\le \max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};\\
\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\} &\ge \min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};\\
\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\} &= \max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};\\
\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\} &= \min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\};\\
\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\} &\le \max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\},
\end{aligned}$$

which further implies that

$$\frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}} \geq \frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}};$$
$$\frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}} \geq \frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}};$$
$$\frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}} \geq \frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}.$$

Hence, we have

$$\frac{\min\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{B}}(x_i)\right\}+\min\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{B}}(x_i)\right\}+\min\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{B}}(x_i)\right\}}{\max\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{B}}(x_i)\right\}+\max\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{B}}(x_i)\right\}+\max\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{B}}(x_i)\right\}} \geq \frac{\min\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{C}}(x_i)\right\}+\min\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{C}}(x_i)\right\}+\min\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{C}}(x_i)\right\}}{\max\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{C}}(x_i)\right\}+\max\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{C}}(x_i)\right\}+\max\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{C}}(x_i)\right\}}. \tag{4.3.4}$$

Similarly, we have

$$\min\{\mu_{\tilde{B}}(x_i),\, \mu_{\tilde{C}}(x_i)\} \geq \min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};$$
$$\max\{\mu_{\tilde{B}}(x_i),\, \mu_{\tilde{C}}(x_i)\} = \max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};$$
$$\min\{\nu_{\tilde{B}}(x_i),\, \nu_{\tilde{C}}(x_i)\} = \min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};$$
$$\max\{\nu_{\tilde{B}}(x_i),\, \nu_{\tilde{C}}(x_i)\} \leq \max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};$$
$$\min\{\pi_{\tilde{B}}(x_i),\, \pi_{\tilde{C}}(x_i)\} \geq \min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\};$$
$$\max\{\pi_{\tilde{B}}(x_i),\, \pi_{\tilde{C}}(x_i)\} = \max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\},$$

which implies that

$$\frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}} \geq \frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}},$$
$$\frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}} \geq \frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}},$$
$$\frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}} \geq \frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}.$$

Hence, we have

$$\frac{\min\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{B}}(x_i)\right\}+\min\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{B}}(x_i)\right\}+\min\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{B}}(x_i)\right\}}{\max\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{B}}(x_i)\right\}+\max\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{B}}(x_i)\right\}+\max\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{B}}(x_i)\right\}} \geq \frac{\min\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{C}}(x_i)\right\}+\min\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{C}}(x_i)\right\}+\min\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{C}}(x_i)\right\}}{\max\left\{\mu_{\tilde{A}}(x_i),\mu_{\tilde{C}}(x_i)\right\}+\max\left\{\nu_{\tilde{A}}(x_i),\nu_{\tilde{C}}(x_i)\right\}+\max\left\{\pi_{\tilde{A}}(x_i),\pi_{\tilde{C}}(x_i)\right\}} \tag{4.3.5}$$

From equation (4.3.4) and (4.3.5), we have $S_{NTV}(\tilde{A},\, \tilde{B}) \geq S_{NTV}(\tilde{A},\, \tilde{C})$ and $S_{NTV}(\tilde{B},\, \tilde{C}) \geq S_{NTV}(\tilde{A},\, \tilde{C})$.

Therefore, $S_{NTV}(\tilde{A},\, \tilde{B})$ is a valid similarity measure between IFSs $\tilde{A}$ and $\tilde{B}$. $\square$

Further, we associate some weights depending upon importance of the elements of the universal set to define the weighted form of the similarity measure (4.3.3).
Let $w = (w_1,\, w_2,\, \ldots,\, w_n)$ be the weight vector of the elements $x_i,\ i = 1,\, 2,\, \ldots,\, n$.
We propose the following weighted similarity measure:

$$S'_{NTV}(\tilde{A},\, \tilde{B}) = \sum_{i=1}^{n} w_i \left(\frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} + \min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\} + \min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} + \max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\} + \max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}} \right), \tag{4.3.6}$$

where $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.
Remark: If $w = (1/n,\, 1/n, \ldots, 1/n)$, then the weighted similarity measure (4.3.6) reduces to the similarity measure (4.3.3).

Next, we consider two IVIFSs as

$$\tilde{A}_* = \left\{ \left\langle x,\, \left[\mu^L_{\tilde{A}_*}(x),\, \mu^U_{\tilde{A}_*}(x)\right],\, \left[\nu^L_{\tilde{A}_*}(x),\, \nu^U_{\tilde{A}_*}(x)\right] \right\rangle | x \in X \right\}$$

and

$$\tilde{B}_* = \left\{ \left\langle x,\, \left[\mu^L_{\tilde{B}_*}(x),\, \mu^U_{\tilde{B}_*}(x)\right],\, \left[\nu^L_{\tilde{B}_*}(x),\, \nu^U_{\tilde{B}_*}(x)\right] \right\rangle | x \in X \right\}.$$

Analogous to the 'NTV' similarity measure for IFS in (4.3.3), we propose the following similarity measure for IVIFSs:

$$S_{NTV}(\tilde{A}_*,\, \tilde{B}_*) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{M_L(\mu,\, \nu) + M_U(\mu,\, \nu)}{N_L(\mu,\, \nu) + N_U(\mu,\, \nu)} \right), \tag{4.3.7}$$

and the weighted form of the similarity measure (4.3.7) is given by

$$S'_{NTV}(\tilde{A}_*,\, \tilde{B}_*) = \sum_{i=1}^{n} w_i \left(\frac{M_L(\mu,\, \nu) + M_U(\mu,\, \nu)}{N_L(\mu, \nu) + N_U(\mu,\, \nu)} \right), \tag{4.3.8}$$

where

$$M_L(\mu,\, \nu) = \min\left\{\mu^L_{\tilde{A}_*}(x_i), \mu^L_{\tilde{B}_*}(x_i)\right\} + \min\left\{\nu^L_{\tilde{A}_*}(x_i),\, \nu^L_{\tilde{B}_*}(x_i)\right\} + \min\left\{\pi^L_{\tilde{A}_*}(x_i),\, \pi^L_{\tilde{B}_*}(x_i)\right\},$$

$$N_L(\mu,\, \nu) = \max\left\{\mu^L_{\tilde{A}_*}(x_i),\, \mu^L_{\tilde{B}_*}(x_i)\right\} + \max\left\{\nu^L_{\tilde{A}_*}(x_i),\, \nu^L_{\tilde{B}_*}(x_i)\right\} + \max\left\{\pi^L_{\tilde{A}_*}(x_i),\, \pi^L_{\tilde{B}_*}(x_i)\right\},$$

$$M_U(\mu,\, \nu) = \min\left\{\mu^U_{\tilde{A}_*}(x_i),\, \mu^U_{\tilde{B}_*}(x_i)\right\} + \min\left\{\nu^U_{\tilde{A}_*}(x_i),\, \nu^U_{\tilde{B}_*}(x_i)\right\} + \min\left\{\pi^U_{\tilde{A}_*}(x_i),\, \pi^U_{\tilde{B}_*}(x_i)\right\},$$

$$N_U(\mu,\,\nu) = \max\left\{\mu^U_{\tilde{A}_*}(x_i),\,\mu^U_{\tilde{B}_*}(x_i)\right\} + \max\left\{\nu^U_{\tilde{A}_*}(x_i),\,\nu^U_{\tilde{B}_*}(x_i)\right\} + \max\left\{\pi^U_{\tilde{A}_*}(x_i),\,\pi^U_{\tilde{B}_*}(x_i)\right\}.$$

Theorem 4.3.2: Similarity measure $S_{NTV}(\tilde{A}_*,\,\tilde{B}_*)$ is a valid similarity measure.

Proof. The proof of the theorem follows on the similar lines as the proof of theorem 4.3.1. □

4.4 Entropy Measures based on Proposed Similarity Measures

In this section, we introduce entropy measures based on the proposed similarity measures for IFSs and IVIFSs, respectively. We first recall some entropy formulas for IFSs.

For an IFS $\tilde{A} = \{\langle x_i,\,\mu_{\tilde{A}}(x_i),\,\nu_{\tilde{A}}(x_i)\rangle | x_i \in X\}$, Szmidt and Kacprzyk (2001) defined two kind of cardinalities of $\tilde{A}$. The least cardinality or min-sigma-count of $\tilde{A}$ given by

$$\min \sum \mathrm{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i),$$

and the biggest cardinality or max-sigma-count of $\tilde{A}$ given by

$$\max \sum \mathrm{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i) + \pi_{\tilde{A}}(x_i).$$

Using these two cardinalities, Szmidt and Kacprzyk (2001) proposed an entropy measure for $\tilde{A}$ as

$$E_{SK}(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n} \frac{\max \mathrm{count}(\tilde{A}_i \cap \tilde{A}_i^c)}{\max \mathrm{count}(\tilde{A}_i \cup \tilde{A}_i^c)}, \tag{4.4.1}$$

where for each i, $\tilde{A}_i$ denote the single-element IFS corresponding to the element x_i in X, and described as $\tilde{A}_i = \{\langle x_i,\,\mu_{\tilde{A}}(x_i),\,\nu_{\tilde{A}}(x_i)\rangle\}$, Also,

$$\tilde{A}_i \cap \tilde{A}_i^c = \{\langle x_i,\,\min\{\mu_{\tilde{A}}(x_i),\,\nu_{\tilde{A}}(x_i)\},\,\max\{\mu_{\tilde{A}}(x_i),\,\nu_{\tilde{A}}(x_i)\}\rangle\},$$

$$\tilde{A}_i \cup \tilde{A}_i^c = \{\langle x_i,\,\max\{\mu_{\tilde{A}}(x_i),\,\nu_{\tilde{A}}(x_i)\},\,\min\{\mu_{\tilde{A}}(x_i),\,\nu_{\tilde{A}}(x_i)\}\rangle\}.$$

For an IFS $\tilde{A}$, Wang *et al.* (1997) gave a different entropy formula

$$E_{WL}(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n} \frac{\min\{\mu_{\tilde{A}}(x_i),\ \nu_{\tilde{A}}(x_i)\} + \pi_A(x_i)}{\max\{\nu_{\tilde{A}}(x_i),\ \mu_{\tilde{A}}(x_i)\} + \pi_A(x_i)}. \tag{4.4.2}$$

Hung and Liu *et al.* (2005) introduced fuzzy entropy for a vague sets. Using the equivalence of two theories of vague sets and IFSs [Bustince and Burillo (1996b)], Wei *et al.* (2011) transform the Hung and Liu (2005) fuzzy entropy for a vague set to fuzzy entropy for an IFS $\tilde{A}$ as

$$E_{HL}(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n} \frac{1 - |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i)| + \pi_A(x_i)}{1 + |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i| + \pi_A(x_i)}. \tag{4.4.3}$$

Wei *et al.* (2011) also proved that all these entropies given by (4.4.1), (4.4.2) and (4.4.3) are equivalent. In fuzzy set theory, Kosko (1990) gave the idea to drive entropies from the distance and similarity measures. Xuecheng (1992) found various entropies from the similarity measures for the fuzzy sets by the relation $E(\tilde{A}) = S(A,\ A^c)$.

Similarly, we derive entropies for IFSs and IVIFSs from the proposed similarity measures (4.3.3) and (4.3.7) as follows:

$$\begin{aligned} E_T(\tilde{A}) &= S_{NTV}(\tilde{A},\ \tilde{A}^c) \\ &= \tfrac{1}{n}\sum_{i=1}^{n}\left(\frac{\min\left\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\right\} + 0.5\pi_{\tilde{A}}(x_i)}{\max\left\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\right\} + 0.5\pi_{\tilde{A}}(x_i)}\right) \end{aligned} \tag{4.4.4}$$

and

$$E_T(\tilde{A}_*) = \tfrac{1}{n}\sum_{i=1}^{n}\left(\frac{\min\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} + \min\left\{\mu^U_{\tilde{A}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i)\right\} + 0.5\left(\pi^L_{\tilde{A}_*}(x_i) + \pi^U_{\tilde{A}_*}(x_i)\right)}{\max\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} + \max\left\{\mu^U_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} + 0.5\left(\pi^L_{\tilde{A}_*}(x_i) + \pi^U_{\tilde{A}_*}(x_i)\right)}\right). \tag{4.4.5}$$

Theorem 4.4.1: $E_T(\tilde{A})$ is a valid information measure for the intuitionistic fuzzy set.

Proof. In order to prove that the entropy (4.4.4) is a valid measure, we prove all the four properties ($E1$) to ($E4$) as listed by Szmidt and Kacprzyk (2001).

($E1$) If $\tilde{A}$ is a crisp set, then either $\mu_{\tilde{A}}(x_i) = 1,\ \nu_{\tilde{A}}(x_i) = 0,\ \pi_{\tilde{A}}(x_i) = 0$ or $\mu_{\tilde{A}}(x_i) = 0,\ \nu_{\tilde{A}}(x_i) = 1, \pi_{\tilde{A}}(x_i) = 0,\ \forall\, x_i \in X$.
From this we have $S(\tilde{A},\ \tilde{A}^c) = 0 \Rightarrow E_T(\tilde{A}) = 0$.

Conversely, if $E_T(\tilde{A}) = 0$, then $\min\left\{\mu_{\tilde{A}}(x_i),\, \nu_{\tilde{A}}(x_i)\right\} + 0.5 \times \pi_{\tilde{A}}(x_i) = 0,\ \forall x_i \in X$;
which implies either $\mu_{\tilde{A}}(x_i) = 1, \nu_{\tilde{A}}(x_i) = 0,\ \pi_{\tilde{A}}(x_i) = 0$ or
$\mu_{\tilde{A}}(x_i) = 0,\ \nu_{\tilde{A}}(x_i) = 1,\ \pi_{\tilde{A}}(x_i) = 0,\ \forall\, x_i \in X$;
$\Rightarrow \tilde{A}$ is a crisp set.

$(E2)$ Let $\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i),\ \forall x_i \in X$

$\Leftrightarrow \mu_{\tilde{A}^c}(x_i) = \nu_{\tilde{A}}(x_i) = \mu_{\tilde{A}}(x_i),\ \nu_{\tilde{A}^c}(x_i) = \mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i)$,

$\Leftrightarrow \tilde{A}^c = \tilde{A} \Leftrightarrow S_{NTV}(\tilde{A},\, \tilde{A}^c) = 1 \Leftrightarrow E_T(\tilde{A}) = 1$.

$(E3)$ It is easy to verify that, $S_{NTV}(\tilde{A},\, \tilde{A}^c) = S_{NTV}(\tilde{A}^c,\, A) \Leftrightarrow E_T(\tilde{A}) = E_T(\tilde{A}^c)$.

$(E4)$ Suppose that $\mu_{\tilde{B}}(x_i) \le \nu_{\tilde{B}}(x_i)$ for each $x_i \in X$, then $\tilde{A} \subseteq \tilde{B}$, i.e.,
$\mu_{\tilde{A}}(x_i) \le \mu_{\tilde{B}}(x_i),\ \nu_{\tilde{A}}(x_i) \ge \nu_{\tilde{B}}(x_i)$;
$\Rightarrow \mu_{\tilde{A}}(x_i) \le \mu_{\tilde{B}}(x_i) \le \nu_{\tilde{B}}(x_i) \le \nu_{\tilde{A}}(x_i)$;
$\Rightarrow \tilde{A} \subseteq \tilde{B} \subseteq \tilde{B}^c \subseteq \tilde{A}^c$.

Therefore, we have $S_{NTV}(\tilde{A}, \tilde{A}^c) \le S_{NTV}(\tilde{B}, \tilde{A}^c) \le S_{NTV}(\tilde{B}, \tilde{B}^c)$.

Similarly, if $\mu_{\tilde{A}}(x_i) \ge \mu_{\tilde{B}}(x_i), \nu_{\tilde{A}}(x_i) \le \nu_{\tilde{B}}(x_i)$, for $\mu_{\tilde{B}}(x_i) \ge \nu_{\tilde{B}}(x_i)$,
then we have $\nu_{\tilde{A}}(x_i) \le \nu_{\tilde{B}}(x_i) \le \mu_{\tilde{B}}(x_i) \le \mu_{\tilde{A}}(x_i)$,

$\Rightarrow \tilde{A}^c \subseteq \tilde{B}^c \subseteq \tilde{B} \subseteq \tilde{A}$,

$\Rightarrow S_{NTV}(\tilde{A}^c,\, \tilde{A}) \le S_{NTV}(\tilde{B}^c,\, \tilde{A}) \le S_{NTV}(\tilde{B}^c,\, \tilde{B})$,

$\Rightarrow S_{NTV}(\tilde{A},\, \tilde{A}^c) \le S_{NTV}(\tilde{A},\, \tilde{B}^c) \le S_{NTV}(\tilde{B},\, \tilde{B}^c)$,

$\Rightarrow E_T(\tilde{A}) = S_{NTV}(\tilde{A},\, \tilde{A}^c) \le S_{NTV}(\tilde{B},\, \tilde{B}^c) = E_T(\tilde{B})$,

$\Rightarrow E_T(\tilde{A}) \le E_T(\tilde{B})$.

Since $E_T(\tilde{A})$ satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IFSs. □

Theorem 4.3.2: $E_T(\tilde{A}_*)$ is a valid information measure for the interval-valued intuitionistic fuzzy set.

Proof. In order to prove that the entropy (4.4.5) is a valid measure, we prove all the four properties $(E1)$ to $(E4)$ as listed by Liu *et al.* (2005).

$(E1)$ Let $\tilde{A}_*$ be a crisp set. Then either we have
$[\mu^L_{\tilde{A}_*}(x_i), \mu^U_{\tilde{A}_*}(x_i)] = [1,\, 1], [\nu^L_{\tilde{A}_*}(x_i),\, \nu^U_{\tilde{A}_*}(x_i)] = [0,\, 0]\ \&\ [\pi^L_{\tilde{A}_*}(x_i),\, \pi^U_{\tilde{A}_*}(x_i)] = [0,\, 0]$

or

$[\mu^L_{\tilde{A}_*}(x_i), \mu^U_{\tilde{A}_*}(x_i)] = [0,\, 0], [\nu^L_{\tilde{A}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i)] = [1,\, 1] \,\&\, [\pi^L_{\tilde{A}_*}(x_i), \pi^U_{\tilde{A}_*}(x_i)] = [0,\, 0]$ for each $x_i \in X$.

Hence, we have $S(\tilde{A}_*,\, \tilde{A}^c_*) = 0 \Rightarrow E_T(\tilde{A}_*) = 0$.

Conversely, suppose that $E_T(\tilde{A}_*) = 0$, then we have

$\min\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} + \min\left\{\mu^U_{\tilde{A}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i)\right\} + 0.5\left(\pi^L_{\tilde{A}_*}(x_i) + \pi^U_{\tilde{A}_*}(x_i)\right) = 0;$

Since each term in the above equation is non-negative, therefore,

$\min\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} = 0, \min\left\{\mu^U_{\tilde{A}_*}(x_i),\, \nu^U_{\tilde{A}_*}(x_i)\right\} = 0$

and $\pi^L_{\tilde{A}_*}(x_i) + \pi^U_{\tilde{A}_*}(x_i) = 0$ for each $x_i \in X$;

which further implies that $\tilde{A}_*$ is a crisp set.

$(E2)$ If $[\mu^L_{\tilde{A}_*}(x),\, \mu^U_{\tilde{A}_*}(x)] = [\nu^L_{\tilde{A}_*}(x),\, \nu^U_{\tilde{A}_*}]$ for each $x_i \in X$, then from equation (4.4.5) we obtain $E_T(\tilde{A}_*) = 1$.

Conversely, if we suppose that $E_T(\tilde{A}_*) = 1$, then we get

$$\min\left\{\mu^L_{\tilde{A}_*}(x_i),\, \nu^L_{\tilde{A}_*}(x_i)\right\} + \min\left\{\mu^U_{\tilde{A}_*}(x_i),\, \nu^U_{\tilde{A}_*}(x_i)\right\}$$
$$= \max\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} + \max\left\{\mu^U_{\tilde{A}_*}(x_i),\, \nu^L_{\tilde{A}_*}(x_i)\right\};$$

which implies that $[\mu^L_{\tilde{A}_*}(x),\, \mu^U_{\tilde{A}_*}(x)] = [\nu^L_{\tilde{A}_*}(x),\, \nu^U_{\tilde{A}_*}],\, \forall\, x_i \in X$.

$(E3)$ It is easy to verify that, $S_{NTV}(\tilde{A}_*,\, \tilde{A}^c_*) = S_{NTV}(\tilde{A}^c_*,\, A_*) \Rightarrow E_T(\tilde{A}_*) = E_T(\tilde{A}^c_*)$.

$(E4)$ Let $\tilde{A}_*$ is less fuzzy than $\tilde{B}_*$, i.e., $\tilde{A}_* \subseteq \tilde{B}_*$

$\Rightarrow \mu^L_{\tilde{A}_*}(x_i) \leq \mu^L_{\tilde{B}_*}(x_i), \mu^U_{\tilde{A}_*}(x_i) \leq \mu^U_{\tilde{B}_*}(x_i) \,\&\, \nu^L_{\tilde{A}_*}(x_i) \geq \nu^L_{\tilde{B}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i) \geq \nu^U_{\tilde{B}_*}(x_i)$

for $\mu^L_{\tilde{B}_*}(x_i) \leq \nu^L_{\tilde{B}_*}(x_i)$ and $\mu^U_{\tilde{B}_*}(x_i) \leq \nu^U_{\tilde{B}_*}(x_i), \forall x_i \in X$.

Then it follows that $\mu^L_{\tilde{A}_*}(x_i) \leq \mu^L_{\tilde{B}_*}(x_i) \leq \nu^L_{\tilde{B}_*}(x_i) \leq \nu^L_{\tilde{A}_*}(x_i)$

and $\mu^U_{\tilde{A}_*}(x_i) \leq \mu^U_{\tilde{B}_*}(x_i) \leq \nu^U_{\tilde{B}_*}(x_i) \leq \nu^U_{\tilde{A}_*}(x_i), \forall x_i \in X$;

$\Rightarrow \tilde{A}_* \subseteq \tilde{B}_* \subseteq \tilde{B}^c_* \subseteq \tilde{A}^c_*$.

Therefore, we have $S_{NTV}(\tilde{A}_*,\, \tilde{A}^c_*) \leq S_{NTV}(\tilde{B}_*,\, \tilde{A}^c_*) \leq S_{NTV}(\tilde{B}_*,\, \tilde{B}^c_*)$.

Similarly, if $\mu^L_{\tilde{A}_*}(x_i) \geq \mu^L_{\tilde{B}_*}(x_i)$, $\mu^U_{\tilde{A}_*}(x_i) \geq \mu^U_{\tilde{B}_*}(x_i)$ and $\nu^L_{\tilde{A}_*}(x_i) \leq \nu^L_{\tilde{B}_*}(x_i)$, $\nu^U_{\tilde{A}_*}(x_i) \leq \nu^U_{\tilde{B}_*}(x_i)$ for $\mu^L_{\tilde{B}_*}(x_i) \geq \nu^L_{\tilde{B}_*}(x_i)$ and $\mu^U_{\tilde{B}_*}(x_i) \geq \nu^U_{\tilde{B}_*}(x_i), \forall x_i \in X$;

which follows that $\nu^L_{\tilde{A}_*}(x_i) \leq \nu^L_{\tilde{B}_*}(x_i) \leq \mu^L_{\tilde{B}_*}(x_i) \leq \mu^L_{\tilde{A}_*}(x_i)$

and $\nu^U_{\tilde{A}_*}(x_i) \leq \nu^U_{\tilde{B}_*}(x_i) \leq \mu^U_{\tilde{B}_*}(x_i) \leq \mu^U_{\tilde{A}_*}(x_i), \forall x_i \in X$;

$\Rightarrow \tilde{A}^c_* \subseteq \tilde{B}^c_* \subseteq \tilde{B}_* \subseteq \tilde{A}_*$;

$\Rightarrow S_{NTV}(\tilde{A}^c_*,\, \tilde{A}_*) \leq S_{NTV}(\tilde{B}^c_*,\, \tilde{A}_*) \leq S_{NTV}(\tilde{B}^c_*,\, \tilde{B}_*)$;

$\Rightarrow S_{NTV}(\tilde{A}_*,\, \tilde{A}^c_*) \leq S_{NTV}(\tilde{A}_*,\, \tilde{B}^c_*) \leq S_{NTV}(\tilde{B}_*,\, \tilde{B}^c_*)$;

$\Rightarrow E_T(\tilde{A}_*) = S_{NTV}(\tilde{A}_*,\, \tilde{A}_*^c) \leq S_{NTV}(\tilde{B}_*,\, \tilde{B}_*^c) = E_T(\tilde{B}_*)$;

$\Rightarrow E_T(\tilde{A}_*) \leq E_T(\tilde{B}_*)$.

Since $E_T(\tilde{A}_*)$ satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IVIFS. □

4.5 Multiple-Criteria Decision Making with IFS and IVIFS

In this section, we present a new method which is based on the proposed weighted similarity measures, where the objective weights are calculated using the proposed entropies to deal with the Multiple-Criteria Decision Making (MCDM) problems under the intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. Ratings of the alternatives, importance/weights of criteria and importance of decision makers in a group decision committee are the three most significant factors which can affect on the results of decision making problems.

Let $A = \{A_1,\, A_2,\, \ldots,\, A_m\}$ be the set of possible alternatives, $D = \{D_1,\, D_2,\, \ldots,\, D_l\}$ be the set of decision makers and $C = \{C_1,\, C_2,\, \ldots,\, C_n\}$ be the set of criteria with which the performance of alternatives are measured. Assume that the weight information of the criteria and the decision makers are completely unknown. Let $([a_{ij},\, b_{ij}]\, ,\, [c_{ij},\, d_{ij}])$ be the interval-valued intuitionistic fuzzy number, where $[a_{ij},\, b_{ij}]$ indicates the degree that alternative A_i satisfies the criterion C_j, $[c_{ij},\, d_{ij}]$ indicates the degree that alternative A_i does not satisfies the criterion C_j and $[a_{ij},\, b_{ij}] \subset [0,\, 1]$, $[c_{ij},\, d_{ij}] \subset [0,\, 1]$ such that $b_{ij} + d_{ij} \leq 1$, $i = 1,\, 2,\, \ldots,\, m$, $j = 1,\, 2,\, \ldots,\, n$.

Now, we propose the following algorithm to solve the above multiple-criteria decision making problem:

Step 1: Determine the weights of decision makers in the decision group.

Assume that decision group contains l decision makers. The importance/weights of the decision makers in the selection committee may not be equal. The importance/weights of decision makers are considered as linguistic variables expressed by interval-valued intuitionistic fuzzy numbers (IVIFNs).

Let $D_k = ([a_k,\, b_k],\, [c_k,\, d_k])$ be an interval-valued intuitionistic fuzzy number for rating of kth decision maker. Then the subjective weight of kth decision maker can be defined as:

$$\lambda_k = \frac{\left(a_k + b_k + (1 - b_k - d_k)\left(\frac{a_k}{a_k+c_k}\right) + (1 - a_k - c_k)\left(\frac{b_k}{b_k+d_k}\right)\right)}{\sum_{k=1}^{l}\left(a_k + b_k + (1 - b_k - d_k)\left(\frac{a_k}{a_k+c_k}\right) + (1 - a_k - c_k)\left(\frac{b_k}{b_k+d_k}\right)\right)} \tag{4.5.1}$$

and $\sum_{k=1}^{l} \lambda_k = 1$. The linguistic variables for the importance of the decision makers are provided in the Table 4.1. If the importance of all the decision makers is same namely extremely importance, the rating of the kth decision maker can be expressed as $([1,\, 1], [0,\, 0][0,\, 0])$. Then the weight of each decision maker will be $1/l$.

Step 2: Construct the aggregated interval-valued intuitionistic fuzzy decision matrix by pulling the individual decision opinions into a group opinions.

Let $D^k = \left(r_{ij}^{(k)}\right)_{m\times n}$ is an interval-valued intuitionistic fuzzy decision matrix for kth $(k = 1,\, 2,\, \ldots,\, l)$ decision maker and $\lambda = \lambda_1,\, \lambda_2,\, \ldots,\, \lambda_l$ is the weight vector for decision makers, $\sum_{k=1}^{l} \lambda_k = 1,\, \lambda_k \in [0,\, 1]$. In group decision-making process, all the individual decision opinions need to be fused into group opinions to construct aggregated interval-valued intuitionistic fuzzy decision matrix. In order to do, we utilize interval-valued intuitionistic fuzzy weighted average (IIFWA) operator due to Xu and Chen (2007) as follows:

$$\begin{aligned} r_{ij} &= IIFWA_\lambda\left(r_{ij}^{(1)},\, r_{ij}^{(2)},\, \ldots,\, r_{ij}^{(l)}\right) \\ &= \left(\left[1 - \prod_{k=1}^{l}(1 - a_{ij}^{(k)})^{\lambda_k},\, 1 - \prod_{k=1}^{l}(1 - b_{ij}^{(k)})^{\lambda_k}\right],\, \left[\prod_{k=1}^{l}(c_{ij}^{(k)})^{\lambda_k},\, \prod_{k=1}^{l}(d_{ij}^{(k)})^{\lambda_k}\right]\right). \end{aligned}$$

The aggregated interval-valued intuitionistic fuzzy decision matrix can be defined as:

$$D = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}$$

Step 3: Determine the aggregated interval-valued intuitionistic fuzzy weights of the criteria using IIFWA operator.

All criteria may not be assumed to be of equal importance. Let W represents a set of grades of importance for given criteria's. In order to obtain W, all the individual decision maker opinions for the importance of each of criterion need to be combined. Let $w_j^{(k)} = \left(\left[a_{ij}^{(k)}, a_{ij}^{(k)}\right], \left[c_{ij}^{(k)}, d_{ij}^{(k)}\right]\right)$ be an IVIFN assigned to criterion C_j by the kth decision maker. Then the aggregated weights of the criteria are calculated using the IIFWA operator due to Xu and Chen (2007) as follows:

$$\begin{aligned} w_j &= IIFWA_\lambda \left(w_j^{(1)}, w_j^{(2)}, \ldots, w_j^{(l)}\right) \\ &= \left(\left[1 - \prod_{k=1}^{l}(1 - a_{ij}^{(k)})^{\lambda_k}, 1 - \prod_{k=1}^{l}(1 - b_{ij}^{(k)})^{\lambda_k}\right], \left[\prod_{k=1}^{l}(c_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(d_{ij}^{(k)})^{\lambda_k}\right]\right). \end{aligned} \tag{4.5.2}$$

The aggregated weights of the criteria can be defined as:

$W = [w_1, w_2, \ldots, w_n]^T$, here $w_j = ([a_j, b_j], [a_j, b_j]), j = 1, 2, \ldots, n.$

Step 4: Construct the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.

After the aggregated weights of criteria and the aggregated interval valued intuitionistic fuzzy decision matrix are determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix can be defined as follows:

$$D' = D \otimes W = \left(r'_{ij}\right)_{m\times n}, \tag{4.5.3}$$

where $r'_{ij} = \left(\left[a'_{ij}, a'_{ij}\right], \left[c'_{ij}, d'_{ij}\right]\right)$ is an element of the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.

Step 5: Determine the objective weights of criteria using the proposed interval-valued intuitionistic fuzzy entropy measure (4.4.5).

Hwang and Yoon (1981) introduced a method based on information entropy to determine the weights of attributes. Rao (2007), Rao and Singh (2012) methods also suggested the calculation of objective weights using entropy. Xu (2004), Xu and Hui (2009) assigns a small weight to an attribute with similar attribute values across alternatives because such attribute does not help in differentiating alternatives. Furthermore, the method requires all elements in a decision matrix to be normalized to the range [0, 1] so that each column of the decision matrix sums to one.

The entropy of the j^{th} criterion C_j, $j = 1, 2, \ldots, n$ for the m available alternatives can be obtained from entropy measure (4.4.5) as follows:

$$E_j = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{\min\{a_{ij}, c_{ij}\} + \min\{b_{ij}, d_{ij}\} + (1-(a_{ij}+b_{ij}+c_{ij}+d_{ij})/2)}{\max\{a_{ij}, c_{ij}\} + \max\{b_{ij}, d_{ij}\} + (1-(a_{ij}+b_{ij}+c_{ij}+d_{ij})/2)}\right)$$

and the attribute weight w_j for each criterion C_j based on entropy value can be defined as

$$w_j = \frac{1-E_j}{n-\sum\limits_{j=1}^{n} E_j}, \; j = 1, 2, \ldots, n.$$

Step 6: Obtain the interval-valued intuitionistic fuzzy positive-ideal solution (IVIFPIS) and the interval-valued intuitionistic fuzzy negative-ideal solution (IVIFNIS).

Let J_1 and J_2 be benefit criteria and cost criteria, respectively. The interval-valued intuitionistic fuzzy positive-ideal solution, denoted as A^+, and the interval-valued intuitionistic fuzzy negative-ideal solution, denoted as A^- , are defined as follows:

$$A^+ = \left(\left(\left[a_1^+, b_1^+\right], \left[c_1^+, d_1^+\right]\right), \left(\left[a_2^+, b_2^+\right], \left[c_2^+, d_2^+\right]\right), \ldots, \left(\left[a_n^+, b_n^+\right], \left[c_n^+, d_n^+\right]\right)\right),$$

$$A^- = \left(\left(\left[a_1^-, b_1^-\right], \left[c_1^-, d_1^-\right]\right), \left(\left[a_2^-, b_2^-\right], \left[c_2^-, d_2^-\right]\right), \ldots, \left(\left[a_n^-, b_n^-\right], \left[c_n^-, d_n^-\right]\right)\right),$$

where for each $j = 1, 2, \ldots, n$,

$$\begin{aligned}\left(\left[a_j^+, b_j^+\right], \left[c_j^+, d_j^+\right]\right) = (&\langle[\max a_{ij}, \max b_{ij}], [\min a_{ij}, \min b_{ij}] \,|j \in J_1\rangle, \\ &\langle[\min a_{ij}, \min b_{ij}], [\max a_{ij}, \max b_{ij}] \,|j \in J_2\rangle)\end{aligned}$$

$$\begin{aligned}\left(\left[a_j^-, b_j^-\right], \left[c_j^-, d_j^-\right]\right) = (&\langle[\min a_{ij}, \min b_{ij}], [\max a_{ij}, \max b_{ij}] \,|j \in J_1\rangle, \\ &\langle[\max a_{ij}, \max b_{ij}], [\min a_{ij}, \min b_{ij}] \,|j \in J_2\rangle).\end{aligned}$$

Step 7: Calculate the similarity of alternatives with the IVIFPIS and IVIFNIS based on proposed weighted similarity measure (4.3.8), respectively as follows:.

The similarity between alternatives can be found based on the proposed weighted similarity measure (4.3.8) as follows:

$$S(A_i, A^+) = \sum_{j=1}^{n} w_j\left(\frac{p+q}{s+t}\right),$$

and

$$S(A_i,\, A^-) = \sum_{j=1}^{n} w_j \left(\frac{p' + q'}{s' + t'} \right),$$

where

$$p = \min\left\{a_{ij},\, a_j^+\right\} + \min\left\{c_{ij},\, c_j^+\right\} + \min\left\{1 - b_{ij} - d_{ij},\, 1 - b_j^+ - d_j^+\right\},$$

$$q = \min\left\{b_{ij},\, b_j^+\right\} + \min\left\{d_{ij},\, d_j^+\right\} + \min\left\{1 - a_{ij} - c_{ij},\, 1 - a_j^+ - c_j^+\right\},$$

$$s = \max\left\{a_{ij},\, a_j^+\right\} + \max\left\{c_{ij},\, c_j^+\right\} + \max\left\{1 - b_{ij} - d_{ij},\, 1 - b_j^+ - d_j^+\right\},$$

$$t = \max\left\{b_{ij},\, b_j^+\right\} + \max\left\{d_{ij},\, d_j^+\right\} + \max\left\{1 - a_{ij} - c_{ij},\, 1 - a_j^+ - c_j^+\right\},$$

$$p' = \min\left\{a_{ij},\, a_j^-\right\} + \min\left\{c_{ij},\, c_j^-\right\} + \min\left\{1 - b_{ij} - d_{ij},\, 1 - b_j^- - d_j^-\right\},$$

$$q' = \min\left\{b_{ij},\, b_j^-\right\} + \min\left\{d_{ij},\, d_j^-\right\} + \min\left\{1 - a_{ij} - c_{ij},\, 1 - a_j^- - c_j^-\right\},$$

$$s' = \max\left\{a_{ij},\, a_j^-\right\} + \max\left\{c_{ij},\, c_j^-\right\} + \max\left\{1 - b_{ij} - d_{ij},\, 1 - b_j^- - d_j^-\right\},$$

$$t' = \max\left\{b_{ij},\, b_j^-\right\} + \max\left\{d_{ij},\, d_j^-\right\} + \max\left\{1 - a_{ij} - c_{ij},\, 1 - a_j^- - c_j^-\right\},$$

Step 8: Calculate the relative closeness coefficient to the interval-valued intuitionistic fuzzy ideal solution.

The relative closeness coefficient of an alternative A_i with respect A^+ and A^- is defined as follows:

$$C_{i^*} = \frac{S(A_i,\, A^+)}{S(A_i,\, A^+) + S(A_i,\, A^-)},\; i = 1,\, 2,\, \ldots,\, m. \tag{4.5.4}$$

Step 9: Rank all the alternatives.

After the relative closeness coefficient of each alternative is determined, alternatives are ranked according to descending order of C_{i^*}'s and select one that has largest rank, denoted by C_{k^*} among the values C_{i^*}, $i = 1,\, 2,\, \ldots,\, m$. Hence, C_{i^*} is the best choice.

Remark 4.5.2: Since the intuitionistic fuzzy set is a particular case of interval-valued intuitionistic fuzzy set, therefore above proposed algorithm for IVIFSs may similarly be outline for IFSs. For this, we will have to make the following changes:

- In step 1, the subjective weight given by the equation (4.5.1) will be replaced by the weight as suggested in Boran *et al.* (2009).
- In step 2 and 3, the interval-valued intuitionistic fuzzy weighted average (IIFWA) operator due to Xu and Chen (2007) will be replaced by the Xu (2007b) intuitionistic fuzzy weighted average (IFWA) operator.
- In step 5, the entropy measure given by the equation (4.4.5) will be replaced by the entropy measure given by the equation (4.4.4).
- In step 5, the weighted similarity measure given by the equation (4.3.8) will be replaced by the weighted similarity measure given by the equation (4.3.6).

Table 4.1: *Importance of Decision Makers with their Weights.*

	DM_1	DM_2	DM_3
Linguistic terms	Very Important	Medium	Important
Weight	0.393	0.236	0.372

Table 4.2: *Linguistic Terms for Rating the Criteria by Decision Makers*

Linguistic terms	IFNs	IVIFNs
Very Important (VI)	(0.90, 0.10)	$([0.90, 0.95], [0.00, 0.05])$
Important (I)	(0.85, 0.10)	$([0.85, 0.90], [0.05, 0.10])$
Medium (M)	(0.50, 0.40)	$([0.50, 0.55], [0.35, 0.40])$
Unimportant (U)	(0.20, 0.70)	$([0.20, 0.25], [0.65, 0.70])$
Very Unimportant (VU)	(0.05, 0.90)	$([0.05, 0.10], [0.85, 0.90])$

4.6 Numerical Examples

Example 4.6.1: An automobile company desires to select the most appropriate supplier for one of the key elements in its manufacturing process. After pre-evaluation, five suppliers (A_1, A_2, A_3, A_4, A_5) have remained as alternatives for further evaluation. In order to evaluate alternative suppliers, a committee of three decision makers DM_1, DM_2 and DM_3 has been formed. Four criteria are considered as:

Table 4.3: *Linguistic Terms for Rating the Alternatives*

Linguistic terms	IFNs	IVIFNs
Extremely Good (EG)/Extremely High (EH)	(0.95, 0.05)	$([0.90, 95.00], [0.00, 0.05])$
Very Very Good (VVG)/Very Very High (VVH)	(0.85, 0.10)	$([0.85, 0.90], [0.05, 0.10])$
Very good (VG)/Very High (VH)	(0.80, 0.15)	$([0.80, 0.85], [0.10, 0.15])$
Good (G)/High (H)	(0.75, 0.20)	$([0.75, 0.80], [0.15, 0.20])$
Medium Good (MG)/Medium High (MH)	(0.60, 0.25)	$([0.60, 0.65], [0.20, 0.25])$
Fair (F)/Medium (M)	(0.50, 0.35)	$([0.50, 0.55], [0.30, 0.35])$
Medium Poor (MP)/Medium Low (ML)	(0.40, 0.55)	$([0.40, 0.45], [0.50, 0.55])$
Poor (P)/Low (L)	(0.30, 0.65)	$([0.30, 0.35], [0.60, 0.65])$
Very Poor (VP)/Very Low (VL)	(0.20, 0.75)	$([0.20, 0.25], [0.70, 0.75])$
Very Very Poor (VVP)/Very Very Low (VVL)	(0.10, 0.85)	$([0.10, 0.15], [0.80, 0.85])$

- X_1: Product quality.
- X_2: Relationship closeness.
- X_3: Delivery performance.
- X_4: Price.

The proposed method is currently applied to solve this problem and the computational procedure is as follows:

Importance degree of the decision makers on group decision is shown in Table 4.1. Linguistic terms used for the ratings of the decision makers and criteria are given in Table 4.2. In order to obtain the weights of the decision makers, equation (4.5.1) is utilized:

$$\lambda_{DM_1} = 0.393,\ \lambda_{DM_2} = 0.372,\ \lambda_{DM_2} = 0.236.$$

Now the aggregated interval-valued intuitionistic fuzzy decision matrix based on the opinions of decision makers is constructed using IIFWA operator. The linguistic terms shown in Table 4.3 are used to rate each alternative supplier with respect to each criterion by three decision makers. The ratings given by the decision makers to five alternatives is shown in Table 4.4.

Table 4.4: *Rating of the Alternatives*

Criteria	Suppliers	Decisions makers			Criteria	Suppliers	Decisions makers		
		DM_1	DM_2	DM_3			DM_1	DM_2	DM_3
X_1	A_1	G	G	G	X_3	A_1	VG	G	VG
	A_2	MG	G	F		A_2	G	MG	MG
	A_3	VVG	VG	VG		A_3	VG	VG	G
	A_4	MG	G	G		A_4	VG	G	G
	A_5	F	MG	MG		A_5	G	G	MG
X_2	A_1	MG	G	MG	X_4	A_1	H	H	H
	A_2	F	MG	G		A_2	MH	M	MH
	A_3	VG	G	VG		A_3	VH	VH	H
	A_4	F	F	MG		A_4	H	MH	MH
	A_5	MP	F	F		A_5	M	MH	M

Table 4.5: *Importance Weight of the Criteria*

Criteria	DM_1	DM_2	DM_3
X_1	VI	VI	I
X_2	I	I	I
X_3	I	I	M
X_4	M	I	M

The aggregated interval-valued intuitionistic fuzzy decision matrix based on aggregation of decision makers opinions is constructed as follows:

$$D = \begin{bmatrix} ([0.750, 0.800], [0.150, 0.120]) & ([0.642, 0.694], [0.187, 0.237]) & ([0.790, 0.840], [0.110, 0.160]) & ([0.750, 0.800], [0.150, 0.200]) \\ ([0.611, 0.664], [0.217, 0.268]) & ([0.634, 0.687], [0.210, 0.262]) & ([0.668, 0.719], [0.178, 0.229]) & ([0.579, 0.629], [0.220, 0.270]) \\ ([0.822, 0.872], [0.076, 0.128]) & ([0.790, 0.840], [0.110, 0.160]) & ([0.783, 0.833], [0.116, 0.167]) & ([0.783, 0.833], [0.116, 0.167]) \\ ([0.700, 0.751], [0.168, 0.218]) & ([0.540, 0.591], [0.258, 0.309]) & ([0.771, 0.822], [0.128, 0.178]) & ([0.668, 0.719], [0.178, 0.229]) \\ ([0.564, 0.614], [0.234, 0.285]) & ([0.463, 0.514], [0.366, 0.418]) & ([0.703, 0.754], [0.167, 0.217]) & ([0.526, 0.576], [0.272, 0.323]) \end{bmatrix}$$

The importance weights of the criteria provided by decision makers can be linguistic terms. These linguistic terms is represented as interval-valued intuitionistic fuzzy numbers in Table 4.5 and opinions of decision makers on criteria are aggregated using equation (4.5.2) to determine the aggregated weights of criteria. The interval-valued intuitionistic fuzzy weights of criteria after aggregation of opinions of decision makers is:

$$W = \begin{bmatrix} ([0.884, 0.936], [0.000, 0.065]) \\ ([0.850, 0.900], [0.050, 0.100]) \\ ([0.766, 0.825], [0.103, 0.167]) \\ ([0.624, 0.685], [0.221, 0.288]) \end{bmatrix}$$

After the weights of the criteria and the rating of the alternatives has been determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix is constructed utilizing equation (4.5.3) as follows:

$$D' = \begin{bmatrix} ([0.663, 0.749], [0.150, 0.251]) & ([0.546, 0.624], [0.227, 0.313]) & ([0.605, 0.693], [0.201, 0.301]) & ([0.468, 0.548], [0.338, 0.430]) \\ ([0.540, 0.621], [0.217, 0.316]) & ([0.539, 0.618], [0.250, 0.336]) & ([0.511, 0.594], [0.263, 0.358]) & ([0.361, 0.431], [0.392, 0.481]) \\ ([0.726, 0.816], [0.076, 0.184]) & ([0.671, 0.756], [0.154, 0.244]) & ([0.600, 0.688], [0.207, 0.306]) & ([0.489, 0.571], [0.311, 0.407]) \\ ([0.619, 0.703], [0.168, 0.268]) & ([0.459, 0.532], [0.295, 0.378]) & ([0.591, 0.678], [0.217, 0.316]) & ([0.417, 0.493], [0.360, 0.451]) \\ ([0.498, 0.575], [0.234, 0.331]) & ([0.394, 0.462], [0.398, 0.476]) & ([0.538, 0.623], [0.252, 0.348]) & ([0.328, 0.395], [0.433, 0.518]) \end{bmatrix}$$

The entropy of the j^{th} criterion X_j, $j = 1, 2, \ldots, 4$ for the available alternatives can be obtained from entropy measure (4.4.5). The objectives weights of criteria based on entropy are $w_1 = 0.359$, $w_2 = 0.230$, $w_3 = 0.303$, $w_4 = 0.108$.

Product quality, relationship closeness and delivery performance are benefit criteria $J_1 = \{X_1, X_2, X_3\}$ and price is cost criteria $J_2 = \{X_4\}$. Then interval-valued intuitionistic fuzzy positive-ideal solution and interval-valued intuitionistic fuzzy negative ideal solution are

$$\begin{aligned} A^+ &= \{([0.726, 0.816], [0.076, 0.184]), ([0.671, 0.756], [0.154, 0.244]), \\ &\quad ([0.605, 0.693], [0.201, 0.301]), ([0.328, 0.395], [0.433, 0.518])\} \end{aligned}$$

and

$$\begin{aligned} A^- &= \{([0.498, 0.575], [0.234, 0.331]), ([0.394, 0.462], [0.398, 0.476]), \\ &\quad ([0.511, 0.594], [0.263, 0.358]), ([0.489, 0.571], [0.311, 0.407])\}. \end{aligned}$$

Similarity of each alternative with the IVIFPIS and IVIFNIS based on proposed weighted similarity measure (4.3.8) is calculated in Table 4.6.

Finally, using equation (4.5.4), the value of relative closeness of each alternative for the final ranking is shown in Table 4.7.

Thus, the preference order of alternatives is A_1, A_2, A_3, A_4 and A_5 according to decreasing order of C_{i^*} is

$$A_3 > A_1 > A_4 > A_2 > A_5.$$

Table 4.6: *Similarities with the IVIFPIS and IVIFNIS*

Alternatives	S^+	S^-
A_1	0.873	0.772
A_2	0.769	0.883
A_3	0.966	0.711
A_4	0.818	0.818
A_5	0.722	0.953

Table 4.7: *Relative Closeness Coefficients*

Alternatives	C_{i^*}
A_1	0.531
A_2	0.465
A_3	0.576
A_4	0.500
A_5	0.431

4.7 Conclusions

The proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form are valid similarity measures. The new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets analogously obtained through the proposed similarity measures are also valid information measures. Further, a new algorithm for MCDM using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies, has been illustrated through a numerical example.

Chapter 5

Reliability Analysis of k-out-of-$n : G$ System Using Triangular Intuitionistic Fuzzy Numbers

5.1 Introduction

In various disciplines of science and engineering, analysis the reliability of a system which is assembled to perform a certain function play an important role. In general, reliability is defined as the probability that an element (that is, a component, subsystem or full system) will accomplish its assigned task within a specified time, which is designated as the interval $t = [0, t_m]$. There is a great interest in evaluating the reliability of k-out-of-$n : G$ (or k-out-of-$n : F$) systems, mainly because such systems are more general than series or parallel systems and some interconnection networks can be modeled using this technique. A system is said to be a k-out-of-$n : G$-system if it works, if and only if at least k out of n components work. A dual concept called k-out-of-$n : F$-system defined as that it fails, if and only if at least k out of n components fail. Based on these two definitions, a system is k-out-of-$n : G$-system if and only if it is $(n-k+1)$-out-of-n : F-system. Likewise, a system is k-out-of-n : F-system if and only if it is $(n-k+1)$-out-of-n : G-system.

It is well known that the conventional reliability analysis has been found to be inadequate to handle uncertainty of failure data and modeling. To overcome this problem, Onisawa and Kacprzyk (1995) used fuzzy set theory in the evaluation of the reliability of a system. From a long period of time, efforts have been made in the design and development of reliable large-scale systems. In that period of time, considerable work has been done by researchers to build a systematic theory of reliability based on the probability theory. Cai *et al.* (1991a) presented the following two fundamental assumptions in the conventional reliability theory, i.e.,

- **Binary state assumption:** The system is precisely defined as functioning or failing; and
- **Probability assumption:** The system behavior is fully characterized in the context of probability measures.

In order to understand the fuzzy states, consider a computer system that consists of three independent processing units. The system is fully functioning when all the three processing units are functioning simultaneously, and is fully failed when all three processing units are failed completely. However, when just one or two processing units are failed, the system will operate in a degraded situation. In this stage, the system is neither fully functioning nor fully failed, but is in some intermediate state. It may be noted that the assumption of the binary state for describing the system failure and success may be no longer appropriate. Consequently, we can fuzzify the definitions for system failure and success, and then characterize them in terms of the fuzzy sets. Now we are naturally in a position to consider the following two assumptions [cf. Cai and Wen (1990), Cai *et al.* (1991a), (1991b), (1993), (1995)]:

- **Fuzzy-state assumption:** The meaning of system failure cannot be precisely defined in a reasonable way. At any time, system may be in one of the following two states: fuzzy success state or fuzzy failure state.
- **Possibility assumption:** The system behavior can be fully characterized in the context of possibility measures.

5.2 Literature Survey

Profust reliability theory is based on the probability and fuzzy-state assumptions. In profust reliability theory, the system success and failure are characterized by fuzzy states, i.e., the meaning of system failure is not defined in a precise way, but in a fuzzy way. Cai and Wen (1990) introduced the fuzzy success state and fuzzy failure state in which a transition between two fuzzy states was regarded as a fuzzy event. With the concept of fuzzy reliability, they made a comparison between two replacement policies, i.e., the block replacement policy under a non-fuzzy environment and the periodic replacement policy without repair at failures under a fuzzy environment. In the work of Cai *et al.* (1991a), the fuzzy system reliability was established based on the binary state and possibility assumptions. However, in the work of Cai *et al.* (1991b), the fuzzy system reliability was established based on the three-state and possibility assumptions. Further, Cai *et al.* (1993) developed the fuzzy system reliability based on the basis of fuzzy state and probability assumptions. Next, Cai *et al.* (1995) also discussed the system reliability for coherent system based on the fuzzy-state and probability assumptions. Cai *et al.* (1993) presented a fuzzy set-based approach to failure rate and reliability analysis, where profust failure rate is defined in the context of statistics. Further, Singer (1990) used a fuzzy set approach for fault tree and reliability analysis in which the relative frequencies of the basic events are considered as fuzzy numbers. Cheng and Mon (1993) used interval of confidence in order to analyze fuzzy system reliability. Chen (1994) presented a new method for fuzzy system reliability analysis using fuzzy number arithmetic operations in which the reliability of each component is considered as fuzzy number and used simplified fuzzy arithmetic operations rather than complicated interval fuzzy arithmetic operations of fuzzy numbers [Cheng and Mon (1993)] or the complicated extended algebraic fuzzy numbers [Singer (1990)]. Mahapatra and Roy (2009) presented a method to analyze the fuzzy reliability of the series and parallel system using triangular intuitionistic fuzzy numbers (TIFNs) arithmetic operations. Yao *et al.* (2008) applied a statistical methodology in fuzzy system reliability analysis.

We studied some basics of k-out-of-n system with identical or non-identical components in section 5.3. In section 5.4, we use the concept of the statistical confidence interval to estimate the reliability of each component of the system. In literature, the domain of the confidence level is taken to be one which is of less practical significance because highest level of confidence of domain experts lies in between [0, 1] according to the experts knowledge. Therefore, in order to handle the problem in a broader sense, the statistical confidence intervals is being converted to a triangular intuitionistic fuzzy numbers. Then we analyze and discussed the reliability of the k-out-of-n : G-system in the intuitionistic fuzzy sense. In section 5.5, we compare the obtained results using proposed methodology and existing methodology with the help of a numerical example.

5.3 Preliminaries

There are several efficient algorithms available for computing the reliability of a non-repairable k-out-of-n system with identical or non-identical components. For more details, we refer to Misra (1992), Rushdi [(1986), (1993)], Dutuit and Rauzy (2001), and Kuo and Zuo (2003). These algorithms are independent of the failure distribution of the components, i.e., the reliability of each component is considered to be known, but they use the independent assumption among the components's failure behavior and in order to evaluate the reliability of k-out-of-n the following assumptions were made:

1. System consists of n mutually statistically independent components.
2. Initially (at time $t = 0$), all components are working and all are new.
3. The system function if and only if there are at least k working components.
4. There is no repair policy.
5. Reliability of each component is known and the components of the system are numbered from 1 to n.
6. Failure time of each component can follow any arbitrary distribution and considered the following two cases:
    - (i) Identical components: all components are identical and follow the same failure distribution.
    - (ii) Non-identical components: Non-identical components: all or some of the components are non-identical and may follow different failure distributions

5.3.1 Independent Identically Distributed k-out-of-n System

Consider a system with n independent and identically distributed (i.i.d.) components, and the system reliability R can be determined by component reliability p_i, $i = 1, 2, \ldots, n$. We write R a function of $p_1, p_2, \ldots, p_n$ as

$$R = \phi(p_1, p_2, \ldots, p_n),$$

where the structure function ϕ is decided by the structure of the system.

In a k-out-of-$n : G$-system with i.i.d. components, the number of working components follows the binomial distribution with parameter (n, p). Then the reliability of the k-out-of-$n : G$-system with exactly i components work is:

$$\text{Prob(exactly i components work)} = \begin{pmatrix} n \\ i \end{pmatrix} p^i q^{n-i}.$$

Thus, reliability of the k-out-of-n : G-system is equal to the probability that the number of working components is greater than or equal to k:

$$R_G(n, p) = \sum_{i=k}^{n} \begin{pmatrix} n \\ i \end{pmatrix} p^i q^{n-i}.$$

The reliability of a k-out-of-n : F-system with independently and identically distributed (i.i.d.) components is equal to the probability that the number of failing components is less than or equal to $k - 1$.

$$R_F(n, p) = \sum_{i=0}^{k-1} \begin{pmatrix} n \\ i \end{pmatrix} p^{n-i} q^i,$$

As a k-out-of-n : F-system is equivalent to a $n-k+1$-out-of-n : G-system, equation 5.3.1 is equivalent to

$$\sum_{j=n-k+1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} p^j q^{n-j}.$$

If we denote $R_G(n, k; t)$ the reliability of a k-out-of-n : G-system and $R_F(n, n - k + 1; t)$ the reliability of a k-out-of-n : F-system, then we have

$$R_G(n, k; t) = R_F(n, n - k + 1).$$

Both series and parallel systems are special cases of the k-out-of-$n : F$ (or k-out-of-$n : G$)-system. A series system is equivalent to 1-out-of-$n : F$ (or n-out-of-$n : G$) system, while a parallel system is equivalent to n-out-of-$n : F$ (or 1-out-of-$n : G$) system.

In particular, the reliability of the series system is given by

$$R(t) \quad = \prod_{i=1}^{n} p_i.$$

The reliability of the parallel system is given by

$$R(t) = 1 - \prod_{i=1}^{n}(1 - p_i).$$

5.3.2 A Non-i.i.d. k-out-of-n System

For the general case with non-identical components, computing the system reliability is somewhat more difficult. There are several algorithms to compute the reliability of a k-out-of-n system with non-identical components [Kuo and Zuo (2003)]. We consider a well known algorithm that was originally proposed by Barlow and Heidtmann (1984) and Rushdi [(1986), (1993)]. We also utilize the iterative implementation provided in the algorithm given Dutuit and Rauzy (2001). The iteratively implemented algorithm has $O(n(nk+1))$ computational complexity and requires less memory than algorithms by Kuo and Zuo (2003). The algorithm is based on the following recursive relationship. Let $H(k, n)$ be the probability of at least r components out of the n components are good. Then the reliability of k-out-of-$n : G$-system with non-identical components is given by

$$R_G(k, n) \quad = \quad H(k, n),$$

where

$$H(k, n) = \begin{cases} p_n \cdot H(k-1, n-1) + q_n \cdot H(k, n-1) & \text{if } 1 \leq k \leq n, \\ 1 & \text{if } k = 0, n \geq 0, \\ 0 & \text{if } k = n+1, n \geq 0. \end{cases}$$

Although $H(r, n)$ is a two-dimensional array, at any given time, we need to store only a few of these values. In the following iterative algorithm, only $k+1$ values of H are stored in the one-dimensional array K.

Algorithm 5.1

$K[0] = 1;$

for $j = 1$ to k do $K[j] = 0;$

done

for $i = 1$ *to* n

for $j = k$ down to 1 *do*

$K[j] = p_i \cdot K[j-1] + q_i \cdot K[j]$

done

done

At the end of the algorithm, for $1 \leq j \leq k$, the reliability results for a j-out-of-n system will be accumulated in $K[j]$. Hence, the reliability of a k-out-of-n system is equivalent to $K[k]$.

5.4 Reliability Analysis using TIFNs

In this section, we presented an intuitionistic fuzzy statistical approach for evaluating the reliability of a k-out-of-$n : G$-system with independent and non-i.i.d. components, where the reliability of the components are unknown.

Since the values of reliability of the components are not fixed as they are extracted from various sources such as historical records, reliability databases, and system reliability experts opinion, therefore uncertainty in these values is an undeniable fact. For example, based on an independent sample, the intervals between consequent failures are measured of the i^{th} component and the result is $\{45, 230, 105, 150, 115\}$. Then $\lambda_i = 5/45 + 230 + 105 + 150 + 115 = 0.0077519$ and reliability of the component associated with the exponential distribution at time $t = 30$ is 0.79250. But, if we have new observation of failure like 30 hour, then $\lambda_i = 6/45{+}230{+}105{+}150{+}115{+}30 = 0.0088889$ and the reliability of the component at $t = 30$ is 0.76593 which is very different. If we use the point estimate $\bar{R}_i$ to estimate R_i from the statistical data in the past, then we don't know the probability of the error $\bar{R}_i - R_i$. Moreover, the reliability of the system may fluctuate around the estimated value $\bar{R}_i$ during a time interval.

It may be noted that the use of point estimation technique to estimate the reliability of the components is not suitable for such real cases. Therefore, it is more desirable to use interval estimation to obtain (statistical confidence interval) the probability distribution of the error between the estimated value $\bar{R}_i$ and the actual value R_i.

The $(1-\gamma)\%$ confidence interval of R_i is

$$[\bar{R}_i - t_{n_i-1}(\gamma_1)\frac{s_i}{\sqrt{n_i}},\ \bar{R}_i + t_{n_i-1}(\gamma_2)\frac{s_i}{\sqrt{n_i}}],\ i = 1,\ 2, \ldots, n, \tag{5.4.1}$$

where $\gamma_1 + \gamma_2 = \gamma$, $0 < \gamma_1, \gamma_2, \gamma < 1$ and $s_i^2 = \frac{1}{(n_i-1)}\sum_{j=1}^{n_i}(R_{ij} - \bar{R}_i)^2$.

Let T be a t-distributed random variable with $n_i - 1$ degree of freedom. Then $t_{n_i-1}(\gamma_k)$ satisfies the condition $p(T \geq t_{n_i-1}(\gamma_k)) = \gamma_k$, $k = 1, 2$.

The decision maker not only chooses γ_1 and γ_2 to satisfy the condition $\gamma_1 + \gamma_2 = \gamma$, $0 < \gamma_1, \gamma_2, \gamma < 1$, but also satisfies the following conditions:

$$0 < \bar{R}_i - t_{n_i-1}(\gamma_1)\frac{s_i}{\sqrt{n_i}} < 1$$

and

$$0 < \bar{R}_i + t_{n_i-1}(\gamma_2)\frac{s_i}{\sqrt{n_i}} < 1,\ i = 1, 2, \ldots, n.$$

Yao *et al.* (2008) transferred the statistical confidence intervals into the triangular fuzzy numbers. Through these triangular fuzzy numbers, fuzzy reliability of the system is computed at zero degree of hesitation between the membership functions. Moreover, the domain of the confidence level is taken to be one, that is, $\alpha = 1$. Therefore, the results computed by fuzzy numbers have not practically significance, because highest level of confidence of domain experts lies in between $[0, 1]$ according to the experts knowledge. Therefore, we could not consider this problem using fuzzy point of view only. In our approach, we transferred the statistical confidence interval into triangular intuitionistic fuzzy number to overcome the above-mention shortcoming by considering some degree of hesitation between the degree of membership and non-membership functions.

Therefore, we transferred the confidence interval in equation (5.4.1) to the intuitionistic fuzzy numbers as follows:

$$\tilde{R}_i = \left\langle \left(\bar{R}_i - t_{n_i-1}(\gamma_1)\frac{s_i}{\sqrt{n_i}},\ \bar{R}_i,\ \bar{R}_i + t_{n_i-1}(\gamma_2)\frac{s_i}{\sqrt{n_i}} \right); \mu_i,\ \nu_i \right\rangle. \tag{5.4.2}$$

The α-level sets of $\tilde{R}_i, i = 1, 2, \ldots, n$ generates the following pair of intervals:

$$\left(\tilde{R}_i(\alpha_\mu) = [R_i^l(\alpha_\mu);\, R_i^u(\alpha_\mu)],\, \tilde{R}_i(\alpha_\nu) = [R_i^l(\alpha_\nu),\, R_i^u(\alpha_\nu)]\right),$$

where

$$R_i^l(\alpha_\mu) = \bar{R}_i - \left(1 - \frac{\alpha_\mu}{\mu_i}\right) t_{n_i-1}(\gamma_1)\frac{s_i}{\sqrt{n_i}},$$

$$R_i^u(\alpha_\mu) = \bar{R}_i + \left(1 - \frac{\alpha_\mu}{\mu_i}\right) t_{n_i-1}(\gamma_2)\frac{s_i}{\sqrt{n_i}},$$

$$R_i^l(\alpha_\nu) = \bar{R}_i - \left(1 - \frac{\alpha_\nu}{(1-\nu_i)}\right) t_{n_i-1}(\gamma_1)\frac{s_i}{\sqrt{n_i}},$$

$$R_i^u(\alpha_\nu) = \bar{R}_i + \left(1 - \frac{\alpha_\nu}{(1-\nu_i)}\right) t_{n_i-1}(\gamma_2)\frac{s_i}{\sqrt{n_i}},$$

for all $\alpha_\mu \in [0,\, \mu_i]$, $\alpha_\nu \in [0,\, 1-\nu_i]$.

Finally, the intuitionistic fuzzy reliability of the k-out-of-$n : G$-system is calculated by the algorithm 5.1 given in section 5.3.2, for both left and right end points of the α-level sets for different values of α. By the decomposition theorem, we constructed intuitionistic fuzzy reliability of the k-out-of-$n : G$-system as

$$\tilde{R}_s = \left[\bigcup_{0 \le \alpha_\mu \le \mu_s} [R_s^l(\alpha_\mu),\, R_s^u(\alpha_\mu)];\ \bigcup_{0 \le \alpha_\nu \le 1-\nu_s} [R_s^l(\alpha_\nu),\, R_s^u(\alpha_\nu)]\right].$$

5.5 Numerical Example

Example 5.5.1: Consider the following statistical data (Yao *et al.* (2008)) for each component in Table 5.1 of the k-out-of-$n : G$-system consisting three non-i.i.d. components.

Table 5.1: *Statistical Data*

Components	Sample size	Sample mean	Sample standard deviation
C_1	$n_1 = 10$	$\bar{R}_1 = 0.80$	$s_1 = 0.02$
C_2	$n_2 = 20$	$\bar{R}_2 = 0.75$	$s_2 = 0.03$
C_3	$n_2 = 15$	$\bar{R}_3 = 0.90$	$s_3 = 0.01$

Let $\gamma = 0.02$, $\gamma_1 = 0.011$ and $\gamma_2 = 0.009$. Then from the table of the t-distribution with $n_i - 1$ degrees of freedom, $i = 1, 2, 3$, we get the following data: $t_9(\gamma_1) = 2.7017$, $t_{19}(\gamma_1) = 2.5212$, $t_{14}(\gamma_1) = 2.5921$, $t_9(\gamma_2) = 2.9068$, $t_{19}(\gamma_2) = 2.6034$, $t_{14}(\gamma_2) = 2.6946$.

Using the above statistical information, we found end points of the statistical confidence interval for each component which is given in Table 5.2.

Table 5.2: *Two end points.*

i	Degree of freedom	$\bar{R}_i - t_{n_i-1}(\gamma_1)\frac{s_i}{\sqrt{n_i}}$	$\bar{R}_i + t_{n_i-1}(\gamma_2)\frac{s_i}{\sqrt{n_i}}$
1	9	0.7829	0.8184
2	19	0.7331	0.7675
3	14	0.8933	0.9070

Using the Table 5.2, we construct triangular intuitionistic fuzzy numbers by considering 0.2 degree of hesitation as follows:

$$\tilde{R}_1 = \langle (0.7829,\, 0.80,\, 0.8184)\,; 0.6,\, 0.2\rangle\,,$$
$$\tilde{R}_2 = \langle (0.7331,\, 0.75,\, 0.7675)\,; 0.4,\, 0.4\rangle\,,$$
$$\tilde{R}_3 = \langle (0.8933,\, 0.90,\, 0.9070)\,; 0.7,\, 0.1\rangle\,,$$

The α level sets of $\tilde{R}_i$, $i = 1, 2, 3$ are given by

$$\tilde{R}_1(\alpha_\mu) = [0.7829 + 0.0285\alpha_\mu,\, 0.8184 - 0.0307\alpha_\mu],\ \forall \alpha_\mu \in [0,\, 0.6],$$
$$\tilde{R}_2(\alpha_\mu) = [0.7331 + 0.0423\alpha_\mu,\, 0.7675 - 0.0438\alpha_\mu],\ \forall \alpha_\mu \in [0,\, 0.4],$$
$$\tilde{R}_3(\alpha_\mu) = [0.8933 + 0.0096\alpha_\mu,\, 0.9070 - 0.0100\alpha_\mu],\ \forall \alpha_\mu \in [0,\, 0.7],$$

and

$$\tilde{R}_1(\alpha_\nu) = [0.7829 + 0.0214\alpha_\nu,\, 0.8184 - 0.0230\alpha_\nu],\ \forall \alpha_\nu \in [0,\, 0.8],$$
$$\tilde{R}_2(\alpha_\nu) = [0.7331 + 0.0282\alpha_\nu,\, 0.7675 - 0.0292\alpha_\nu],\ \forall \alpha_\nu \in [0,\, 0.6],$$
$$\tilde{R}_3(\alpha_\nu) = [0.8933 + 0.0074\alpha_\nu,\, 0.9070 - 0.0078\alpha_\nu],\ \forall \alpha_\nu \in [0,\, 0.9].$$

Using the algorithm 5.1 given in section 5.3.2, we obtained intuitionistic (vague) fuzzy reliability of the k-out-of-3 : $G, k = 1, 2, 3$ in both existing methods and proposed method for different values of α and results are shown in Tables 5.3, 5.4 and 5.5, respectively.

Table 5.3: *Results of* 1*-out-of-*3 *System*

α	Crisp	Yao et.al.(2008)			Proposed approach				
		a	b	c	a'	a	b	c	c'
0.0	0.9950	0.9938	0.9950	0.9961	0.9938	0.9938	0.9950	0.9961	0.9961
0.1	0.9950	0.9939	0.9950	0.9960	0.9940	0.9940	0.9950	0.9959	0.9959
0.2	0.9950	0.9941	0.9950	0.9959	0.9943	0.9941	0.9950	0.9958	0.9957
0.3	0.9950	0.9942	0.9950	0.9958	0.9945	0.9943	0.9950	0.9957	0.9955
0.4	0.9950	0.9943	0.9950	0.9957	0.9947	0.9945	0.9950	0.9955	0.9953
0.5	0.9950	0.9944	0.9950	0.9956	0.9949	0.9946	0.9950	0.9954	0.9951
0.6	0.9950	0.9945	0.9950	0.9954	–	0.9948	0.9950	0.9952	–
0.7	0.9950	0.9947	0.9950	0.9953	–	0.9949	0.9950	0.9951	–
0.8	0.9950	0.9948	0.9950	0.9952	–	–	0.9950	–	–
0.9	0.9950	0.9949	0.9950	0.9951	–	–	0.9950	–	–
1.0	0.9950	0.9950	0.9950	0.9950	–	–	0.9950	–	–

Table 5.4: *Results of* 2*-out-of-*3 *System*

α	Crisp	Yao et.al.(2008)			Proposed approach				
		a	b	c	a'	a	b	c	c'
0.0	0.9150	0.9028	0.9150	0.9271	0.9028	0.9028	0.9150	0.9271	0.9271
0.1	0.9150	0.9040	0.9150	0.9259	0.9052	0.9045	0.9150	0.9255	0.9249
0.2	0.9150	0.9053	0.9150	0.9248	0.9076	0.9062	0.9150	0.9239	0.9226
0.3	0.9150	0.9065	0.9150	0.9236	0.9099	0.9079	0.9150	0.9222	0.9202
0.4	0.9150	0.9078	0.9150	0.9224	0.9123	0.9096	0.9150	0.9206	0.9179
0.5	0.9150	0.9090	0.9150	0.9212	0.9146	0.9113	0.9150	0.9189	0.9155
0.6	0.9150	0.9102	0.9150	0.9199	–	0.9129	0.9150	0.9172	–
0.7	0.9150	0.9114	0.9150	0.9187	–	0.9146	0.9150	0.9155	–
0.8	0.9150	0.9126	0.9150	0.9175	–	–	0.9150	–	–
0.9	0.9150	0.9138	0.9150	0.9162	–	–	0.9150	–	–
1.0	0.9150	0.9150	0.9150	0.9150	–	–	0.9150	–	–

Table 5.5: *Results of* 3*-out-of-*3 *System*

α	Crisp	Yao et.al.(2008)			Proposed approach				
		a	b	c	a'	a	b	c	c'
0.0	0.5400	0.5127	0.5400	0.5697	0.5127	0.5127	0.5400	0.5697	0.5697
0.1	0.5400	0.5154	0.5400	0.5667	0.5181	0.5165	0.5400	0.5655	0.5637
0.2	0.5400	0.5181	0.5400	0.5637	0.5235	0.5203	0.5400	0.5612	0.5577
0.3	0.5400	0.5208	0.5400	0.5607	0.5290	0.5242	0.5400	0.5570	0.5518
0.4	0.5400	0.5235	0.5400	0.5577	0.5345	0.5280	0.5400	0.5528	0.5459
0.5	0.5400	0.5262	0.5400	0.5547	0.5400	0.5319	0.5400	0.5486	0.5401
0.6	0.5400	0.5290	0.5400	0.5518	–	0.5358	0.5400	0.5445	–
0.7	0.5400	0.5317	0.5400	0.5488	–	0.5397	0.5400	0.5404	–
0.8	0.5400	0.5345	0.5400	0.5459	–	–	0.5400	–	–
0.9	0.5400	0.5372	0.5400	0.5429	–	–	0.5400	–	–
1.0	0.5400	0.5400	0.5400	0.5400	–	–	0.5400	–	–

The true membership and false membership functions corresponding to the obtained results (k-out-of-3 : G, $k = 1, 2, 3$ system) are shown in figure 5.3.

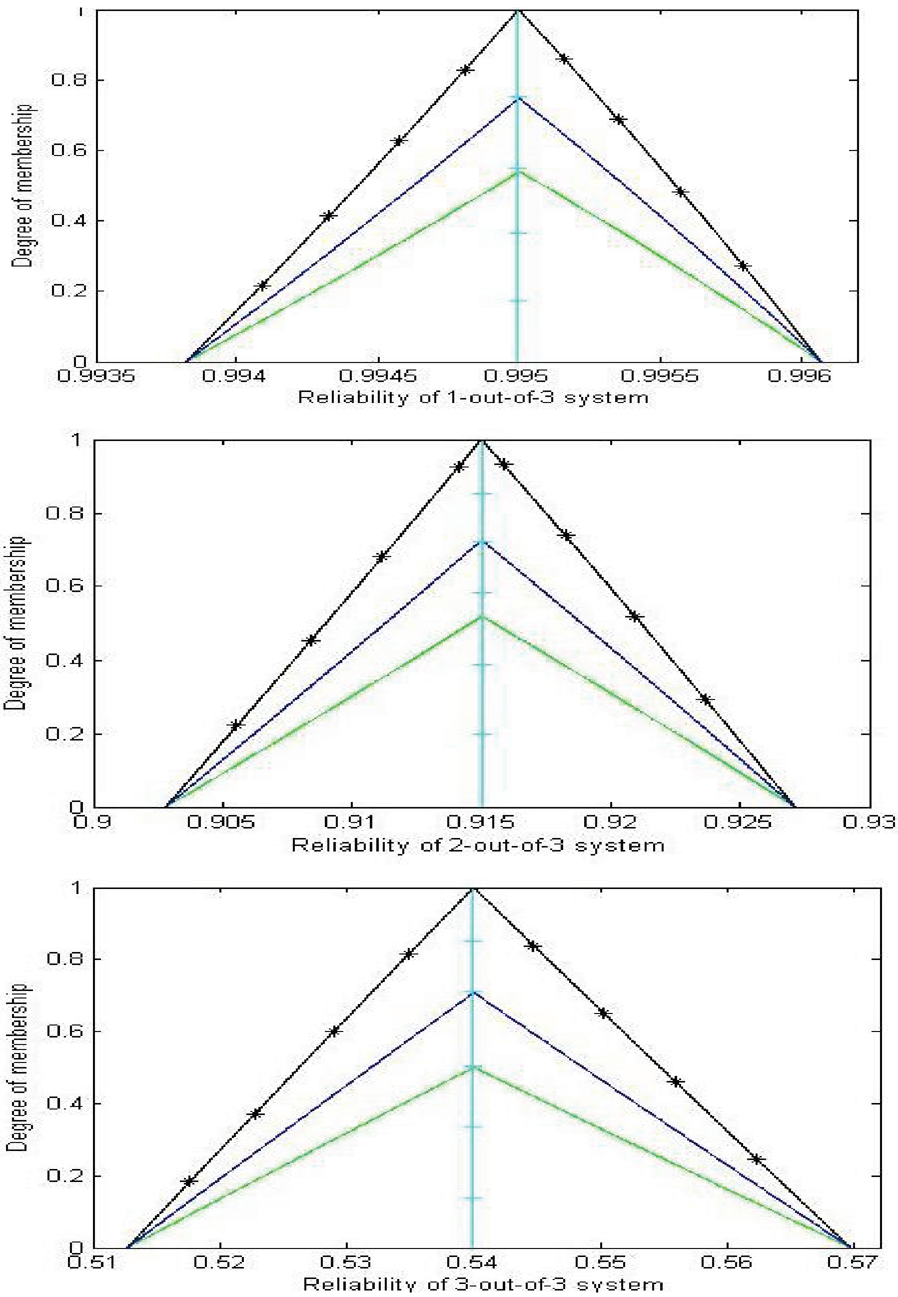

Figure 5.1: *Reliability of* k*-out-of-*3 *System:* $k = 1, 2, 3$

5.5.1 Comparison and Discussion

The intuitionistic fuzzy reliability results using the existing method and proposed method compared as follows:

- Using the point estimate method, the reliability of the series (3-out-of-3 : G) system is equal to 0.54 for all values of α, its means that in this method any vagueness does not consider in the data. Moreover, point estimate method can be suitable where the data are precise and certain, also it does not consider the confidence level of the domain experts.

- Using the table 5.3 in Yao *et al.* (2008) method, it can be be easily seen that the degree of truth membership and false membership correspond to the reliability value 0.9943 are 0.4 and 0.6 respectively. It may be noted that the degree of hesitation has not been considered in the computation. Moreover, Yao *et al.* (2008) do not consider the confidence level of domain experts that lies in the interval [0, 1].

- Using the table 5.3 in the proposed method, it can be seen that the degree of truth membership and false membership values corresponding to the crisp reliability 0.9943 are 0.2 and 0.3 respectively. There is 0.10 degree of hesitation that the value of reliability is 0.9943 which was not considered in Yao *et al.* (2008) method. Moreover, the reliability of the system in view of the Yao *et al.* (2008) is being represented just by one number (which represents the evidences both in favor/against for reliability of the system). On the other hand, the computed reliability of the system by proposed method is being represented by two numbers (which represent the evidence in favor/against and an indeterminacy part for reliability of the system). As the proposed method also considers the confidence level of domain experts ($\alpha \leq 0.8$), therefore, the proposed method is more flexible and realistic.

5.6 Conclusions

The intuitionistic fuzzy reliability of k-out-of-$n : G$-system with independent and non-identically distributed components, where the reliability of the components are unknown, has been analyzed. The reliability of each component has been estimated using statistical confidence interval approach. Considering the highest level of confidence of domain experts that belongs to the interval $[0, 1]$, we converted these statistical confidence interval into triangular intuitionistic fuzzy numbers. The reliability of the k-out-of-$n : G$-system has been calculated and discussed on the basis of these triangular intuitionistic fuzzy numbers with the help of a numerical example.

Chapter 6

Complex Intuitionistic Fuzzy Soft Sets with Distance Measures and Entropies

6.1 Introduction

Ramot *et al.* [(2002), (2003)] introduced a new concept of Complex Fuzzy Set (CFS), where the membership function μ instead of being a real valued function with the range of [0, 1] is replaced by a complex-valued function of the form $r_A(x) \cdot e^{i\Omega_A(x)}$, $(i = \sqrt{-1})$, where $r_A(x)$ is a real valued function such that $r_A(x) \in [0, 1]$ and $\Omega_A(x)$ is a periodic function. The key feature of complex fuzzy sets is the presence of phase and its membership. This gives those complex fuzzy sets wavelike properties which could result in constructive and destructive interference depending on the phase value. Several examples are given by Ramot *et al.* (2003), which demonstrate the utility of these complex fuzzy sets. They also defined several important operations such as complement, union, intersection and discussed fuzzy relations for such Complex Fuzzy Sets (CFSs). On the other hand, Jun *et al.* (2012) used the complex fuzzy set to represent the information with uncertainty and periodicity, where they introduced a product-sum aggregation operator based prediction (PSAOP) method to find the solution of the multiple periodic factor prediction (MPFP) problems. Further, Chen *et al.* (2011) proposed a neurofuzzy system architecture to implement the complex fuzzy rule as a practical application of the concept of complex fuzzy logic.

Alkouri and Salleh (2012) introduced the concept of Complex Intuitionistic Fuzzy Set (CIFS) to represent the information which is happening repeatedly over a period of time. Further, as an application, Alkouri and Salleh (2013) presented an example of suppler selection model which is based on the distance measure of complex intuitionistic fuzzy sets.

Molodtsov (1999) pointed out that the important existing theories viz. probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory etc., which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. The inadequacy of the parametrization tools of these theories make them very limited and difficult. In order to overcome the above stated difficulties, Molodtsov (1999) introduced the concept of Soft Sets for dealing with uncertainties in parameterized form. Later on Maji *et al.* [(2001), (2004a), (2004b)] extended Soft Sets to Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets (IFSSs). Pei and Miao (2005) and Chen *et al.* (2005) have studied and extended the work of Maji *et al.* [(2002), (2003)]. Also, Majumdar and Samanta (2010) have further generalized the concept of fuzzy soft sets.

In this chapter, we introduced the concept of Complex Intuitionistic Fuzzy Soft Sets (CIFSSs) along with their basic operations in section 6.2. New Distance measures for CIFSSs have been obtained on the basis of some well known distance measures and a general way to find the entropies of complex intuitionistic fuzzy soft sets have also been proposed in section 6.3. An application in the area of multi-criteria decision making problem on the basis of the proposed CIFSSs has also been suggested in section 6.4. Finally, the conclusion is provided in section 6.5.

6.2 Complex Intuitionistic Fuzzy Soft Sets

In this section, we introduce the concept of complex intuitionistic fuzzy soft sets with its definition, various operations and properties. Let X be the universal set, E be the set of parameters under consideration and $\mathcal{CIFS}(X)$ denotes the set of all complex intuitionistic fuzzy subsets of X.

Definition 6.3.1 (Complex Intuitionistic Fuzzy Soft Set): A complex Intuitionistic Fuzzy Soft Set (CIFSS) may be represented by the set of ordered pairs as

$$\langle \tilde{F},\, E\rangle = \{\langle \varepsilon,\, \tilde{F}(\varepsilon)|\varepsilon \in E,\, \tilde{F}(\varepsilon) \in \mathcal{CIFS}(X)\},$$

where $\tilde{F} : E \to \mathcal{CIFS}(X)$ such that $\tilde{F}(\varepsilon) = \phi$ (*i.e.*, $\mu_{\tilde{F}(\varepsilon)}(x) = 0$ and $\nu_{\tilde{F}(\varepsilon)}(x) = 1$ for all $x \in X$), if $\varepsilon \notin E$.

Definition 6.3.2 (Operations on CIFSSs): Suppose that $\langle \tilde{F},\, E\rangle$ and $\langle \tilde{G},\, E\rangle$ are two CIFSSs over the universal set X, then we define the following operations:

- **Union:** $\langle \tilde{F},\, E\rangle \cup \langle \tilde{G},\, E\rangle = \langle \tilde{H},\, E\rangle$,
  where
  $$\tilde{H}(\varepsilon) = \Big\{\Big\langle x,\, \mu_{\tilde{F}(\varepsilon)}(x) \diamond \mu_{\tilde{G}(\varepsilon)}(x),\, \nu_{\tilde{F}(\varepsilon)}(x) * \nu_{\tilde{G}(\varepsilon)}(x)\Big\rangle|x \in X,\, \varepsilon \in E\Big\};$$
- **Intersection:** $\langle \tilde{F},\, E\rangle \cap \langle \tilde{G},\, E\rangle = \langle \tilde{H},\, E\rangle$,
  where
  $$\tilde{H}(\varepsilon) = \Big\{\Big\langle x,\, \mu_{\tilde{F}(\varepsilon)}(x) * \mu_{\tilde{G}(\varepsilon)}(x),\, \nu_{\tilde{F}(\varepsilon)}(x) \diamond \nu_{\tilde{G}(\varepsilon)}(x)\Big\rangle|x \in X,\, \varepsilon \in E\Big\};$$
- **Complement:** $(\tilde{F},\, E)^c = (\tilde{F}^c,\, \neg E)$, where $\tilde{F}^c : \neg E \to \mathcal{IFS}(X)$ is mapping given by
  $$\begin{aligned}\tilde{F}^c(\neg\varepsilon) &= \Big\{\Big\langle x,\, \nu_{\tilde{F}(\neg\neg\varepsilon)}(x),\, \mu_{\tilde{F}(\neg\neg\varepsilon)}(x)\Big\rangle|x \in X\Big\}\\ &= \Big\{\Big\langle x,\, \nu_{\tilde{F}(\varepsilon)}(x),\, \mu_{\tilde{F}(\varepsilon)}(x)\Big\rangle|x \in X\Big\},\ \forall\neg\varepsilon \in \neg E;\end{aligned}$$
- **Inclusion:** $\langle \tilde{F},\, E\rangle \subseteq \langle \tilde{G},\, E\rangle$, if and only if $\mu_{\tilde{F}(\varepsilon)}(x) \leq \mu_{\tilde{G}(\varepsilon)}(x)$ and $\nu_{\tilde{F}(\varepsilon)}(x) \geq \nu_{\tilde{G}(\varepsilon)}(x)$, $\forall x \in X$ and $\varepsilon \in E$;
- **Equality:** $\langle \tilde{F},\, E\rangle = \langle \tilde{G},\, E\rangle$, if and only if $\mu_{\tilde{F}(\varepsilon)}(x) = \mu_{\tilde{G}(\varepsilon)}(x)$ and $\nu_{\tilde{F}(\varepsilon)}(x) = \nu_{\tilde{G}(\varepsilon)}(x)$, $\forall x \in X$ and $\varepsilon \in E$.

where the $\diamond$ and $*$ are s-norm and t-norm operators, respectively.

In order to propose the intuitionistic entropy of complex intuitionistic fuzzy soft sets, we need to introduce some important properties of complex intuitionistic fuzzy soft sets.

Definition 6.3.3 (Sharpness of CIFSSs): Let E be the set of parameters and suppose that $\langle \tilde{F}, E\rangle$ and $\langle \tilde{G}, E\rangle$ are two complex intuitionistic fuzzy soft sets over the universal set X, then we say that $\langle \tilde{G}, E\rangle$ is a sharpened version of $\langle \tilde{F}, E\rangle$, i.e., $\langle \tilde{F}, E\rangle \preceq \langle \tilde{G}, E\rangle$ if and only if $\mu_{\tilde{F}(\varepsilon)}(x) \leq \mu_{\tilde{G}(\varepsilon)}(x)$ and $\nu_{\tilde{F}(\varepsilon)}(x) \leq \nu_{\tilde{G}(\varepsilon)}(x)$ (i.e., $r_{\tilde{F}(\varepsilon)}(x) \leq r_{\tilde{G}(\varepsilon)}(x)$ and $k_{\tilde{F}(\varepsilon)}(x) \leq k_{\tilde{G}(\varepsilon)}(x)$, for the amplitude terms and for the phase terms $\omega^r_{\tilde{F}(\varepsilon)}(x) \leq \omega^r_{\tilde{G}(\varepsilon)}(x)$ and $\omega^k_{\tilde{F}(\varepsilon)}(x) \leq \omega^k_{\tilde{G}(\varepsilon)}(x)$), $\forall x \in X$ and $\forall \varepsilon \in E$.

Definition 6.3.4 (Transformation from CIFSS to Complex Fuzzy Soft Set): To every element $f \in \{c | c \in \mathbb{C},\, |c| \leq 1\}^{\{a|a\in\mathbb{C},\, |a|\leq 1\}\times\{b|b\in\mathbb{C},\, |b|\leq 1\}}$,
we associate a mapping

$$f_\alpha : \mathcal{CIFSS}(X) \to \mathcal{CFSS}(X);\;\; \alpha \in [0,\, 1],$$

given by

$$f_\alpha : \langle \tilde{F}, E\rangle \to \langle F_\alpha, E\rangle,$$

where F_α is defined as follows:

$$\begin{aligned} F_\alpha(\varepsilon) &= f_\alpha(\tilde{F}(\varepsilon)) \\ &= f_\alpha\left(\{\langle x,\, \mu_{\tilde{F}(\varepsilon)}(x),\, \nu_{\tilde{F}(\varepsilon)}(x)\rangle | x \in X\}\right) \\ &= \{\langle x,\, \mu_{F(\varepsilon)}(x) = r_{F(\varepsilon)}(x) \cdot e^{i2\pi\omega^r_{F(\varepsilon)}(x)}\rangle | x \in X\},\, \forall \varepsilon \in E. \end{aligned}$$

Here

$$r_{F(\varepsilon)}(x) = \left(r_{\tilde{F}(\varepsilon)}(x) + \alpha \cdot \left(1 - r_{\tilde{F}(\varepsilon)}(x) - k_{\tilde{F}(\varepsilon)}(x)\right)\right)$$

and

$$\omega^r_{F(\varepsilon)}(x) = \left[\left(\frac{\omega^r_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right) + \alpha \cdot \left(1 - \frac{\omega^r_{\tilde{F}(\varepsilon)}(x) + \omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right)\right].$$

The proposed operator f_α defined in definition 6.3.4 is to assign a complex intuitionistic fuzzy soft set to a complex fuzzy soft set. The following theorem provides the properties of the operator f_α:

Theorem 6.3.1: If $\alpha,\, \beta \in [0,\, 1]$ and $\xi,\, \tilde{\xi} \in \mathcal{CIFSS}(X)$, then the following holds:

(i) if $\alpha \leq \beta$, then $f_\alpha(\xi) \subseteq f_\beta(\xi)$;

(ii) if $\xi \subseteq \tilde{\xi}$, then $f_\alpha(\xi) \subseteq f_\alpha(\tilde{\xi})$;

(iii) $f_\alpha\left(f_\beta(\xi)\right) = f_\beta(\xi)$;

(iv) $(f_\alpha(\xi^c))^c = f_{1-\alpha}(\xi)$. $\square$

Proof. Let $\xi = \langle \tilde{F},\, E\rangle = [a_{ij}]_{m\times n}$ and $\tilde{\xi} = \langle \tilde{G},\, E\rangle = [b_{ij}]_{m\times n}$ are two complex intuitionistic fuzzy soft sets.

Let $f_\alpha\left(\langle \tilde{F},\, E\rangle\right) = \langle F_\alpha,\, E\rangle,$ where $\tilde{F}(\varepsilon) = \{\langle x,\, \mu_{\tilde{F}(\varepsilon)},\, \nu_{\tilde{F}(\varepsilon)}\rangle | x \in X\},$ and

$$\begin{aligned} F_\alpha(\varepsilon) &= f_\alpha(\tilde{F}(\varepsilon)) \\ &= f_\alpha\left(\{\langle x,\, \mu_{\tilde{F}(\varepsilon)}(x),\, \nu_{\tilde{F}(\varepsilon)}(x)\rangle | x \in X\}\right) \\ &= \{\langle x,\, r_{F_\alpha(\varepsilon)}(x)\cdot e^{i2\pi\omega^r_{F_\alpha(\varepsilon)}(x)}\rangle | x \in X\},\, \forall \varepsilon \in E. \end{aligned}$$

Here

$$\begin{aligned} r_{F_\alpha(\varepsilon)}(x) &= \left(r_{\tilde{F}(\varepsilon)}(x) + \alpha\cdot\left(1 - r_{\tilde{F}(\varepsilon)}(x) - k_{\tilde{F}(\varepsilon)}(x)\right)\right); \\ \omega^r_{F_\alpha(\varepsilon)}(x) &= \left[\left(\frac{\omega^r_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right) + \alpha\cdot\left(1 - \frac{\omega^r_{\tilde{F}(\varepsilon)}(x) + \omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right)\right]. \end{aligned}$$

Let $f_\beta\left(\langle \tilde{F},\, E\rangle\right) = \langle F_\beta,\, E\rangle,$ where $\tilde{F}(\varepsilon) = \{\langle x,\, \mu_{\tilde{F}(\varepsilon)},\, \nu_{\tilde{F}(\varepsilon)}\rangle | x \in X\},$ and

$$\begin{aligned} F_\beta(\varepsilon) &= f_\beta(\tilde{F}(\varepsilon)) \\ &= f_\beta\left(\{\langle x,\, \mu_{\tilde{F}(\varepsilon)}(x),\, \nu_{\tilde{F}(\varepsilon)}(x)\rangle | x \in X\}\right) \\ &= \{\langle x,\, r_{F_\beta(\varepsilon)}(x)\cdot e^{i2\pi\omega^r_{F_\beta(\varepsilon)}(x)}\rangle | x \in X\},\, \forall \varepsilon \in E. \end{aligned}$$

Here

$$\begin{aligned} r_{F_\beta(\varepsilon)}(x) &= \left(r_{\tilde{F}(\varepsilon)}(x) + \beta\cdot\left(1 - r_{\tilde{F}(\varepsilon)}(x) - k_{\tilde{F}(\varepsilon)}(x)\right)\right); \\ \omega^r_{F_\beta(\varepsilon)}(x) &= \left[\left(\frac{\omega^r_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right) + \beta\cdot\left(1 - \frac{\omega^r_{\tilde{F}(\varepsilon)}(x) + \omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right)\right]. \end{aligned}$$

(1) In the following, we have to prove that $\mu_{F_\alpha(\varepsilon)}(x) \leq \mu_{F_\beta(\varepsilon)}(x)$, i.e, $r_{F_\alpha(\varepsilon)}(x) \leq r_{F_\beta(\varepsilon)}(x)$ and $\omega^r_{F_\alpha(\varepsilon)}(x) \leq \omega^r_{F_\beta(\varepsilon)}(x), \forall x \in X$ and $\varepsilon \in E$.
Since $\alpha \leq \beta$, therefore, we have $r_{F_\alpha(\varepsilon)}(x) \leq r_{F_\beta(\varepsilon)}(x)$ and $\omega^r_{F_\alpha(\varepsilon)}(x) \leq \omega^r_{F_\beta(\varepsilon)}(x)$,
$\Rightarrow \mu_{F_\alpha(\varepsilon)}(x) \leq \mu_{F_\beta(\varepsilon)}(x)$, $\forall x \in X$ and $\varepsilon \in E$.
Hence, $f_\alpha(\xi) \subseteq f_\beta(\xi)$.

(2) Let $f_\alpha(\tilde{\xi}) = f_\alpha\left(\langle \tilde{G}, E\rangle\right) = \langle G_\alpha, E\rangle$, where $\tilde{G}(\varepsilon) = \{\langle x, \mu_{\tilde{G}(\varepsilon)}, \nu_{\tilde{G}(\varepsilon)}\rangle | x \in X\}$,

$$\begin{aligned} G_\alpha(\varepsilon) &= f_\alpha(\tilde{G}(\varepsilon)) \\ &= f_\alpha\left(\{\langle x, \mu_{\tilde{G}(\varepsilon)}(x), \nu_{\tilde{G}(\varepsilon)}(x)\rangle | x \in X\}\right) \\ &= \{\langle x, \mu_{G_\alpha(\varepsilon)}(x) = r_{G_\alpha(\varepsilon)}(x) \cdot e^{i2\pi\omega^r_{G_\alpha(\varepsilon)}(x)}\rangle | x \in X\}, \forall \varepsilon \in E, \end{aligned}$$

Here

$$\begin{aligned} r_{G_\alpha(\varepsilon)}(x) &= \left(r_{\tilde{G}(\varepsilon)}(x) + \alpha \cdot \left(1 - r_{\tilde{G}(\varepsilon)}(x) - k_{\tilde{G}(\varepsilon)}(x)\right)\right) \\ \omega^r_{G_\alpha(\varepsilon)}(x) &= \left[\left(\frac{\omega^r_{\tilde{G}(\varepsilon)}(x)}{2\pi}\right) + \alpha \cdot \left(1 - \frac{\omega^r_{\tilde{G}(\varepsilon)}(x) + \omega^k_{\tilde{G}(\varepsilon)}(x)}{2\pi}\right)\right]. \end{aligned}$$

In the following, we have to prove that $\mu_{F_\alpha(\varepsilon)}(x) \le \mu_{G_\alpha(\varepsilon)}(x)$, i.e.,
$r_{F_\alpha(\varepsilon)}(x) \le r_{G_\alpha(\varepsilon)}(x)$ and $\omega^r_{F_\alpha(\varepsilon)}(x) \le \omega^r_{G_\alpha(\varepsilon)}(x)$, $\forall x \in X$ and $\varepsilon \in E$.
Since $\xi \subseteq \tilde{\xi}$, therefore $\mu_{\tilde{F}(\varepsilon)}(x) \le \mu_{\tilde{G}(\varepsilon)}(x)$ and $\nu_{\tilde{F}(\varepsilon)}(x) \ge \nu_{\tilde{G}(\varepsilon)}(x)$, i.e.,
$r_{\tilde{F}(\varepsilon)}(x) \le r_{\tilde{G}(\varepsilon)}(x)$ and $k_{\tilde{F}(\varepsilon)}(x) \ge k_{\tilde{G}(\varepsilon)}(x)$, for the amplitude terms and for the phase terms $\omega^r_{\tilde{F}(\varepsilon)}(x) \le \omega^r_{\tilde{G}(\varepsilon)}(x)$ and $\omega^k_{\tilde{F}(\varepsilon)}(x) \ge \omega^k_{\tilde{G}(\varepsilon)}(x)$,$\forall x \in X$ and $\varepsilon \in E$.
Thus, $(1-\alpha)\cdot r_{\tilde{F}(\varepsilon)}(x) \le (1-\alpha)\cdot r_{\tilde{G}(\varepsilon)}(x)$, $\alpha \cdot k_{\tilde{F}(\varepsilon)}(x) \ge \alpha \cdot k_{\tilde{G}(\varepsilon)}(x)$, and
$\frac{(1-\alpha)}{2\pi}\cdot\omega^r_{\tilde{F}(\varepsilon)}(x) \le \frac{(1-\alpha)}{2\pi}\cdot\omega^r_{\tilde{G}(\varepsilon)}(x)$, $\frac{\alpha}{2\pi}\cdot\omega^k_{\tilde{F}(\varepsilon)}(x) \ge \frac{\alpha}{2\pi}\cdot\omega^k_{\tilde{G}(\varepsilon)}(x)$,
which implies
$\alpha + (1-\alpha)\cdot r_{\tilde{F}(\varepsilon)}(x) - \alpha\cdot k_{\tilde{G}(\varepsilon)}(x) \le \alpha + (1-\alpha)\cdot r_{\tilde{G}(\varepsilon)}(x) - \alpha \cdot k_{\tilde{F}(\varepsilon)}(x)$ and

$$\begin{aligned} &\alpha + \left[\frac{(1-\alpha)}{2\pi}\cdot\omega^r_{\tilde{F}(\varepsilon)}(x) - \frac{\alpha}{2\pi}\cdot\omega^k_{\tilde{G}(\varepsilon)}(x)\right] \\ &\qquad\qquad \le \alpha + \left[\frac{(1-\alpha)}{2\pi}\cdot\omega^r_{\tilde{G}(\varepsilon)}(x) - \frac{\alpha}{2\pi}\cdot\omega^k_{\tilde{F}(\varepsilon)}(x)\right], \end{aligned}$$

$\Rightarrow r_{F_\alpha(\varepsilon)}(x) \le r_{G_\alpha(\varepsilon)}(x)$ and $\omega^r_{F_\alpha(\varepsilon)}(x) \le \omega^r_{G_\alpha(\varepsilon)}(x)$.
Thus, we have $\mu_{F_\alpha(\varepsilon)}(x) \le \mu_{G_\alpha(\varepsilon)}(x)$. Hence, $f_\alpha(\xi) \subseteq f_\alpha(\tilde{\xi})$.

(3) $f_\alpha(f_\beta(\xi)) = f_\alpha(\langle F_\beta, E\rangle) = \langle (F_\beta)_\alpha, E\rangle$, where $(F_\beta)_\alpha(\varepsilon) = f_\alpha(F_\beta(\varepsilon)) = f_\alpha(f_\beta(\tilde{F}(\varepsilon)))$, $\forall \varepsilon \in E$.
Next, we have to prove that $f_\alpha(f_\beta(\tilde{F}(\varepsilon))) = (f_\beta(\tilde{F}(\varepsilon))$.
We have

$$\begin{aligned} f_\beta(\tilde{F}(\varepsilon)) &= f_\beta\left(\{\langle x, \mu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(x)\rangle | x \in X\}\right) \\ &= \{\langle x, r_{F_\beta(\varepsilon)}(x) \cdot e^{i2\pi\omega^r_{F_\beta(\varepsilon)}(x)}\rangle | x \in X\}, \forall \varepsilon \in E. \qquad (6.2.1) \end{aligned}$$

Here

$$\begin{aligned} r_{F_\beta(\varepsilon)}(x) &= \left(r_{\tilde{F}(\varepsilon)}(x) + \beta \cdot \left(1 - r_{\tilde{F}(\varepsilon)}(x) - k_{\tilde{F}(\varepsilon)}(x)\right)\right) \\ \omega^r_{F_\beta(\varepsilon)}(x) &= \left[\left(\frac{\omega^r_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right) + \beta \cdot \left(1 - \frac{\omega^r_{\tilde{F}(\varepsilon)}(x) + \omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right)\right]. \end{aligned}$$

Thus

$$\begin{aligned} f_\alpha(f_\beta(\tilde{F}(\varepsilon))) &= f_\alpha(\{\langle x,\, r_{F_\beta(\varepsilon)}(x) \cdot e^{i2\pi\omega^r_{F_\beta(\varepsilon)}(x)}\rangle | x \in X\}) \\ &= \{\langle x,\, r_{F_\beta(\varepsilon)}(x) \cdot e^{i2\pi\omega^r_{F_\beta(\varepsilon)}(x)}\rangle | x \in X\},\ \forall \varepsilon \in E. \qquad (6.2.2) \end{aligned}$$

From the equation (6.2.1) and (6.2.2), we have $f_\alpha\left(f_\beta(\xi)\right) = f_\beta(\xi)$.

(4) Let $\xi^c = \langle \tilde{F},\, E\rangle^c = \langle \tilde{F}^c,\, \neg E\rangle$,

where $\tilde{F}^c(\varepsilon) = \{\langle x,\, \nu_{\tilde{F}(\neg\varepsilon)}(x),\, \mu_{\tilde{F}(\neg\varepsilon)}(x)\rangle | x \in X\}$, $\forall \varepsilon \in E$.

Then we have

$f_\alpha(\xi^c) = f_\alpha(\langle \tilde{F}^c,\, \neg E\rangle) = \langle (F^c)_\alpha,\, \neg E\rangle$, where

$$\begin{aligned} (F^c)_\alpha(\varepsilon) &= f_\alpha(\tilde{F}^c(\varepsilon)) \\ &= f_\alpha\left(\{\langle x,\, \nu_{\tilde{F}(\neg\varepsilon)}(x),\, \mu_{\tilde{F}(\neg\varepsilon)}(x)\rangle | x \in X\}\right) \\ &= \{\langle x,\, (1 - r_{F_\alpha(\neg\varepsilon)}(x)) \cdot e^{i2\pi\bar{\omega}^r_{F_\alpha(\neg\varepsilon)}(x)}\rangle | x \in X\},\ \forall \varepsilon \in \neg E. \end{aligned}$$

Here

$$\begin{aligned} 1 - r_{F_\alpha(\neg\varepsilon)}(x) &= \left(k_{\tilde{F}(\neg\varepsilon)}(x) + \alpha \cdot \left(1 - r_{\tilde{F}(\neg\varepsilon)}(x) - k_{\tilde{F}(\neg\varepsilon)}(x)\right)\right) \\ \bar{\omega}^r_{F_\alpha(\neg\varepsilon)}(x) &= \left[\left(\frac{\omega^k_{\tilde{F}(\neg\varepsilon)}(x)}{2\pi}\right) + \alpha \cdot \left(1 - \frac{\omega^r_{\tilde{F}(\neg\varepsilon)}(x) + \omega^k_{\tilde{F}(\neg\varepsilon)}(x)}{2\pi}\right)\right]. \end{aligned}$$

Thus, we have $(f_\alpha(\xi^c))^c = \langle (F^c)_\alpha,\, \neg E\rangle^c = \langle ((F^c)_\alpha)^c,\, \neg\neg E\rangle = \langle ((F^c)_\alpha)^c,\, E\rangle$, $\forall \varepsilon \in E$.

$$\begin{aligned} ((F^c)_\alpha)^c(\varepsilon) &= (\{\langle x,\, (1 - r_{F_\alpha(\neg\varepsilon)}(x)) \cdot e^{i2\pi\bar{\omega}^r_{F_\alpha(\neg\varepsilon)}(x)}\rangle | x \in X\})^c \\ &= \{\langle x,\, r_{F_\alpha(\neg\neg\varepsilon)}(x) \cdot e^{i2\pi\omega^r_{F_\alpha(\neg\neg\varepsilon)}(x)}\rangle | x \in X\},\ \forall \varepsilon \in \neg\neg E, \\ &= \{\langle x,\, r_{F_\alpha(\varepsilon)}(x) \cdot e^{i2\pi\omega^r_{F_\alpha(\varepsilon)}(x)}\rangle | x \in X\},\ \forall \varepsilon \in E. \end{aligned}$$

Here

$$\begin{aligned} r_{F_\alpha(\varepsilon)}(x) &= \left(1 - k_{\tilde{F}(\varepsilon)}(x) - \alpha \cdot \left(1 - r_{\tilde{F}(\varepsilon)}(x) - k_{\tilde{F}(\varepsilon)}(x)\right)\right); \\ \omega^r_{F_\alpha(\varepsilon)}(x) &= 1 - \left[\left(\frac{\omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right) + \alpha \cdot \left(1 - \frac{\omega^r_{\tilde{F}(\varepsilon)}(x) + \omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right)\right]. \end{aligned}$$

Since $f_{1-\alpha}(\xi) = f_{1-\alpha}(\langle \tilde{F}, E\rangle) = \langle F_{1-\alpha}, E\rangle$, where

$$\begin{aligned}
F_{1-\alpha}(\varepsilon) &= f_{1-\alpha}(\tilde{F}(\varepsilon)) \\
&= f_{1-\alpha}\left(\{\langle x, \mu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(x)\rangle | x \in X\}\right) \\
&= \{\langle x, r_{F_{1-\alpha}(\varepsilon)}(x) \cdot e^{i\omega^r_{F_{1-\alpha}(\varepsilon)}(x)}\rangle | x \in X\}, \forall \varepsilon \in E,
\end{aligned}$$

Here

$$\begin{aligned}
r_{F_{1-\alpha}(\varepsilon)}(x) &= \left(1 - k_{\tilde{F}(\varepsilon)}(x) - \alpha \cdot \left(1 - r_{\tilde{F}(\varepsilon)}(x) - k_{\tilde{F}(\varepsilon)}(x)\right)\right) \\
\omega^r_{F_{1-\alpha}(\varepsilon)}(x) &= \left[\left(\frac{\omega^r_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right) + (1-\alpha) \cdot \left(1 - \frac{\omega^r_{\tilde{F}(\varepsilon)}(x) + \omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right)\right] \\
&= 1 - \left[\left(\frac{\omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right) + \alpha \cdot \left(1 - \frac{\omega^r_{\tilde{F}(\varepsilon)}(x) + \omega^k_{\tilde{F}(\varepsilon)}(x)}{2\pi}\right)\right].
\end{aligned}$$

Thus, $((F^c)_\alpha)^c(\varepsilon) = F_{1-\alpha}(\varepsilon)$. Consequently, $(f_\alpha(\xi^c))^c = f_{1-\alpha}(\xi)$.

□

6.3 Distance Measures and Entropies of Complex Intuitionistic Fuzzy Soft Set

Based on various well known distance functions, we introduce some distance measures between complex intuitionistic fuzzy soft sets and propose a general way to find the entropies of complex intuitionistic fuzzy soft set. We also give the structure of intuitionistic entropy of complex intuitionistic fuzzy soft sets by extending the structure of intuitionistic entropy on intuitionistic fuzzy soft sets Jiang *et al.* (2013).

6.3.1 Distance Measures for Complex Intuitionistic Fuzzy Soft Sets

The axiomatic definition of the distance measure for complex intuitionistic fuzzy soft sets has been reframed and proposed as follows:

Definition 6.4.1.1 (Distance Measures on CIFSSs): Suppose that $\xi = \langle \tilde{F},\, E\rangle$ and $\tilde{\xi} = \langle \tilde{G},\, E\rangle$ are two complex intuitionistic fuzzy soft sets over the universal set X. A real valued non-negative function

$$d : \mathcal{CIFSS}(X) \times \mathcal{CIFSS}(X) \to [0,\, 1]$$

is called distance measure on $\mathcal{CIFSS}(X)$, if d satisfies the following properties:

$(D1)$ $0 \leq d(\xi,\, \tilde{\xi}) \leq 1$;

$(D2)$ $d(\xi,\, \tilde{\xi}) = d(\tilde{\xi},\, \xi)$;

$(D3)$ $d(\xi,\, \tilde{\xi}) = 0$ if and only if $\xi = \tilde{\xi}$;

$(D4)$ for any $\eta = \langle \tilde{H},\, E\rangle = [c_{ij}]_{m\times n} \in \mathcal{CIFSS}(X)$, $d(\xi,\, \tilde{\xi}) + d(\tilde{\xi},\, \eta) \geq d(\xi,\, \eta)$.

Now we extend and write the Hamming and Euclidean distance measures between two CIFSSs ξ and $\tilde{\xi}$.

- Hamming Distance:

$$\begin{aligned} d_h(\xi,\, \tilde{\xi}) \;&=\; \frac{1}{4}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[|r_{\tilde{F}(\varepsilon_j)}(x_i) - r_{\tilde{G}(\varepsilon_j)}(x_i)| + |k_{\tilde{F}(\varepsilon_j)}(x_i) - k_{\tilde{G}(\varepsilon_j)}(x_i)| \\ &\qquad + \frac{1}{2\pi}\Big(|\omega^{r}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{r}_{\tilde{G}(\varepsilon_j)}(x_i)| + |\omega^{k}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{k}_{\tilde{G}(\varepsilon_j)}(x_i)|\Big)\Big] \end{aligned}$$

- Normalized Hamming Distance:

$$\begin{aligned} d_h^n(\xi,\, \tilde{\xi}) \;&=\; \frac{1}{4mn}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[|r_{\tilde{F}(\varepsilon_j)}(x_i) - r_{\tilde{G}(\varepsilon_j)}(x_i)| + |k_{\tilde{F}(\varepsilon_j)}(x_i) - k_{\tilde{G}(\varepsilon_j)}(x_i)| \\ &\qquad + \frac{1}{2\pi}\Big(|\omega^{r}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{r}_{\tilde{G}(\varepsilon_j)}(x_i)| + |\omega^{k}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{k}_{\tilde{G}(\varepsilon_j)}(x_i)|\Big)\Big] \end{aligned}$$

- Euclidean Distance:

$$\begin{aligned} d_e^n(\xi,\, \tilde{\xi})^2 \;&=\; \frac{1}{4}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[(r_{\tilde{F}(e_j)}(x_i) - r_{\tilde{G}(\varepsilon_j)}(x_i))^2 + (k_{\tilde{F}(\varepsilon_j)}(x_i) - k_{\tilde{G}(\varepsilon_j)}(x_i))^2 \\ &\qquad + \frac{1}{4\pi^2}\Big((\omega^{r}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{r}_{\tilde{G}(\varepsilon_j)}(x_i))^2 + (\omega^{k}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{k}_{\tilde{G}(\varepsilon_j)}(x_i))^2\Big)\Big] \end{aligned}$$

- Normalized Euclidean Distance:

$$\begin{aligned} d_e^n(\xi,\, \tilde{\xi})^2 \;&=\; \frac{1}{4mn}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[(r_{\tilde{F}(e_j)}(x_i) - r_{\tilde{G}(\varepsilon_j)}(x_i))^2 + (k_{\tilde{F}(\varepsilon_j)}(x_i) - k_{\tilde{G}(\varepsilon_j)}(x_i))^2 \\ &\qquad + \frac{1}{4\pi^2}\Big((\omega^{r}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{r}_{\tilde{G}(\varepsilon_j)}(x_i))^2 + (\omega^{k}_{\tilde{F}(\varepsilon_j)}(x_i) - \omega^{k}_{\tilde{G}(\varepsilon_j)}(x_i))^2\Big)\Big] \end{aligned}$$

6.3.2 Entropies on Complex Intuitionistic Fuzzy Soft Sets

Here, we present the axiomatic definition for the entropy of a complex intuitionistic fuzzy soft sets. The following conditions give the intuitive idea for the degree of fuzziness of a complex intuitionistic fuzzy soft set, i.e., for the entropy of a complex intuitionistic fuzzy soft set:

(i) It will be null when the complex intuitionistic fuzzy soft set is a complex fuzzy soft set;

(ii) It will be maximum if the complex intuitionistic fuzzy soft set is completely intuitionistic;

(iii) An intuitionistic entropy of a complex intuitionistic fuzzy soft set will be equal to its complement;

(iv) If the degree of membership and the degree of non-membership of each element increase, the sum will do so as well, and therefore, this complex intuitionistic fuzzy soft set becomes less fuzzy, and therefore the entropy should decrease.

In view of the above stated points and the definition of entropy for an intuitionistic fuzzy soft set given in Jiang *et al.* (2013), we propose the following definition for the entropy of a complex intuitionistic fuzzy soft set:

Definition 6.4.2.1 (Intuitionistic Entropy of CIFSSs): A real-valued function $H : \mathcal{CIFSS} \to \mathbb{R}^+$ is called an intuitionistic entropy on $\mathcal{CIFSS}(X)$, if H has the following properties:

- P1 (*Sharpness*): $H(\xi) = 0$ iff ξ is a complex fuzzy soft set;
- P2 (*Maximality*): Let $\xi = \langle \tilde{F}, E \rangle = [a_{ij}]_{m\times n}$, $H(\xi) = mn$ iff $\mu_{\tilde{F}(\varepsilon)}(x) = \nu_{\tilde{F}(\varepsilon)}(x) = 0$, $\forall x \in X$ and $\varepsilon \in E$;
- P3 (*Symmetry*): $H(\xi) = H(\xi^c)$, for all $\xi \in \mathcal{CIFSS}(X)$;
- P4 (*Resolution*): if ξ^* is sharpened version of ξ that is, if $\xi \preceq \xi^*$, then $H(\xi) \geq H(\xi^*)$.

In the following theorem, we prove $P2$ property of the definition 6.4.2.1.

Theorem 6.4.2.1: $H(\xi)$ is maximum if and only if $\xi = \langle \tilde{F}, E\rangle = [a_{ij}]_{m\times n} = [0]_{m\times n}$, that is, $a_{ij} = \mu_{\tilde{F}(\varepsilon_j)}(x_i) = \nu_{\tilde{F}(\varepsilon_j)}(x_i) = 0$, $\forall \varepsilon_j \in E$, $x_i \in X$, where $0 \leq i \leq m$ and $0 \leq j \leq n$.

Proof. Necessary part: Let $\xi = \langle \tilde{F}, E\rangle = [a_{ij}]_{m\times n} = [0]_{m\times n}$. Let $\tilde{\xi} = \langle \tilde{G}, E\rangle$ be any complex intuitionistic fuzzy soft set. Since $\mu_{\tilde{G}(\varepsilon_j)}(x_i) \geq 0$ and $\nu_{\tilde{G}(\varepsilon_j)}(x_i) \geq 0$, $\forall \varepsilon_j \in E$, $x_i \in X$, where $0 \leq i \leq m$ and $0 \leq j \leq n$, therefore, by the definition 6.3.3, we have $\xi \preceq \tilde{\xi}$. Thus, $H(\xi) \geq H(\tilde{\xi})$ by the property $P4$ of the definition 6.4.2.1 for all $\tilde{\xi}$, then $H(\xi)$ is maximum.

Sufficient part: Let $H(\xi)$ is maximum. We assume that $\xi = \langle \tilde{F}, E\rangle = [a_{ij}]_{m\times n} \neq [0]_{m\times n}$, then there is a $\varepsilon_j \in E$ and $x_i \in X$ such that $\mu_{\tilde{F}(\varepsilon_j)}(x_i) \neq 0$ and $\nu_{\tilde{F}(\varepsilon_j)}(x_i) \neq 0$, where $0 \leq i \leq m$ and $0 \leq j \leq n$. We construct the following complex intuitionistic fuzzy soft set $\tilde{\xi} = \langle \tilde{G}, E\rangle$ with $\mu_{\tilde{G}(\varepsilon_j)}(x_i) = \frac{1}{2}\mu_{\tilde{F}(\varepsilon_j)}(x_i)$ and $\nu_{\tilde{G}(\varepsilon_j)}(x_i) = \frac{1}{3}\nu_{\tilde{F}(\varepsilon_j)}(x_i)$ for all $\varepsilon_j \in E$ and $x_i \in X$, then by the definition 6.3.3 we have $H(\tilde{\xi}) \geq H(\xi)$ which contradicts the hypothesis $H(\xi)$ is maximum. Therefore, $\xi = [0]_{m\times n}$.

Definition 6.4.2.2: Let $D = \{(x, y) \in [0, 1] \times [0, 1] | x + y \leq 1\}$ and construct $\psi_D : D \to [0, 1]$, which satisfies the following conditions:

(i) $\psi_D(x, y) = 1$ if and only if $x + y = 1$;

(ii) $\psi_D(x, y) = 0$ if and only if $x = 0 = y$;

(iii) $\psi_D(x, y) = \psi_D(y, x)$;

(iv) if $x \leq x'$ and $y \leq y'$, then $\psi_D(x, y) \leq \psi_D(x', y')$.

Theorem 6.4.2.2: Let $\xi = \langle \tilde{F}, E\rangle = [a_{ij}]_{m\times n} \in \mathcal{CIFSS}(X)$ and $H : \mathcal{CIFSS}(X) \to \mathbb{R}^+$ such that

$$H(\xi) \quad = \quad \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1 - \psi(r_{\tilde{F}(\varepsilon_j)}(x_i), k_{\tilde{F}(\varepsilon_j)}(x_i))\Big) + \Big(1 - \psi(\omega^{r}_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi, \omega^{k}_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big)\Big],$$

where ψ satisfies the conditions (i)–(iv) of definition 6.4.2.2, then H is an intuitionistic entropy on $\mathcal{CIFSS}(X)$.

Proof.

1. $H(\xi) = 0$ if and only if, for all $\varepsilon_j \in E$, $x_i \in X$,
$\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-\psi(r_{\tilde{F}(\varepsilon_j)}(x_i),\, k_{\tilde{F}(\varepsilon_j)}(x_i))\Big)+\Big(1-\psi(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big)\Big]=0,$
$\Leftrightarrow \psi(r_{\tilde{F}(\varepsilon_j)}(x_i),\, k_{\tilde{F}(\varepsilon_j)}(x_i)) = 1$ and $\psi(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi) = 1,$
$\Leftrightarrow r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i) = 1$ and $\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i) = 2\pi,\ \Leftrightarrow \xi \in CFSS(X).$
Thus, H satisfies property $P1$ of the definition 6.4.2.1.

2. $H(\xi) = mn$ if and only if, for all $\varepsilon_j \in E$, $x_i \in X$,
$\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-\psi(r_{\tilde{F}(\varepsilon_j)}(x_i),\, k_{\tilde{F}(\varepsilon_j)}(x_i))\Big)+\Big(1-\psi(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big)\Big]=mn,$
$\Leftrightarrow r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i) = 0$ and $\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i) = 0,$
$\Leftrightarrow r_{\tilde{F}(\varepsilon_j)}(x_i),\, k_{\tilde{F}(\varepsilon_j)}(x_i) = 0$ and $\omega^r_{\tilde{F}(\varepsilon_j)}(x_i),\, \omega^k_{\tilde{F}(\varepsilon_j)}(x_i) = 0,$
which implies ξ is completely intuitionistic. Thus, H satisfies property $P2$ of the definition 6.4.2.1.

3. Since $\xi^c = \langle \tilde{F},\, E\rangle^c = \langle \tilde{F}^c,\, \neg E\rangle$,
where $\tilde{F}^c = \{\langle x_i,\, \nu_{\tilde{F}(\varepsilon)}(x_i),\, \mu_{\tilde{F}(\varepsilon)}(x_i)\rangle | x_i \in X\},\ \forall \neg\varepsilon \in \neg E,$
then

$$\begin{aligned}
H(\xi) &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-\psi(r_{\tilde{F}(\varepsilon_j)}(x_i),\, k_{\tilde{F}(\varepsilon_j)}(x_i))\Big)+\Big(1-\psi(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big)\Big]\\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-\psi(k_{\tilde{F}(\varepsilon_j)}(x_i),\, r_{\tilde{F}(\varepsilon_j)}(x_i))\Big)+\Big(1-\psi(\omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi,\, \omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big)\Big]\\
&= H(\xi^c).
\end{aligned}$$

Thus, H satisfies property $P3$ of the definition 6.4.2.1.

4. Let $\tilde{\xi} = \langle \tilde{F},\, E\rangle = [b_{ij}]_{m\times n}$, if $\xi \leq \tilde{\xi}$, then we have
$r_{\tilde{F}(\varepsilon_j)}(x_i) \leq r_{\tilde{G}(\varepsilon_j)}(x_i)$ and $k_{\tilde{F}(\varepsilon_j)}(x_i) \leq k_{\tilde{G}(\varepsilon_j)}(x_i)$, for the amplitude terms and for the phase terms $\omega^r_{\tilde{F}(\varepsilon_j)}(x_i) \leq \omega^r_{\tilde{G}(\varepsilon_j)}(x_i)$ and $\omega^k_{\tilde{F}(\varepsilon_j)}(x_i) \leq \omega^k_{\tilde{G}(\varepsilon_j)}(x_i)),\ \forall x_i \in X$ and $\forall \varepsilon_j \in E$.
Thus, we have
$\psi(r_{\tilde{F}(\varepsilon_j)}(x_i),\, k_{\tilde{F}(\varepsilon_j)}(x_i)) \leq \psi(r_{\tilde{G}(\varepsilon_j)}(x_i),\, k_{\tilde{G}(\varepsilon_j)}(x_i)),$
and
$\psi(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi) \leq \psi(\omega^r_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi).$
$\Rightarrow$
$\Big(1-\psi(r_{\tilde{F}(\varepsilon_j)}(x_i),\, k_{\tilde{F}(\varepsilon_j)}(x_i))\Big) \geq \Big(1-\psi(r_{\tilde{G}(\varepsilon_j)}(x_i),\, k_{\tilde{G}(\varepsilon_j)}(x_i))\Big),$
and
$\Big(1-\psi(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big) \geq \Big(1-\psi(\omega^r_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi,\, \omega^k_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi)\Big).$

$\Rightarrow H(\tilde{\xi}) \leq H(\xi)$,

Thus, H satisfies property $P4$ of the definition 6.4.2.1.

Therefore, H is an intuitionistic entropy of complex intuitionistic fuzzy soft set. $\square$

Burillo and Bustince (1996a) gave some expressions for intuitionistic entropy of intuitionistic fuzzy soft sets. Jiang *et al.* (2013) extended these expressions for intuitionistic entropy of intuitionistic fuzzy soft sets. On similar pattern, we are extending these expressions for intuitionistic entropy of complex intuitionistic fuzzy soft sets.

Let $\xi = \langle \tilde{F}, E \rangle = [a_{ij}]_{m\times n} \in\in \mathcal{CIFSS}(X)$. It is easy to verify that the following expressions are the intuitionistic entropies of ξ :

$$\begin{aligned}
H_1(\xi) &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-(r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i))\Big)+\Big(1-(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i))/2\pi\Big)\Big];\\
H_2(\xi) &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-(r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i))^n\Big)\\
&\quad+\Big(1-\Big((\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i))/2\pi\Big)^n\Big)\Big],\ n=2,\,3,\ldots;\\
H_3(\xi) &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-(r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i))\cdot e^{1-(r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i))}\Big),\\
&\quad+\Big(1-(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i))/2\pi\cdot e^{1-(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i))/2\pi}\Big)\Big];\\
H_4(\xi) &= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1-(r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i))\cdot sin\Big((\pi/2)\Big(r_{\tilde{F}(\varepsilon_j)}(x_i)+k_{\tilde{F}(\varepsilon_j)}(x_i)\Big)\Big)\Big)\\
&\quad+\Big(1-\Big(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i)\Big)/2\pi\Big)\cdot sin\Big(\Big(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)+\omega^k_{\tilde{F}(\varepsilon_j)}(x_i)\Big)/4\Big)\Big].
\end{aligned}$$

In the following definition, we introduce a function from $\mathcal{CIFSS}(X)$ to $\mathbb{R}^+$, which is an extension of the $H_{\phi,\,\phi'}$-function from $\mathcal{IFSS}(X)$ to $\mathbb{R}^+$ given in Jiang *et al.* (2013), which is also an extension of the $H_{\phi,\,\phi'}$-function from $\mathcal{FSS}(X)$ to $\mathbb{R}^+$ given in Burillo and Bustince (1996a).

Definition 6.4.2.3: Let $\phi,\,\phi' : [0,\,1] \to [0,\,1]$ be such that if $x+y \leq 1$, then $\phi(x)+\phi'(y) \leq 1$, with $x,\,y \in [0,\,1]$. We define function $H_{\phi,\,\phi'}(\cdot)$ of the complex intuitionistic fuzzy soft set $\xi = \langle \tilde{F}, E \rangle = [a_{ij}]_{m\times n} \in \mathcal{CIFSS}(X)$ to $\mathbb{R}^+$ as follows:

$$H_{\phi,\,\phi'}(\xi) = mn - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(\phi(r_{\tilde{F}(\varepsilon_j)}(x_i)) + \phi'(k_{\tilde{F}(\varepsilon_j)}(x_i))\Big) + \Big(\phi(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi) + \phi'(\omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big)\Big]$$

Obviously, $0 \le H_{\phi,\,\phi'}(\xi) \le mn$ for all $\xi = [a_{ij}]_{m\times n}$ belonging to $\mathcal{CIFSS}(X)$.

Theorem 6.4.2.3: Let $\xi = \langle \tilde{F}, F \rangle = [a_{ij}]_{m\times n}$, $\tilde{\xi} = \langle \tilde{G}, E \rangle = [b_{ij}]_{m\times n} \in \mathcal{CIFSS}(X)$, then the function $H_{\phi,\,\phi'}$ satisfies the following property.

$$H_{\phi,\,\phi'}(\xi \cup \tilde{\xi}) + H_{\phi,\,\phi'}(\xi \cap \tilde{\xi}) = H_{\phi,\,\phi'}(\xi) + H_{\phi,\,\phi'}(\tilde{\xi}).$$

Proof. By the definition 6.4.2.3, we have the following

$$H_{\phi,\,\phi'}(\xi \cup \tilde{\xi}) = mn - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(\phi(r_{\tilde{F}(\varepsilon_j)}(x_i) \diamond r_{\tilde{G}(\varepsilon_j)}(x_i)) + \phi'(k_{\tilde{F}(\varepsilon_j)}(x_i) * k_{\tilde{F}(\varepsilon_j)}(x_i))\Big) + \phi\Big(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi \cup \omega^r_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi\Big) + \phi'\Big(\omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi \cap \omega^k_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi\Big)\Big]$$

and

$$H_{\phi,\,\phi'}(\xi \cap \tilde{\xi}) = mn - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(\phi(r_{\tilde{F}(\varepsilon_j)}(x_i) * r_{\tilde{G}(\varepsilon_j)}(x_i)) + \phi'(k_{\tilde{F}(\varepsilon_j)}(x_i) \diamond k_{\tilde{F}(\varepsilon_j)}(x_i))\Big) + \phi\Big(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi \cup \omega^r_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi\Big) + \phi'\Big(\omega^k_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi \cap \omega^k_{\tilde{G}(\varepsilon_j)}(x_i)/2\pi\Big)\Big]$$

Thus, we have

$$H_{\phi,\,\phi'}(\xi \cup \tilde{\xi}) + H_{\phi,\,\phi'}(\xi \cap \tilde{\xi}) = H_{\phi,\,\phi'}(\xi) + H_{\phi,\,\phi'}(\tilde{\xi}).$$

□

It may be noted that there are $H_{\phi,\phi'}$-functions which are not intuitionistic entropies, e.g.,

$$H_{\phi,\,\phi'}(\xi) = mn - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(r_{\tilde{F}(\varepsilon_j)}(x_i) + k_{\tilde{F}(\varepsilon_j)}(x_i)\Big) + \frac{1}{2\pi}\Big(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i) + \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)\Big)\Big]$$

On the other hand, it may also be easily verified that there are entropies which are not $H_{\phi,\phi'}$- functions, e.g.,

$$H_{\phi,\,\phi'}(\xi) = mn - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(r_{\tilde{F}(\varepsilon_j)}(x_i) + k_{\tilde{F}(\varepsilon_j)}(x_i)\Big)^2 + \Big(\frac{1}{2\pi}\Big(\omega^r_{\tilde{F}(\varepsilon_j)}(x_i) + \omega^k_{\tilde{F}(\varepsilon_j)}(x_i)\Big)\Big)^2\Big]$$

Also, there are entropies which are also $H_{\phi,\phi'}$-functions, e.g.,

$$H_{\phi,\,\phi'}(\xi) = mn - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1 - \Big(r_{\tilde{F}(\varepsilon_j)}(x_i) + k_{\tilde{F}(\varepsilon_j)}(x_i)\Big)\Big) + \Big(1 - \frac{1}{2\pi}\Big(\omega^{r}_{\tilde{F}(\varepsilon_j)}(x_i) + \omega^{k}_{\tilde{F}(\varepsilon_j)}(x_i)\Big)\Big)\Big]$$

Next, we introduce a property which defines entropies in a general way, as follows:

Theorem 6.4.2.4: If $\phi : [0,\,1] \to [0,\,1]$ satisfies the following conditions:

(i) ϕ is increasing,

(ii) $\phi(x) = 0$ if and only if $x = 0$,

(iii) $\phi(x) + \phi(y) = 1$ if and only if $x + y = 1$,

then $\phi(x) + \phi(y)$ satisfies the conditions (i)–(iv) of the ψ function in definition 6.4.2.2. □

We denote the $H_{\phi,\,\phi'}$-function as $H_{\phi,\,\phi}$-function if $\phi = \phi'$. The following theorem characterizes the intuitionistic entropy of complex intuitionistic fuzzy soft sets in a general way:

Theorem 6.4.2.5: Let $H : \mathcal{CIFSS} \to \mathbb{R}^+$, $\phi : [0,\,1] \to [0,\,1]$ and $\xi = \langle \tilde{F},\,E\rangle = [a_{ij}]_{m\times n} \in \mathcal{CIFSS}(X)$. H is an intuitionistic entropy as well as $H_{\phi,\,\phi}$-function if and only if

$$H(\xi) \;=\; \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\Big[\Big(1 - \Big(\phi(r_{\tilde{F}(\varepsilon_j)}(x_i)) + \phi(k_{\tilde{F}(\varepsilon_j)}(x_i))\Big)\Big) + \Big(1 - \Big(\phi(\omega^{r}_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi) + \phi(\omega^{k}_{\tilde{F}(\varepsilon_j)}(x_i)/2\pi)\Big)\Big)\Big],$$

where ϕ satisfies the conditions(i)–(iii) of theorem 6.4.2.4. □

Based on the definitions and properties stated above, it may be easily verified that there is a correspondence between the proposed entropies and the distance measures of complex intuitionistic fuzzy soft sets.

Theorem 6.4.2.6: Let $\xi = \langle \tilde{F},\,E\rangle = [a_{ij}]_{m\times n} \in \mathcal{CIFSS}(X)$ and $f_\alpha(\xi) = \{\xi_\alpha\}_{\alpha\in[0,\,1]}$ be the family of complex fuzzy soft sets associated to ξ by the operator f_α defined in definition 4. Then

(i) $H_1(\xi) = 2 \cdot d^h(\xi, \xi_\alpha)$;

(ii) $H_1(\xi) = 2 \cdot d^h(\xi_0, \xi_1)$;

(iii) $d^h(\xi_\alpha, \xi_\beta) = (\alpha - \beta) \cdot H_1(\xi)$.

□

6.4 Application in Multi-criteria Decision Making problems

Suppose that a car dealer X decides to purchase cars from a car company Y. The car company provides some information to car dealer on four models of cars with different manufacturing dates for each model. So, car dealer X wants to select four models Car1, Car2, Car3 and Car4 with its manufacturing date simultaneously. Suppose that a team of experts (decision makers) agreed that five parameters should be considered in the selection process. They can be: reliability, maximum payload, purchasing cost, maximum speed and durability. But these parameters will get affected and changed if the production date is different for the same model of cars. The decision made by the expert team will also depend on the knowledge and experience of its members. The best way to represent this kind of information may be by using CIFSS, in which for each car model, the experts have different opinions and mentalities. For instance, suppose that at least 60% experts believe that the Car1 is suitable at the first parameter and not more than 15% of the experts that the Car1 is poor at the first parameter, in this way we can calculate the amplitude terms for both membership and non-membership functions, respectively in CIFSS. The phase terms that represent the production date for first parameter of the Car1 can be calculated as follows: if at least 70% experts believe that the production date of $car1$ is suitable at the first parameter and not more than 20% of them believe that the production date of $car1$ is poor. Therefore, the information based on experts about $car1$ on the first parameter can be represented in form of CIFSS as $\langle 0.6 \cdot e^{2\pi 0.70}, 0.15 \cdot e^{2\pi 0.20} \rangle$. In this way, all data can be obtained in the form of CIFSS, where both amplitude and phase terms can represent the information on experts's decision which happens periodically. Assume that the expert team had suggested an ideal car (i.e., a car that is in demand) before getting the characteristic information from car maker Y. The aim of the expert team is to select a suitable car listed by car maker Y that is most likely to be the ideal car. Using this information, a car dealer X can take decision to purchase cars from the car company Y that are in demand in the market to gain profit.

6.5 Conclusions

The introduced concept of Complex Intuitionistic Fuzzy Soft Sets (CIFSSs) which is a parametric tool has been well proposed and studied in detail along with its important properties and fundamental operations. Based on various well known distance measures, some new distance measures for CIFSSs have been obtained and extended to find the entropies of complex intuitionistic fuzzy soft sets. A correspondence between the proposed entropies and the distance measures of complex intuitionistic fuzzy soft sets has been well established. An application in the area of Multi-criteria Decision Making problem on the basis of the proposed CIFSSs, distance measures and information measures has also been suggested.

Chapter 7

Conclusions

The conclusions are summarized as under

- A new R-norm intuitionistic fuzzy entropy and a weighted R-norm intuitionistic fuzzy directed divergence measure have been proposed with their proof of validity. Further, after empirical study on the proposed information measures we find that R-norm fuzzy intuitionistic fuzzy entropy is a decreasing function of R, while the weighted R-norm intuitionistic fuzzy directed divergence measure is increasing function of R as well as λ. The proposed intuitionistic fuzzy information measures have found many applications in the field of pattern recognition and image processing.

- The estimators of regression coefficients have also been obtained with the help of fuzzy entropy for the restricted/unrestriced fuzzy linear regression model by assigning some weights in the distance function. Some numerical examples illustrating the outcomes of the model have been provided. Further, simulation study over the obtained estimators has been conducted to compare their performance. It has been observed that the restricted estimator is better than unrestricted estimator in the sense of absolute bias as well as mean square error matrix. Thus, whenever some prior information is available in terms of exact linear restrictions on regression coefficients, it is advised to use restricted estimator $\tilde{\boldsymbol{\beta}}$ in place of unrestricted estimator $\hat{\boldsymbol{\beta}}$.

- The proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form are valid similarity measures. The new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets analogously obtained through the proposed similarity measures are also valid information measures. Further, a new algorithm for MCDM using the proposed weighted similarity measures, in which the weights have been calculated using the proposed entropies, has been illustrated through a numerical example.

- The intuitionistic fuzzy reliability of k-out-of-$n : G$ system with independent and non-identically distributed components, where the reliability of the components are unknown, has been computed. The reliability of each component has been estimated using statistical confidence interval approach. Considering the highest level of confidence of domain experts that belongs to the interval $[0, 1]$, we converted these statistical confidence interval into triangular intuitionistic fuzzy numbers. The reliability of the k-out-of-$n : G$ system has been calculated and discussed on the basis of these triangular intuitionistic fuzzy numbers with the help of a numerical example. On similar pattern, the intuitionistic fuzzy reliability of the real-time repairable k-out-of-n system may be computed.

- The concept of complex intuitionistic fuzzy soft sets which is a parametric tool have been well proposed with their important properties and operations. Based on some well known distance measures, new distance measures for CIFSSs have been obtained and are used to propose the entropies of complex intuitionistic fuzzy soft sets.

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