RELIABILITY PROPERTIES OF RESIDUAL LIFE TIME, INACTIVITY TIME AND DECISION MAKING IN INTUITIONISTIC FUZZY ENVIRONMENT

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SUPERVISOR'S CERTIFICATE

This is to certify that the thesis entitled, "***Reliability Properties of Residual Life Time, Inactivity Time and Decision Making in Intuitionistic Fuzzy Environment***" which is being submitted by ***Neeraj Gandotra*** in fulfillment for the award of degree of ***Doctor of Philosophy*** in ***Mathematics*** by the ***Jaypee University of Information Technology, Waknaghat*** is the record of candidate's own work carried out by him under our supervision. This work has not been submitted partially or wholly to any other University or Institute for the award of this or any other degree or diploma.

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Chapter 1

Introduction

1.1 Background and Motivation

During the last few years, global competition in the market place has become more complicated. Customers generally prefer the product which has higher quality and reliability (durability and maintainability). Engineering systems, components and devices are not perfect. A car battery go dead, a floppy disk drive go bad, a TV remote control quit functioning, a stereo amplifier quit, an automatic engine starter fail and a house roof leak. In order to achieve high product reliability, one has to understand the definition of the term reliability.

Additionally, in order to select a product, the role of human behaviour which is influenced by some interrelating factors is an important factor in a consumer decision making process. The external characteristics such as price, brand, capability etc. are also concerned in making a decision. Everyone make decision on their beliefs, and the beliefs depend on the information gathered from experiments, experts and experience.

Reliability refers "Quality over Time". Reliability is the probability of a product performing its intended function over its specified duration of usage and under specified working conditions, in a manner that meets or exceeds customers expectations. In reliability we study various reliability notations such as the failure rate, reverse failure rate, the mean residual life and the likelihood ratio order etc. The concepts of residual life time and inactivity time are extensively used in reliability theory for modelling life time data. Block et. al. (1998), Chandra & Roy (2001), Li & Zhang (2003), Li & Lu (2003), Li & Zuo (2004), Misra et al. (2008) and Pellerey & Petakos (2002) studied reliability properties of residual life/inactivity time. The stochastic comparisons of residual life time and inactivity time in series and parallel systems is discussed by Li and Lu (2003) and Li & Zhang (2003).

Decision making generally depends on two factors : the amount of information associated with an alternative and the reliability of information. Fuzzy set theory, developed by Zadeh (1965) has capability to describe the uncertain situations, containing ambiguity and vagueness. Fuzziness is likely to be present in our decisions, in our thoughts, in processing the information and particularly in our language. Among various extensions of fuzzy sets such as intuitionistic fuzzy set (IFS), vague set, interval-valued fuzzy set, IFSs are found to be more consistent with human behaviour. Intuitionistic fuzzy set introduced by Atanassov (1986) has been found to be more useful to deal with ambiguity/vagueness/imprecision. Intuitionistic fuzzy set is characterized by two functions - the degree of membership function and a non membership function. It may be noted that the sum of membership function and non membership function must be smaller than or equal to one. The theory of intuitionistic fuzzy sets has been used to build soft decision making models that can accommodate imprecise information and analyze the extent of agreement in a group of experts.

1.2 Probabilistic Descriptions in Reliability and Life Testing

In reliability and life testing, the probabilistic descriptions includes distribution functions, survival functions, densities, hazard rates, mean residual lives and total time on test transforms. Upon existence of any of these functions, we can obtain any other function theoretically. We include below some definitions which are standard in the literature [cf. Albert W. Marshall & Ingram Olkin (2007)].

I. Distribution Functions and Survival Functions

If X is a continuous random variable function with probability density function $f(x)$, then the function

$$F_X(t) = P\{X \leq t\} = \int_{-\infty}^{t} f(x)\, dx, \quad -\infty < x < t$$

is called distribution function of the random variable X. Distribution functions are also known as "cumulative distribution functions".

A distribution function F is non-decreasing and right continuous ($\lim_{z \downarrow t} F(z) = F(t)$). Moreover, $\lim_{z \to -\infty} F(z) = 0$ and $\lim_{z \to \infty} F(z) = 1$. Any function with these properties is a distribution function for some random variable.

The basic quantity employed to describe time-to-event phenomena is the survival function. If X is a continuous random variable with probability density function $f(x)$, then the survival function $\bar{F}(t)$ is defined by

$$\bar{F}(t) = P\{X > t\} = \int_{t}^{\infty} f(x)\, dx$$

The survival function is sometimes called the "survivor function" or the "reliability". It is found that for non-negative random variables, the survival function is more meaningful and takes a more convenient form than the better known distribution function. Thus, we have the following relationship between reliability function and distribution function

$$\bar{F}(t) = 1 - F(t)$$

Differentiating both sides with respect to t, we get

$$\frac{d}{dt}\bar{F}(t) = -f(t)$$

II. Probability Mass Functions and Density Functions

For any random variable, the distribution function and survival function always exist. Suppose that the random variable X take only a finite or countable number of values, e.g., X might be the number of trials required to obtain "heads" in repeated tosses of a coin. Then X is said to be discrete random variable.

If $x_1, x_2, \ldots$ is the set of possible values of X and $p(x_i) = P\{X = x_i\}, i = 1, 2, 3, \ldots$, then

$$F(x) = \sum_{x_i \le x} p(x_i),$$

and p is called the probability mass function of X. It may be noted that the discrete distribution functions are step functions.

On the other hand, when X takes on all values in some (possibly infinite) interval of the real line, then X is said to be continuous random variable. It is often possible to write F as

$$F(x) = \int_{-\infty}^{x} f(z)dz,$$

for all real x. Then f is called a probability density of X (or F).

III. Hazard Functions and Hazard Rates

The function R defined on $(-\infty, \infty)$ by $R(x) = -\log \bar{F}(x)$ is called the hazard function of F, or of X. For a non-negative random variable, $R(0-) = 0$, R

is increasing, and $\lim_{x\to\infty} R(x) = \infty$; any function with these properties is a hazard function.

If F is an absolutely continuous distribution function with density$f$, then the function r defined on $(-\infty, \infty)$ by

$$\begin{aligned} r(x) &= \tfrac{f(x)}{\bar{F}(x)}, & \text{if } \bar{F}(x) > 0 \\ &= \infty, & \text{if } \bar{F}(x) = 0 \end{aligned}$$

is called a hazard rate of F, or of X.

When F is absolutely continuous and the hazard function R is differentiable, then its derivative is a hazard rate.

Therefore, $r(x) = \lim_{\delta \downarrow 0} \frac{P\{x<X\leq x+\delta | X>x\}}{\delta}$.

Hence, $\delta\, r(x) \approx P\{x < X \leq x + \delta | X > x\}$.

Thus, it is the probability that the item will fail in the next δ time unit given that the item is functioning properly in time x.

In other words, failure rate or hazard rate function is defined as the conditional probability, given survival up to time x, of death or failure in the next small increment δ of time.

IV. Reverse Hazard Functions and Reverse Hazard Rates

The reverse hazard function is defined in a manner similar to the hazard function $R(x) = -\log \bar{F}(x)$, but with the distribution function F replacing the survival function $\bar{F}$.

The function S defined on $(-\infty, \infty)$ by $S(x) = \log F(x)$ is called the reverse hazard function of F, or of X. If F is an absolutely continuous distribution function with density f, then a function s defined on $(-\infty, \infty)$ by

$$s(x) = \frac{f(x)}{F(x)}$$

is called a reverse hazard rate of F, or of X.

V. The Residual Life Distribution

The distribution of remaining life for an unfailed item of age t is often of interest. Let F be a distribution function such that $F(0) = 0$. the residual life distribution F_t of F at t is defined for all $t \geq 0$ such that $\bar{F}(t) > 0$ by

$$\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}, \quad x \geq 0.$$

If F has density f, then F_t has density f_t and hazard rate r_t given by

$$f_t(x) = \frac{f(x+t)}{\bar{F}(t)}, \quad x \geq 0,$$

$$r_t(x) = \frac{f(x+t)}{\bar{F}(x+t)} = r(x+t), \quad x \geq 0.$$

Thus, the residual life distribution F_t is a conditional distribution of the remaining life given survival up to time t.

VI. The Mean Residual Life Function

In order to understand the concept of residual life distribution, it is necessary to define "mean" or "expectation" of a random variable. Suppose that the random variable X has the distribution function F and that the integral

$$\int_{-\infty}^{\infty} |x| \; dF(x)$$

exists (is finite). Then, the expected value of X exists and is given by integral

$$E(X) = \int_{-\infty}^{\infty} x \, dF(x).$$

The expected value of X is also called the mean of X, or the expectation of X, and is often denoted by μ.

The mean residual life function $m(t)$ is the mean of residual life distribution F_t as a function of t. More explicitly, when F has finite mean μ and $F(x) = 0$, for

$x < 0$, the mean residual life function is given by

$$m(t) = \int_0^\infty \frac{\bar{F}(x+t)}{\bar{F}(t)}\,dx = \int_t^\infty \frac{\bar{F}(z)}{\bar{F}(t)}\,dz = \int_t^\infty \frac{(t-z)}{\bar{F}(t)}\,dF(z)$$

for t such that $\bar{F}(t) > 0$, and is equal to 0 if $\bar{F}(t) = 0$.

VII. Equilibrium Distribution

Let F be a distribution function with finite mean μ such that $F(x) = 0$ for $x < 0$, and let

$$f_1(x) = \frac{\bar{F}(x)}{\mu} = \frac{\bar{F}(x)}{\int_0^\infty \bar{F}(z)\,dz}, \quad x \geq 0,$$

$$= 0, \; x < 0.$$

The density function f_1 arises in the context of renewal theory where the corresponding distribution is called the equilibrium distribution or the stationary renewal distribution.

VIII. Moment Generating Functions and Laplace Transforms

The function mgf(p) $= E(e^{pX}) = \sum_{j-0}^{\infty} \frac{p^j EX^j}{j!}$ is called the moment generating function of X. The moment generating function is finite for all p in some interval of the form $(-\infty, a)$ where $a \geq 0$. In case $a > 0$ and r is a positive integer, the r^{th} derivative of the moment generating function evaluated at $p = 0$ yields the r^{th} moment:

$$E(X^r) = \frac{d^r}{dp^r} mgf(p)|_{p=0}.$$

The laplace transform φ of X is defined as

$$\varphi(p) = E(e^{-pX}).$$

For non-negative random variables, the laplace transform exists for all $p \geq 0$ and may exists for some or all values of $p < 0$.

The moment generating function mgf(p) of X is related to the laplace transform through the equation mgf$(p) = \varphi(-p)$.

IX. Parametric Families of Life Distributions

The Exponential Distribution:

For exponential distribution, the parameter $\kappa > 0$ is a scale parameter and

$$\bar{F}(x) = e^{-\kappa x}, \quad x \geq 0;$$

$$f(x) = \kappa e^{-\kappa x}, \quad x \geq 0.$$

For the exponential distribution, we have

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} = \bar{F}(x),$$

Thus, exponential distribution is the conditional probability of surviving an additional period of x, given survival up to time t, is the same as the unconditional probability of survival to time x.

The Gamma Distribution:

For gamma distribution, we have the scale parameter $\kappa > 0$ and shape parameter $\tau > 0$ and

$$f(x|\kappa,\tau) = \kappa^{\tau} x^{\tau-1} e^{-\kappa x} / \Gamma(\tau), \quad x \geq 0.$$

For $\tau \doteq 1$, above equation reduces to the exponential distribution.

The Weibull Distribution:

For Weibull distribution, we have the scale parameter $\kappa > 0$ and shape parameter $\tau > 0$. The survival function for Weibull distribution has a simple form given by

$$\bar{F}(x) = e^{-(\kappa x)^{\tau}}, \quad x \geq 0.$$

Therefore, the density function for Weibull distribution is given by

$$f(x) = \tau\kappa(\kappa x)^{\tau-1} e^{-(\kappa x)^{\tau}}, \quad x \geq 0.$$

The Gompertz Distribution:

For Gompertz distribution, we have the scale parameter $\kappa \geq 0$ and frailiy parameter $\xi \geq 0$. The survival function for Gompertz distribution has a simple form given by

$$\bar{F}(x) = e^{-\xi(e^{\kappa x}-1)}, \quad x \geq 0.$$

Therefore, the density function for Gompertz distribution is given by

$$f(x) = \kappa\xi e^{\kappa x-\xi(e^{\kappa x}-1)}, \quad x \geq 0$$

X. Residual Life Time and Inactivity Time:

The residual life of X with age/time $t \geq 0$ is given by

$$X_t = (X - t \,|\, X > t), \; t \geq 0,$$

and inactivity time of X at time $t \geq 0$ is given by

$$X_{(t)} = (t - X \,|\, X \leq t), \; t \geq 0.$$

For a fixed $t \geq 0$, the survival functions of X_t and $X_{(t)}$ are given by

$$S_{R,t}(x) = P(X_t > x) = \begin{cases} 1 & \text{if } x < 0 \\ \frac{\bar{F}(x+t)}{\bar{F}(t)}, & \text{if } x \geq 0 \end{cases},$$

and

$$S_{I,t}(x) = P(X_{(t)} > x) = \begin{cases} 1 & \text{if } x < 0 \\ \frac{F(t-x)}{F(t)}, & \text{if } 0 \leq x < t \\ 0 & \text{if } x \geq t \end{cases},$$

respectively. We denote $F_{R,t}(x) = 1 - S_{R,t}(x)$ and $F_{I,t}(x) = 1 - S_{I,t}(x)$ be the corresponding cumulative distribution functions.

XI. Systems of Components

The state of a system is determined completely by the states of the components. Therefore, $\phi = \phi(x)$; where $x = (x_1, x_2, \ldots, x_n)$. The function $\phi(x)$ is called the structure function of the system. The number of components n in the system is called the order of the system.

Series System:

A series system functions if and only if all of its components function. The structure function of a series system is given by

$$\phi(x) = \prod_{i=1}^{n} x_i = \min(x_1, x_2, \ldots, x_n)$$

Parallel System:

A parallel system functions if and only if at least one component functions. The structure function of a parallel system is given by

$$\phi(x) = \coprod_{i=1}^{n} x_i = \max(x_1, x_2, \ldots, x_n)$$

Also,

$$\coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n} (1 - x_i).$$

The concepts related to residual life time and inactivity time are extensively used in the reliability theory for modelling life time data.

1.3 Intuitionistic Fuzzy Sets: Preliminaries

The theory of Intuitionistic Fuzzy Sets is well suited to dealing with imprecise or uncertain decision information, image edge detection, medical diagnosis, pattern recognition, human expressions like perception, knowledge and behaviour, by reflecting and modelling the hesitancy present in real life situations and decision making. In this section, we present the basics of intuitionistic fuzzy sets and intuitionistic fuzzy numbers which is well known in literature.

Definition 1 *Atanassov's (1986,1989) intuitionistic fuzzy set (IFS) over a finite non empty fixed set* X *, is a set* $\widetilde{A} = \{< x, \mu_{\tilde{A}}(x), \gamma_{\tilde{A}}(x) > | \, x \in X\}$ *which assigns to each element* $x \in X$ *to the set* $\tilde{A}$*, which is a subset of* X *having the degree of membership* $\mu_{\tilde{A}}(x) : X \rightarrow [0,1]$ *and degree of non-membership* $\gamma_{\tilde{A}}(x) : X \rightarrow [0,1]$ *, satisfying* $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ *, for all* $x \in X$*.*

For each IFS in X, a hesitation margin $\pi_{\tilde{A}}(x)$, which is the intuitionistic fuzzy index of element x in the IFS $\tilde{A}$, defined by $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \gamma_{\tilde{A}}(x)$, denotes a measure of non-determinancy. We denote $IFS(X)$, the set of all the IFSs on X.

Definition 2 *An intuitionistic fuzzy subset* $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle : x \in X\}$ *of the real line* $\mathbb{R}$ *is called an intuitionistic fuzzy number if the following axioms hold:*

(i) $\tilde{A}$ *is normal, i.e., there exist a real numbers* b *(called the mean value of* $\tilde{A}$*) such that* $\mu_{\tilde{A}}(b) = 1$ *and* $\nu_{\tilde{A}}(b) = 0$*.*

(ii) The membership function $\mu_{\tilde{A}}$ *is fuzzy-convex i.e.*

$$\mu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda) \cdot x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \, \mu_{\tilde{A}}(x_2)\} \;\; \forall x_1, \, x_2 \in X, \;\; \lambda \in [0, \, 1]$$

(iii) The non-membership function $\nu_{\tilde{A}}$ *is fuzzy-concave i.e.*

$$\nu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda) \leq \max\{\nu_{\tilde{A}}(x_1), \, \nu_{\tilde{A}}(x_2)\} \;\; \forall x_1, \, x_2 \in X, \;\; \lambda \in [0, \, 1].$$

(iv) The membership and the non-membership function of $\tilde{A}$ *satisfying the condition* $0 \leq f_1(x) + g_1(x) \leq 1, \quad 0 \leq f_2(x) + g_2(x) \leq 1$ *have the following form:*

$$\mu_{\tilde{A}}(x) = \begin{cases} f_1(x) & \text{for } a_1 \leq x \leq a_2, \\ 1 & \text{for } x = a_2, \\ f_2(x) & \text{for } a_2 \leq x \leq a_3, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{\tilde{A}}(x)=\begin{cases} g_1(x) & for\ a_1^{'}\le x\le a_2,\\ 0 & for\ x=a_2,\\ g_2(x) & for\ a_2\le x\le a_3^{'}\\ 1 & otherwise\end{cases}$$

respectively, where $f_1(x)$ and $f_2(x)$ are strictly increasing and decreasing functions in $[a_1,\ a_2]$ and $[a_2,\ a_3]$,; and $g_1(x)$ and $g_2(x)$ are strictly decreasing and increasing functions in $[a_1^{'},\ a_2]$ and $[a_2,\ a_3^{'}]$, respectively. Symbolically the intuitionistic fuzzy number is represented as $\tilde{A}_{IFN}=\left(a_1,a_2,a_3;a_1^{'},a_2,a_3^{'}\right)$.

Shu & Cheng (2006) defined triangular intuitionistic fuzzy numbers (TIFNs) which have a greater capability to handle more ample and flexible information than triangular(trapezoidal) fuzzy numbers.

Definition 3 *Triangular intuitionistic fuzzy number $\tilde{\chi}\ =<\ (\underline{t},t,\bar{t});u_{\tilde{\chi}},w_{\tilde{\chi}}\ >$ (TIFN) is a special intuitionistic fuzzy set, whose membership function and non-membership function are defined by Atanassov's (1999) as follows:*

$$\mu_{\tilde{\chi}}(x)=\begin{cases} u_{\tilde{\chi}}(x-\underline{t})/(t-\underline{t}) & if\ \underline{t}\le x<t\\ u_{\tilde{\chi}} & if\ \ x=t\\ u_{\tilde{\chi}}(\bar{t}-x)/(\bar{t}-t) & if\ \ t<x\le\bar{t}\\ 0 & if\, x<\underline{t}\ or\, x>\bar{t}\end{cases}$$

and

$$\nu_{\tilde{\chi}}(x)=\begin{cases} [t-x+\ w_{\tilde{\chi}}(x-\underline{t})]/(t-\underline{t}) & if\ \ \underline{t}\le x<t\\ w_{\tilde{\chi}} & if\ \ \ x=t\\ [x-t+w_{\tilde{\chi}}(\bar{t}\ -x)]/(\bar{t}-t) & if\ \ t<x\le\bar{t}\\ 1 & if\ \ x<\ \underline{t}\ \ or\ \ x>\bar{t}\end{cases}$$

respectively, where the values $u_{\tilde{\chi}}$ and $w_{\tilde{\chi}}$ represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy
$0\le u_{\tilde{\chi}}\le 1,\quad 0\le w_{\tilde{\chi}}\le 1,\quad 0\le u_{\tilde{\chi}}+w_{\tilde{\chi}}\le 1.$

Let $\pi_{\tilde{\chi}}(x) = 1 - \mu_{\tilde{\chi}}(x) - \gamma_{\tilde{\chi}}(x)$, which is called as intuitionistic fuzzy index of an element x in $\tilde{\chi}$. It is the degree of indeterminacy membership of the element x in $\tilde{\chi}$. The TIFN $\tilde{\chi} =< (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ is called as a positive TIFN, denoted by $\tilde{\chi} > 0$, if $t \geq 0$ and one of the three values $\underline{t}$, t and $\bar{t}$ is not equal to zero. Similarly, if $\bar{t} \leq 0$ and one of the three values $\underline{t}$, t and $\bar{t}$ is not equal to zero, then the TIFN $\tilde{\chi} =< (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ is called as a negative TIFN, denoted by $\tilde{\chi} < 0$.

The concept of intuitionistic trapezoidal fuzzy numbers (ITFNs) was introduced by Wang (2008) and it may be noted that intuitionistic trapezoidal fuzzy numbers (ITFNs) express more flexible and abundant information than trapezoidal fuzzy numbers.

Definition 4 *Intuitionistic trapezoidal fuzzy number (ITFN)* $\tilde{\chi} = \{(a, b, c, d)\, ;\, \mu_{\tilde{\chi}},\, \gamma_{\tilde{\chi}}\}$ *is a special intuitionistic fuzzy set, whose membership function and non-membership function have been defined as follows:*

$$\mu_{\tilde{\chi}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\mu_{\tilde{\chi}} & \text{if } a \leq x \leq b, \\ \mu_{\tilde{\chi}} & \text{if } b \leq x \leq c, \\ \frac{(d-x)}{(d-c)}\mu_{\tilde{\chi}} & \text{if } c < x \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_{\tilde{\chi}}(x) = \begin{cases} \frac{(b-x)+\gamma_{\tilde{\chi}}(x-a)}{(b-a)} & \text{if } a_1 \leq x \leq b, \\ \gamma_{\tilde{\chi}} & \text{if } b \leq x \leq c, \\ \frac{(x-c)+\gamma_{\tilde{\chi}}(d_1-x)}{(d_1-c)}\mu_{\tilde{\chi}} & \text{if } c < x \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 \leq \mu_{\tilde{\chi}} \leq 1$ and $0 \leq \gamma_{\tilde{\chi}} \leq 1$. Also, $\mu_{\tilde{\chi}} + \gamma_{\tilde{\chi}} \leq 1$ for all $a, b, c, d \in R$. The values $\mu_{\tilde{\chi}}$ and $\gamma_{\tilde{\chi}}$ represent the maximum membership degree and minimum non-membership degree, respectively.

1.4 Measures of 'Useful' Fuzzy Information and Directed Divergence

Though in many practical situations of probabilistic nature, subjective considerations play its own role, Shannon entropy does not take into account the effectiveness or importance of the events. Belis & Guiasu (1968) considered a qualitative aspect of information called 'useful' information by implementing a utility distribution given by $U = (u_1, u_2, u_3, \ldots, u_n)$, where $u_i > 0$, for each i and is utility or importance of an event x_i whose probability of occurrence is p_i. Also, it is assumed that u_i is independent of p_i. It has also been suggested that the occurrence of an event removes two types of uncertainty - the quantitative type related to its probability of occurrence and the qualitative type related to its utility (importance) for fulfillment of some goal set by the experimenter. In view of this, they proposed the following 'useful' information measure as

$$H(U;P) \;=\; -\sum u_i p_i \log p_i.$$

In case $u_i = 1\ \forall i$, the above equation reduces to $H(P) = -\sum p_i \log p_i$, which is well known Shannon's Entropy (1948).

Bhaker and Hooda (1993) obtained the generalized mean value characterization of the useful information measures for incomplete probability distributions:

$$H(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}$$

and

$$H_\alpha(P;U) = \frac{1}{1-\alpha} \log \frac{\sum_{i=1}^{n} u_i p_i^\alpha}{\sum_{i=1}^{n} u_i p_i}; \;\; \alpha \neq 1, \;\; \alpha > 0.$$

Zadeh (1968) was the first to quantify the uncertainty associated with a fuzzy event in the context of a discrete probabilistic framework, who defined the weighted entropy of A with respect to (X, P) as

$$H(A, P) = -\sum_{i=1}^{n} \mu_A(x_i) p_i \log p_i,$$

where μ_A is the membership function of A and p_i is the probability of occurrence of x_i. It may be noted that the situation contains the different types of uncertainties, e.g., randomness, ambiguity and vagueness. $H(A, P)$ of a fuzzy event with respect to P is less than Shannon's entropy which is of P alone.

Let P and Q be two probability distributions of a random variable X having utility distribution U. Bhaker and Hooda (1993) characterized the following measure of 'useful' directed divergence:

$$D(P : Q : U) = \frac{\sum_{i=1}^{n} u_i p_i \log \frac{p_i}{q_i}}{\sum_{i=1}^{n} u_i p_i}.$$

1.5 Literature Survey

Block et. al. (1998), Chandra & Roy (2001), Li & Zhang (2003), Li & Lu (2003), Li & Zuo (2004), Misra et al. (2008) and Pellerey & Petakos (2002) studied reliability properties of residual life/inactivity time. The stochastic comparisons of residual life time and inactivity time in series and parallel systems is discussed by Li & Lu (2003) and Li & Zhang (2003).

Let $\eta_f(x) = -f'(x)/f(x),\ x \in S$, and $\eta_g(x) = -g'(x)/g(x),\ x \in S$ denote the eta functions of random variable X and Y respectively. Glaser (1980) demonstrated that the eta functions play a vital role in the study of the failure rates.

Li & Zhang (2003) proved that if X and Y are independent and identically distributed then, for all $t \geq 0$, $(\max(X,Y))_t \leq_{st} \max(X_t, Y_t)$; similar results are also proved for inactivity time. Li & Lu (2003) strengthen the results of Li & Zhang (2003) and proved that if X and Y are independent and identically distributed then, for all $t \geq 0$,

(i) $(\max(X,Y))_t \leq_{lr} \max(X_t, Y_t)$;

(ii) $(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t)}, Y_{(t)})$; and

(iii) $\min(X_{(t)}, Y_{(t)}) \leq_{lr} (\min(X,Y))_{(t)}$.

Li & Lu (2003) also proved that, if X and Y are independent (not necessarily identical distributed) then, for all $t \geq 0$,

(i) $(\max(X,Y))_t \leq_{fr} \max(X_t, Y_t)$;

(ii) $(\max(X,Y))_{(t)} \leq_{fr} \max(X_{(t)}, Y_{(t)})$; and

(iii) $\min(X_{(t)}, Y_{(t)}) \leq_{fr} (\min(X,Y))_{(t)}$.

By the method of ascertainment, the concept of weighted distribution has been introduced by C.R. Rao in 1965. Weighted distribution have been widely used as a tool in various practical problems in the selection of appropriate models for observed data drown without a proper frame, analysis of data relating to human populations and wild life management, investigation of human heredity, line transcend sampling and renewal theory, study of statistical ecology, albinism and reliability modelling.

Let $f(\cdot)$ be the probability density function of original random variable X. Let $w_1(\cdot) : \mathbb{R} \rightarrow [0, \infty)$, where $\mathbb{R} = (-\infty, \infty)$ and the recovered random variable be X_{w_1} with the probability density function given by

$$f_{w_1}(x) = \frac{w_1(x) f(x)}{w_1}; \ x \in \mathbb{R};$$

where $w_1 = E(w_1(X)) > 0$.

The random variable X_{w_1} is called the weighted version of X and its distribution in relation to that of X is called the weighted distribution of X with weighted function $w_1(.)$.

Jain et. al. (1989), Nanda & Jatin (1999), Misra et. al. (2008), Barlow et. al (1981), and Bartoszewicz et. al (2006) have studied the reliability properties of weighted distributions in relation to corresponding reliability measures of parent distributions.

Equilibrium distribution which is also known as integrated tail function originated in the context of renewal processes and acting as the limiting distribution of the forward recurrence time in a renewal process. Equilibrium distribution has wide applications in insurance, financial investment, reliability, stochastic processes, repairable systems and many areas of applied probability such as renewal risk model, tail distributions of ladder heights of random walk, limiting distributions of waiting time and busy model of queueing model etc [cf. Embrechts et. al. (1997), Asmussen (2000) and Rolski et. al. (1999)]. For a detailed survey of equilibrium distribution one may refer Abouammoh et. al. [(1993),(2000)], Mugdadi and Ahmad (2005), Bon and Illayk (2005), Su and Tang (2003), Bon and Illayk (2002), Mi (1988), Bhattacharjee et. al. (2000), and Li and Xu (2008).

In literature many researchers provides characterization of stochastic orders in terms of ordering of equilibrium distributions.
Whitt (1985) proved that

$$X \leq_{hr(mrl,hmrl)} Y \Leftrightarrow \tilde{X} \leq_{lr(hr,st)} \tilde{Y}.$$

Bon and Illayk (2005) proved that if X_1 and X_2 are independent DMRL random variables, then

$$\min(\tilde{X}_1, \tilde{X}_2) \leq_{lr} \widetilde{\min(X_1, X_2)}.$$

Li and Xu (2008) proves that $X \leq_{rh} Y \Rightarrow \tilde{X} \leq_{st} \tilde{Y}$; and shows that reverse implication may not be true. Additionally, it has also been proved that if $\tilde{X} \leq_{rh} \tilde{Y}$, then

$$\widetilde{\min_{1\leq i\leq n} X_i} \leq_{rh} \widetilde{\min_{1\leq i\leq n} Y_i}.$$

The theory of stochastic orders provides various tools for the stochastic comparison of probability distributions. For a detailed study on the theory of stochastic orders one may refer Muller and Stoyan (2002) and Shaked and Shanthikumar (2007). Some of these orders are moment generating function (or exponential) order and Laplace transform order and their residual life and inactivity time (or reversed residual life) (cf. Ahmed and Kayid (2004), Elbatal (2007), Kayid (2011) and Kayid and Alamoudi (2013)).

Belzunce et al (1999), Ahmed and Kayid (2004) and Elbatal (2007) studied several preservation properties of the Laplace transform ordering of residual lives/inactivity times under the reliability operations of convolutions, mixtures and weak convergence. Further, Kayid (2011) and Kayid and Alamoudi (2013) established the preservation properties of the moment generating function ordering of residual lives/inactivity times under the reliability operations of convolutions and mixtures.

Zadeh (1965) introduced the concept of fuzzy set, which is capable of representing human knowledge, perception etc. As an extension of fuzzy set, Atanassov (1986,1989) introduced the concept of intuitionistic fuzzy set (IFS), which is found to be more useful in capturing the vague, incomplete or uncertain information that includes some degree of hesitation and applicable in various fields of research. Gau and Buehrer in 1993, introduced the concept of vague set. Grattan-Guinness (1976), Jahn (1975) and Sambuc (1975) introduced the theory of interval valued fuzzy set , which is well known generalization of ordinary fuzzy set. Among various extensions of fuzzy sets such as IFS, vague set, interval-valued fuzzy set, IFSs are found to be more consistent with human behaviour. Many researchers such as Chen & Tan (1994), Hong and Choi (2000), Szmidt & Kacprzyk (2002) shown great interest in IFS theory and its applications in decision making.

In literature, Burillo, Bustince & Mohedano (1994), Liu & Shi (2000) , Grzegorzewski (2003), Shih, Su & Yao (2009) etc. have proposed various research works on intuitionistic fuzzy numbers.

Shu & Cheng (2006) defined triangular intuitionistic fuzzy numbers (TIFNs) which have a greater capability to handle more ample and flexible information than triangular fuzzy numbers.

The concept of intuitionistic trapezoidal fuzzy numbers (ITFNs) was introduced by Wang (2008) and it may be noted that intuitionistic trapezoidal fuzzy numbers (ITFNs) express more flexible and abundant information than trapezoidal fuzzy numbers.

Shannon (1948) entropy does not take into account the effectiveness or importance of the events. Belis & Guiasu (1968) considered a qualitative aspect of information called 'useful' information by implementing a utility distribution given by $U = (u_1, u_2, u_3, \ldots, u_n)$, where $u_i > 0$, for each i and is utility or importance of an event x_i whose probability of occurrence is p_i.

Chapter 2

Reliability Properties of Residual Life Time and Inactivity Time of Series and Parallel System

2.1 Introduction

Let X and Y be two statistically independent random variables with an absolutely continuous distribution function $F(\cdot)$ and $G(\cdot)$, survival function $\overline{F}(\cdot) = 1 - F(\cdot)$ and $\overline{G}(\cdot) = 1 - G(\cdot)$ and probability density function $f(\cdot)$ and $g(\cdot)$ respectively. Suppose that

$$\{x \in \mathbb{R} : f(x) > 0\} = \{x \in \mathbb{R} : g(x) > 0\} = (0, \infty) = S \text{ (say)},$$

where $\mathbb{R} = (-\infty, \infty)$.

Let X and Y denote the lifetimes of two components, say C_1 and C_2. A series (parallel) system comprising of components C_1 and C_2 functions if and only if all (at least one) of its component function(s). Clearly, $\min(X, Y)$ and $\max(X, Y)$ are respectively the lifetime of series and parallel systems comprising of components C_1 and C_2; here $\min(X,Y)\,(\max(X,Y))$ denotes the minimum (maximum) of X and Y respectively. The residual life of X with age/time $t \geq 0$ is given by

$$X_t = (X - t \mid X > t), \; t \geq 0,$$

and inactivity time of X at time $t \geq 0$ is given by

$$X_{(t)} = (t - X \mid X \leq t), \; t \geq 0.$$

For a fixed $t \geq 0$, the survival functions of X_t and $X_{(t)}$ are given by

$$S_{R,t}(x) = P(X_t > x) = \begin{cases} 1 & \text{if } x < 0 \\ \frac{\overline{F}(x+t)}{\overline{F}(t)}, & \text{if } x \geq 0 \end{cases},$$

and

$$S_{I,t}(x) = P(X_{(t)} > x) = \begin{cases} 1 & \text{if } x < 0 \\ \frac{F(t-x)}{F(t)}, & \text{if } 0 \leq x < t \\ 0 & \text{if } x \geq t \end{cases},$$

respectively. We denote $F_{R,t}(x) = 1 - S_{R,t}(x)$ and $F_{I,t}(x) = 1 - S_{I,t}(x)$ as the corresponding cumulative distribution functions.

For reliability engineers, the study of reliability properties of series and parallel systems is of great importance. Block et. al. (1998), Chandra & Roy (2001), Pellerey & Petakos (2002), Li & Zhang (2003), Li & Lu (2003), Li & Zuo (2004) and Misra et al. (2008) studied reliability properties of residual life/inactivity time. The stochastic comparisons of residual life time and inactivity time in series and parallel systems is discussed by Li & Lu (2003) and Li & Zhang (2003). It may be noted that

- the residual life of series (parallel) system having components X and Y is $(\min(X,Y))_t$ $((\max(X,Y))_t)$;

- the inactivity time of series (parallel) system having components X and Y is $(\min(X,Y))_{(t)}$ $((\max(X,Y))_{(t)})$;

- the lifetime of the series (parallel) system having residual lives X_t and Y_t is $\min(X_t, Y_t)$ $(\max(X_t, Y_t))$;

- the lifetime of series (parallel) system having inactivity times $X_{(t)}$ and $Y_{(t)}$ is $\min(X_{(t)}, Y_{(t)})((\max(X_{(t)}, Y_{(t)}))$.

Let $\eta_f(x) = -f'(x)/f(x),\ x \in S$ and $\eta_g(x) = -g'(x)/g(x),\ x \in S$ denote the eta functions of random variable X and Y respectively. Glaser (1980) demonstrated that the eta functions play a vital role in the study of the failure rates. We use the terms increasing and decreasing instead of non-decreasing and non-increasing, respectively. Next, we include below some definitions of stochastic orders which are standard in the literature [cf. Shaked & Shanthikumar (2007)].

Definition 2.1.1:

The random variable X is said to be smaller than random variable Y in the

(a) likelihood ratio (lr) ordering $(X \leq_{lr} Y)$ if $\frac{g(x)}{f(x)}$ increases in $x \in S$;

(b) reversed failure rate (rfr) ordering $(X \leq_{rfr} Y)$ if $\frac{G(x)}{F(x)}$ increases in $x \in S$;

(c) usual stochastic (st) ordering $(X \leq_{st} Y)$ if $\overline{F}(x) \leq \overline{G}(x)$, for all $x \in \mathbb{R}$.

Now we present some notions of ageing (cf. Barlow and Proschan (1981)):

Definition 2.1.2:

The random variable X is said to have

(d) increasing failure rate (IFR) if the failure rate function $\frac{f(x)}{\overline{F}(x)}$ is increasing in $x \in S$;

(e) decreasing failure rate (DFR) if the failure rate function $\frac{f(x)}{\overline{F}(x)}$ is decreasing in $x \in S$;

(f) decreasing reversed failure rate (DRFR) if the reversed failure rate function $\frac{f(x)}{F(x)}$ is decreasing in $x \in S$.

Li & Zhang (2003) proved that if X and Y are independent and identically distributed, then for all $t \geq 0$, $(\max(X,Y))_t \leq_{st} \max(X_t, Y_t)$. Similar results for inactivity time have also been proved. Li & Lu (2003) strengthen the results of Li & Zhang (2003) and proved that if X and Y are independent and identically distributed, then for all $t \geq 0$,

(i) $(\max(X,Y))_t \leq_{lr} \max(X_t, Y_t)$;

(ii) $(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t)}, Y_{(t)})$; and

(iii) $\min(X_{(t)}, Y_{(t)}) \leq_{lr} (\min(X,Y))_{(t)}$.

Li & Lu (2003) also proved that, if X and Y are independent (not necessarily identical distributed), then for all $t \geq 0$,

(i) $(\max(X,Y))_t \leq_{fr} \max(X_t, Y_t)$;

(ii) $(\max(X,Y))_{(t)} \leq_{fr} \max(X_{(t)}, Y_{(t)})$; and

(iii) $\min(X_{(t)}, Y_{(t)}) \leq_{fr} (\min(X,Y))_{(t)}$.

In section 2.2 of the chapter, we obtain some new results on stochastic comparisons of residual life time and inactivity time in series and parallel systems. Assuming that X and Y are independent, but not necessarily identical distributed and letting $X \leq_{rfr} Y$, $\eta_f < 0$ and $\eta_g > 0$, (or $Y \leq_{rfr} X$, $\eta_f > 0$ and $\eta_g > 0$), we proved that the parallel system of used components, i.e., $\max(X_t, Y_t)$, is better than the used parallel system, i.e., $(\max(X,Y))_t$, in the sense of likelihood ratio order. Further, assuming X and Y are independent, but not necessarily identical distributed and letting $X \leq_{lr} Y$, (or $Y \leq_{lr} X$), we proved that for any $t \geq 0$,

$$(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t)}, Y_{(t)});$$

and

$$\min(X_{(t)}, Y_{(t)}) \leq_{lr} (\min(X,Y))_{(t)}\,.$$

In section 2.3, we prove various ageing properties of used/inactive parallel/series systems and the parallel/series system of used/inactive components. Finally, some examples are provided to support the obtained results of sections 2.2 and 2.3 in section 2.4 by taking Weibull and Gompertz distributions into consideration.

2.2 Stochastic Comparison

Li & Lu (2003) proved that if X and Y are independent and identical distributed then for any $t \geq 0$, $(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t),}Y_{(t)})$. They also proved that if X and Y are independent, but not necessarily identically distributed, then for any $t \geq 0$, $(\max(X,Y))_{(t)} \leq_{fr} \max(X_{(t),}Y_{(t)})$. In the following theorem, we find the sufficient conditions for $(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t),}Y_{(t)})$ to hold when X and Y are independent, but not necessarily identically distributed.

Theorem 2.2.1:

If $X \leq_{lr} Y$ or $Y \leq_{lr} X$ then for any $t \geq 0$, $(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t),}Y_{(t)})$.

Proof:

Let $t \geq 0$ be fixed. Let $H_{1,t}(x)$ and $h_{1,t}(x)$ denote respectively the cumulative distribution function and probability density function of random variable $(\max(X,Y))_{(t)}$. Then for $0 \leq x \leq t$,

$$\begin{aligned} H_{1,t}(x) &= P[(\max(X,Y))_{(t)} \leq x] \\ &= \frac{F(t)G(t) - F(t-x)G(t-x)}{F(t)G(t)}, \end{aligned} \tag{2.2.1}$$

and

$$h_{1,t}(x) = \frac{F(t-x)g(t-x) + f(t-x)G(t-x)}{F(t)G(t)}. \tag{2.2.2}$$

Let $M_{1,t}(x)$ and $m_{1,t}(x)$ denote the cumulative distribution function and probability density function of random variable $\max(X_{(t),}Y_{(t)})$. For $0 \le x \le t$, we have

$$\begin{aligned} M_{1,t}(x) &= P\left((\max(X_{(t)},Y_{(t)})) \le x\right) \\ &= \left(\frac{F(t) - F(t-x)}{F(t)}\right)\left(\frac{G(t) - G(t-x)}{G(t)}\right), \end{aligned} \tag{2.2.3}$$

and

$$m_{1,t}(x) = \frac{(F(t) - F(t-x))\,g(t-x) + (G(t) - G(t-x))\,f(t-x)}{F(t)G(t)}. \tag{2.2.4}$$

For $0 \le x < t$, we consider

$$\begin{aligned} R_{1,t}(x) &= \frac{m_{1,t}(x)}{h_{1,t}(x)} \\ &= \frac{(F(t) - F(t-x))\,g(t-x) + (G(t) - G(t-x))\,f(t-x)}{F(t-x)g(t-x) + f(t-x)G(t-x)} \\ &= -1 + \frac{F(t)\,g(t-x) + G(t)\,f(t-x)}{F(t-x)g(t-x) + f(t-x)G(t-x)}. \end{aligned}$$

For $0 \le x < t$, it is easy to verify that

$$\begin{aligned} R'_{1,t}(x) = \frac{f(t-x)g(t-x)}{[F(t-x)g(t-x)+f(t-x)G(t-x)]^2}\Big[2F(t)g(t-x) + 2G(t)f(t-x) \\ + [\{\eta_f(t-x) - \eta_g(t-x)\}\{G(t)F(t-x) - G(t-x)F(t)\}]\Big]. \end{aligned} \tag{2.2.5}$$

We will prove the assertion for the case $X \le_{lr} Y$. Similarly, the assertion follows for the case $Y \le_{lr} X$. It may be noted that

$$X \le_{lr} Y \Leftrightarrow \ln\left(\frac{g(t)}{f(t)}\right) \text{ is increasing in } t \in (0,\infty) \Leftrightarrow \eta_f(t) \ge \eta_g(t),\ \forall t > 0. \tag{2.2.6}$$

Also,

$$X \le_{lr} Y \Rightarrow X \le_{rfr} Y \Leftrightarrow F(u)G(v) \ge F(v)G(u),\ \forall\, 0 \le u \le v < \infty. \tag{2.2.7}$$

Using (2.2.6) and (2.2.7) in (2.2.5), we conclude that $R'_{1,t}(x) \ge 0,\ \forall\, 0 \le x < t$, i.e., $(\max(X,Y))_{(t)} \le_{lr} \max(X_{(t)}, Y_{(t)})$.

The following corollary is an immediate consequence of Theorem 2.2.1.

Corollary 2.2.1:

If $X =_{st} Y$, then $(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t)}, Y_{(t)})$.

Remark 2.2.1:

The result stated in Corollary 2.2.1 is in by Li and Lu (2003).

Li & Lu (2003) proved that if X and Y are independent and identically distributed, then for any $t \geq 0$, $(\min(X,Y))_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)})$. They also proved that if X and Y are independent, but not necessarily identically distributed, then for any $t \geq 0$, $(\min(X,Y))_{(t)} \geq_{fr} \min(X_{(t)}, Y_{(t)})$. In the following theorem, we find the sufficient conditions for $(\min(X,Y))_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)})$ to hold when X and Y are independent, but not necessarily identically distributed.

Theorem 2.2.2:

If $X \leq_{lr} Y$ or $Y \leq_{lr} X$, then for any $t \geq 0$, $(\min(X,Y))_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)})$.

Proof:

We fix $t \geq 0$. Let $H_{2,t}(x)$ and $h_{2,t}(x)$ denote respectively the cumulative distribution function and probability density function of random variable $(\min(X,Y))_{(t)}$. For $0 \leq x \leq t$, we have

$$\begin{aligned} H_{2,t}(x) &= P\left((\min(X,Y))_{(t)} \leq x\right) \\ &= \frac{(1-F(t-x))(1-G(t-x)) - (1-F(t))(1-G(t))}{1-(1-F(t))(1-G(t))}, \end{aligned} \tag{2.2.8}$$

and

$$h_{2,t}(x) = \frac{(1-F(t-x))g(t-x) + (1-G(t-x))f(t-x)}{1-(1-F(t))(1-G(t))}. \tag{2.2.9}$$

For $0 \leq x \leq t$, let $M_{2,t}(x)$ and $m_{2,t}(x)$ denote respectively the cumulative distribution function and probability density function of random variable $\min(X_{(t)}, Y_{(t)})$.

Then, for $0 \le x \le t$,

$$\begin{aligned} M_{2,t}(x) &= P(\min(X_{(t)}, Y_{(t)}) \le x) \\ &= 1 - \frac{F(t-x)G(t-x)}{F(t)G(t)}, \end{aligned} \tag{2.2.10}$$

and

$$m_{2,t}(x) = \frac{f(t-x)G(t-x) + F(t-x)g(t-x)}{F(t)G(t)}. \tag{2.2.11}$$

For $0 \le x < t$, we consider

$$\begin{aligned} R_{2,t}(x) &= \frac{h_{2,t}(x)}{m_{2,t}(x)} \\ &= \left(\frac{F(t)G(t)}{1-(1-F(t))(1-G(t))}\right)\left(\frac{(1-F(t-x))g(t-x)+(1-G(t-x))f(t-x)}{F(t-x)g(t-x)+f(t-x)G(t-x)}\right) \\ &= A(t)Z_t(x), \end{aligned}$$

where

$$A(t) = \frac{F(t)G(t)}{1-(1-F(t))(1-G(t))},$$

and, for $0 \le x < t$,

$$Z_t(x) = -1 + \frac{g(t-x)+f(t-x)}{F(t-x)g(t-x)+f(t-x)G(t-x)}.$$

Clearly, for $0 \le x < t$,

$$\begin{aligned} Z_t'(x) &= \frac{1}{[F(t-x)g(t-x)+G(t-x)F(t-x)]^2}\Big[-[F(t-x)g(t-x)+f(t-x)G(t-x)] \\ &\quad [g'(t-x)+f'(t-x)] + [g(t-x)+f(t-x)][F(t-x)g'(t-x)+f(t-x)g(t-x) \\ &\quad + f'(t-x)G(t-x)+g(t-x)f(t-x)]\Big] \\ &= \frac{1}{[F(t-x)g(t-x)+G(t-x)f(t-x)]^2}\Big[2g^2(t-x)f(t-x)+2f^2(t-x)g(t-x) \\ &\quad + [F(t-x)\{-g(t-x)f'(t-x)+f(t-x)g'(t-x)\}] + [G(t-x)\{g(t-x)f'(t-x) \\ &\quad - f(t-x)g'(t-x)\}]\Big] \\ &= \frac{f(t-x)g(t-x)}{[F(t-x)g(t-x)+G(t-x)f(t-x)]^2}\Big[2g(t-x)+2f(t-x) \\ &\qquad + [\{\eta_f(t-x)-\eta_g(t-x)\}\{F(t-x)-G(t-x)\}]\Big]. \end{aligned} \tag{2.2.12}$$

We will prove the assertion for the case $X \leq_{lr} Y$. Similarly, the assertion follows for the case $Y \leq_{lr} X$. It may be noted that, as in the proof of the Theorem 2.2.1,

$$X \leq_{lr} Y \Leftrightarrow \eta_f(t) \geq \eta_g(t),\ \forall t > 0. \tag{2.2.13}$$

Also,

$$X \leq_{lr} Y \Rightarrow X \leq_{st} Y \Leftrightarrow F(u) \geq G(u),\ \forall\, 0 \leq u < \infty. \tag{2.2.14}$$

Using (2.2.13) and (2.2.14) in (2.2.12), we conclude that $Z_t'(x) \geq 0,\ \forall\, 0 \leq x < t$, i.e., $(\min(X,Y))_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)})$.

The following corollary is an immediate consequence of Theorem 2.2.2.

Corollary 2.2.2:

If $X =_{st} Y$, then $(\min(X,Y))_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)})$.

Remark 2.2.2:

The result stated in Corollary 2.2.2 is in by Li and Lu (2003).

Li & Lu (2003) proved that if X and Y are independent and identically distributed, then for any $t \geq 0$, $(\max(X,Y))_t \leq_{lr} \max(X_t, Y_t)$. They also proved that if X and Y are independent, but not necessarily identically distributed, then for any $t \geq 0$, $(\max(X,Y))_t \leq_{fr} \max(X_t, Y_t)$. In the following theorem, we find the sufficient conditions for $(\max(X,Y))_t \leq_{lr} \max(X_t, Y_t)$ to hold when X and Y are independent, but not necessarily identically distributed.

Theorem 2.2.3:

If $X \leq_{rfr} Y$, $\eta_f \leq 0$ and $\eta_g \geq 0$ or $Y \leq_{rfr} X$, $\eta_f \geq 0$ and $\eta_g \leq 0$, then for any $t \geq 0$, $(\max(X,Y))_t \leq_{lr} \max(X_t, Y_t)$.

Proof:

Let $t \geq 0$ be fixed. Let $H_{3,t}(x)$ and $h_{3,t}(x)$ denote respectively the cumulative distribution function and probability density function of random variable

$(\max(X,Y))_t$. Then for $x \geq 0$,

$$\begin{aligned} H_{3,t}(x) &= P[(\max(X,Y))_t \leq x] \\ &= \frac{F(t+x)G(t+x) - F(t)G(t)}{1 - F(t)G(t)}, \end{aligned} \tag{2.2.15}$$

and

$$h_{3,t}(x) = \frac{f(t+x)G(t+x) + g(t+x)F(t+x)}{1 - F(t)G(t)}. \tag{2.2.16}$$

Let $M_{3,t}(x)$ and $m_{3,t}(x)$ denote the cumulative distribution function and probability density function of random variable $\max(X_t, Y_t)$.

For $x \geq 0$, we have

$$\begin{aligned} M_{3,t}(x) &= P\left((\max(X_t, Y_t)) \leq x\right) \\ &= \left(\frac{F(t+x) - F(t)}{1 - F(t)}\right)\left(\frac{G(t+x) - G(t)}{1 - G(t)}\right), \end{aligned} \tag{2.2.17}$$

and

$$m_{3,t}(x) = \frac{(G(t+x) - G(t))\, f(t+x) + (F(t+x) - F(t))\, g(t+x)}{(1 - F(t))\,(1 - G(t))}. \tag{2.2.18}$$

For $x \geq 0$, we consider

$$\begin{aligned} R_{3,t}(x) &= \frac{m_{3,t}(x)}{h_{3,t}(x)} \\ &= \left(\frac{1 - F(t)G(t)}{(1 - F(t))\,(1 - G(t))}\right)\left(\frac{(G(t+x) - G(t)f(t+x)) + (F(t+x) - F(t)g(t+x))}{(1 - F(t))\,(1 - G(t))}\right) \\ &= B(t)U_t(x), \end{aligned}$$

where

$$B(t) = \frac{1 - F(t)G(t)}{(1 - F(t))\,(1 - G(t))},$$

and

$$U_t(x) = 1 - \frac{f(t+x)G(t) + g(t+x)F(t)}{f(t+x)G(t+x) + g(t+x)F(t+x)}.$$

For $x \geq 0$, it may be easily verified that

$$\begin{aligned} U_t'(x) = \frac{f(t+x)g(t+x)}{[f(t+x)G(t+x) + g(t+x)F(t+x)]^2} \Big[& 2\,(G(t)f(t+x) + g(t+x)F(t)) \\ & + (G(t+x)F(t) - G(t)F(t+x))\,(\eta_g(t+x) - \eta_f(t+x)) \Big] \end{aligned} \tag{2.2.19}$$

We will prove the assertion for the case $X \leq_{rfr} Y$, $\eta_f \leq 0$ and $\eta_g \geq 0$. Similarly, the assertion follows for the case $Y \leq_{rfr} X$, $\eta_f \geq 0$ and $\eta_g \leq 0$. It may be noted that

$$X \leq_{rfr} Y \Leftrightarrow F(u)G(v) \geq F(v)G(u), \ \forall\, 0 \leq u \leq v < \infty. \tag{2.2.20}$$

Using (2.2.20), $\eta_f \leq 0$ & $\eta_g \geq 0$ in (2.2.19), we conclude that $U_t'(x) \geq 0, \ \forall\, x \geq 0$, i.e., $(\max(X, Y))_t \geq_{lr} \max(X_t, Y_t)$.

2.3 Ageing Properties

In this section, we discuss the various ageing properties of the residual life time and inactivity time in series and parallel systems. The following property proves that if the random variables X and Y have DRFR, then this property is preserved by the random variable $\max(X_t, Y_t)$.

Property 2.3.1:

Suppose that the random variables X and Y have DRFR. Then for any $t \geq 0$, the random variable $\max(X_t, Y_t)$ has DRFR.

Proof:

Fix $t > 0$. Let $\lambda_t(x)$ and $\mu_t(x)$ denote respectively the reversed failure rates of X_t and Y_t and let $M_{4,t}(x)$ denote the cumulative distribution function of $\max(X_t, Y_t)$. Let $F_{R,t}(x)$ and $G_{R,t}(x)$ denote respectively the cumulative distribution functions of X_t and Y_t. Then for $x \geq 0$,

$$\lambda_t(x) = \frac{f(x+t)}{F(x+t) - F(t)}, \ \mu_t(x) = \frac{g(x+t)}{G(x+t) - G(t)}, \text{ and } M_{4,t}(x) = F_{R,t}(x) G_{R,t}(x).$$

X has DRFR implies that $F(x)f'(x) \leq f^2(x), \ \forall x > 0$, which in turn implies that $\lambda_t'(x) \leq 0, \ \forall x > 0$ (i.e., X_t has DRFR or equivalently $\ln(F_{R,t}(x))$ is concave in $x \in (0, \infty)$). Similarly, Y has DRFR implies that Y_t has DRFR (i.e., $\ln(G_{R,t}(x))$

is concave in $x \in (0, \infty)$). Thus if X and Y have DRFR then

$$\ln(M_{4,t}(x)) = \ln(F_{R,t}(x)G_{R,t}(x)) = \ln(F_{R,t}(x)) + \ln(G_{R,t}(x))$$

is concave in $x \in (0, \infty)$, i.e., $\max(X_t, Y_t)$ has DRFR.

In the following property, we prove that if the random variables X and Y have DRFR, then the random variable $(\max(X, Y))_{(t)}$ has IFR.

Property 2.3.2:

Suppose that the random variables X and Y have DRFR. Then for any $t \geq 0$, the random variable $(\max(X, Y))_{(t)}$ has IFR.

Proof:

Fix $t \geq 0$. It is obvious that if the random variables X and Y have DRFR, then $\max(X, Y)$ also has DRFR. Also it is easy to verify that if a non-negative random variable Z has DRFR, then for any $s \geq 0$, the random variable $Z_{(s)} = (s - Z | Z \leq s)$ has IFR. Thus under the hypothesis of the theorem, $\max(X, Y)$ has DRFR, which in turn implies that $(\max(X, Y))_{(t)}$ has IFR.

In the following property, we prove that if the random variables X and Y have IFR, then the random variable $\max(X_{(t)}, Y_{(t)})$ has DRFR.

Property 2.3.3:

Suppose that the random variables X and Y have IFR. Then for any $t \geq 0$, the random variable $\max(X_{(t)}, Y_{(t)})$ has DRFR.

Proof:

Fix $t \geq 0$. It is obvious that if the random variables X and Y have IFR, then random variables $X_{(t)}$ and $Y_{(t)}$ have DRFR. This in turn implies that $\max(X_{(t)}, Y_{(t)})$ has DRFR.

In the following property, we prove that if the random variables X and Y have DRFR, then the random variable $\min(X_{(t)}, Y_{(t)})$ has IFR.

Property 2.3.4:

Suppose that the random variables X and Y have DRFR. Then for any $t \geq 0$, the random variable $\min(X_{(t)}, Y_{(t)})$ has IFR.

Proof:

Fix $t \geq 0$. It is obvious that if random variables X and Y have DRFR, then random variables $X_{(t)}$ and $Y_{(t)}$ have IFR. This in turn implies that $(\min(X,Y))_{(t)}$ has IFR.

The following property proves that if the random variables X and Y have IFR, then this property is preserved by the random variable $\min(X_t, Y_t)$.

Property 2.3.5:

Suppose that the random variables X and Y have IFR. Then for any $t \geq 0$, the random variable $\min(X_t, Y_t)$ has IFR.

Proof:

Fix $t \geq 0$. It is obvious that if random variables X and Y have IFR, then random variables X_t and Y_t have IFR. This in turn implies that $\min(X_t, Y_t)$ has IFR.

The following property proves that if the random variables X and Y have IFR, then this property is preserved by the random variable $(\min(X,Y))_t$.

Property 2.3.6:

Suppose that the random variables X and Y have IFR. Then, for any $t \geq 0$, the random variable $(\min(X,Y))_t$ has IFR.

Proof:

Fix $t \geq 0$. It is obvious that if random variables X and Y have IFR, then $\min(X,Y)$ also has IFR. Also it is easy to verify that if a non-negative random variable Z has IFR, then for any $s \geq 0$, the random variable $Z_s = (Z - s|Z > s)$ has IFR. Thus under the hypothesis of the theorem, $\min(X,Y)$ has IFR, which in turn implies that $(\min(X,Y))_t$ has IFR.

2.4 Examples

Weibull and Gompertz distribution are important life distributions which are used in reliability modelling. In this section, we provide some examples to support the theory developed in Sections 2.2 and 2.3. For the detailed study of these distributions, Marshall and Olkin (2007) may be referred.

Weibull distribution

Consider that the random variable X has Weibull distribution with parameters (α, λ) and with survival function

$$\bar{F}(x) = e^{-(\lambda x)^{\alpha}},\ x > 0,\ \lambda > 0,\ \alpha > 0.$$

The corresponding probability density function is given by

$$f(x) = \alpha\lambda^{\alpha}x^{\alpha-1}e^{-(\lambda x)^{\alpha}},\ x > 0,\ \lambda > 0,\ \alpha > 0,$$

and

$$f'(x) = \alpha\lambda^{\alpha}x^{\alpha-2}e^{-(\lambda x)^{\alpha}}\left((\alpha-1) - \alpha x^{\alpha}\lambda^{\alpha}\right).$$

Clearly, $f'(x) \leq 0$, if $\alpha \leq 1$. Similarly, let Y follows Weibull (β, μ). It may be noted that $f' \leq 0$ and $g' \leq 0 \Leftrightarrow F$ and G are concave $\Rightarrow \ln F$ and $\ln G$ are concave $\Leftrightarrow X$ and Y have DRFR. Hence, if X and Y follows Weibull(α, λ), $\alpha \leq 1$, and Weibull(β, μ), $\beta \leq 1$, then the sufficient conditions of Property 2.3.1, 2.3.2 and 2.3.4 are satisfied.

It is well known that the random variable X, which follows Weibull(α, λ), has IFR if $\alpha \geq 1$ (Barlow and Proschan (1981)). Therefore, if X and Y follows Weibull(α, λ) $\alpha \geq 1$ and Weibull (β, μ), $\beta \geq 1$, then the sufficient conditions of Property 2.3.3, 2.3.5 and 2.3.6 are satisfied.

In order to observe when $X \leq_{lr} Y$ $(Y \leq_{lr} X)$, we consider

$$\frac{g(x)}{f(x)} = \frac{\beta}{\alpha}\frac{\mu^{\beta}}{\lambda^{\alpha}}\psi(x),$$

where $\psi(x) = x^{\beta-\alpha} e^{(\lambda x)^\alpha} e^{-(\mu x)^\beta}$.

Further,

$$\begin{aligned}\psi'(x) &= x^{\beta-\alpha-1} e^{(\lambda x)^\alpha} e^{-(\mu x)^\beta} \left((\beta-\alpha) + \alpha\lambda^\alpha x^\alpha - \beta\mu^\beta x^\beta\right) \\ &= x^{\beta-\alpha-1} e^{(\lambda x)^\alpha} e^{-(\mu x)^\beta} \left(\alpha(\lambda^\alpha x^\alpha - 1) + \beta(1-\mu^\beta x^\beta)\right).\end{aligned}$$

It may be noted that if we take $\alpha = \beta,\ \lambda \geq \mu$ $(\alpha = \beta,\ \lambda \leq \mu)$, then $\psi'(x) \geq 0$ $(\psi'(x) \leq 0)$, i.e., $X \leq_{lr} Y$ $(Y \leq_{lr} X)$.

Hence, if X and Y follows Weibull(α, λ) and Weibull(β, μ) respectively such that $\alpha = \beta$, it is clear from the above arguments (cf. Theorems 2.2.1 and 2.2.2) that,

$$(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t),} Y_{(t)}),$$

and

$$(\min(X,Y))_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)}).$$

Gompertz distribution

If we consider the random variable X which has Gompertz distribution with scale parameter λ and frailiy parameter ξ, i.e., X follows Gompertz(λ, ξ). Then, the random variable X has survival function

$$\bar{F}(x) = e^{-\xi(e^{\lambda x}-1)},\ x \geq 0,\ \lambda \geq 0,\ \xi \geq 0,$$

its probability density function is

$$f(x) = \lambda\xi e^{\lambda x - \xi(e^{\lambda x}-1)},\ x \geq 0,\ \lambda \geq 0,\ \xi \geq 0,$$

and

$$f'(x) = \lambda^2 \xi e^{\lambda x - \xi(e^{\lambda x}-1)}(1 - \xi e^{\lambda x}). \tag{2.4.1}$$

On applying the Maclaurin's series to $e^{\lambda x}$ in expression (2.4.1), we have

$$f'(x) = \lambda^2 \xi e^{\lambda x - \xi(e^{\lambda x}-1)} \left(1 - \xi\left(1 + \lambda x + \frac{(\lambda x)^2}{2!} + \dots\right)\right).$$

If we choose $\xi > 1$, then $f'(x) \leq 0$. Similarly, let Y follows Gompertz(μ, η). It may be noted that

$f' \leq 0$ and $g' \leq 0 \Leftrightarrow F$ and G are concave $\Rightarrow \ln F$ and $\ln G$ are concave $\Leftrightarrow$ X and Y have DRFR.

Clearly, if X and Y follows Gompertz(λ, ξ), $\xi > 1$, and Gompertz(μ, η), $\eta > 1$, respectively, then the sufficient conditions of Property 2.3.1, 2.3.2 and 2.3.4 are satisfied.

In order to observe when $X \leq_{lr} Y$ $(Y \leq_{lr} X)$, we consider for the case when $\lambda = \mu$,

$$\begin{aligned} \frac{g(x)}{f(x)} &= \frac{\eta e^{-\eta(e^{\lambda x}-1)}}{\xi e^{-\xi(e^{\lambda x}-1)}} \\ &= \frac{\eta}{\xi} e^{(e^{\lambda x}-1)(\xi-\eta)}, \end{aligned}$$

which is increasing (decreasing) in x if $\xi \geq \eta$ $(\xi \leq \eta)$. Therefore, if we take $\lambda = \mu$, $\xi \geq \eta$ $(\lambda = \mu,\ \xi \leq \eta)$, then $X \leq_{lr} Y$ $(Y \leq_{lr} X)$.

Hence, if X and Y follows Gompertz(λ, ξ) and Gompertz(μ, η) respectively such that $\lambda = \mu$, it is clear from the above arguments (cf. Theorems 2.2.1 and 2.2.2) that,

$$(\max(X, Y))_{(t)} \leq_{lr} \max(X_{(t),} Y_{(t)}),$$

and

$$(\min(X, Y))_{(t)} \geq_{lr} \min(X_{(t)}, Y_{(t)}).$$

2.5 Conclusions

The stochastic comparison of residual life and inactivity time of series and parallel systems had been studied in the literature when the random variables are independent and identically distributed. In this chapter, such results are extended when the condition of identically distribution is omitted. By assuming that X and Y are independent, but not necessarily identically distributed and letting $X \leq_{lr} Y$, $\eta_f \leq 0$ and $\eta_g \geq 0$, (or $Y \leq_{lr} X$, $\eta_f \geq 0$ and $\eta_g \leq 0$) we proved that the parallel system of used components, i.e., $\max(X_t, Y_t)$, is better than the used parallel system, i.e., $(\max(X,Y))_t$, in the sense of likelihood ratio order. Also, by assuming X and Y are independent, but not necessarily identically distributed and letting $X \leq_{lr} Y$, (or $Y \leq_{lr} X$), we proved that, for any $t \geq 0$,

$$(\max(X,Y))_{(t)} \leq_{lr} \max(X_{(t)}, Y_{(t)});$$

and

$$\min(X_{(t)}, Y_{(t)}) \leq_{lr} (\min(X,Y))_{(t)} .$$

Also, we proved various ageing properties of used/inactive parallel/series systems and the parallel/series system of used/inactive components. The obtained results are supported by well known distributions, such as weibull and gompertz distributions.

Chapter 3

Reliability Properties under Weighing and Equilibrium Distribution

3.1 Introduction

Weighted Distribution and Equilibrium Distribution are important life distributions which are widely used in reliability modelling. By the method of ascertainment, the concept of weighted distribution has been introduced by C.R. Rao (1965). The fundamental concepts of weighted distribution and equilibrium distribution have been presented below. For a detailed study of equilibrium distribution one may refer Abouammoh et. al. [(1993),(2000)], Bhattacharjee et. al. (2000), Bon & Illayk (2002), Su & Tang (2003), Mugdadi & Ahmad (2005), Bon & Illayk (2005) and Li & Xu (2008).

Weighted Distribution:

Weighted distribution has been widely used as a tool in various practical problems in the selection of appropriate models for observed data drown without a proper frame, analysis of data relating to human populations and wild life management, investigation of human heredity, line transcend sampling and renewal theory, study of statistical ecology, albinism and reliability modelling. The properties of weighted random sample corresponding to original random sample is required to study when the observations can be recorded as a weighted random sample with some weight attached to the original random sample.

Let $f(\cdot)$ be the probability density function of original random variable X. Let $w_1(\cdot) : \mathbb{R} \to [0, \infty)$, where $\mathbb{R} = (-\infty, \infty)$ and the recovered random variable be X_{w_1} with the probability density function given by

$$f_{w_1}(x) = \frac{w_1(x) f(x)}{w_1};\ x \in \mathbb{R};$$

where $w_1 = E(w_1(X)) > 0$.

The random variable X_{w_1} is called the weighted version of X and its distribution in relation to that of X is called the weighted distribution of X with weighted function $w_1(.)$.

Equilibrium Distribution:

Equilibrium distribution which is also known as integrated tail function originated in the context of renewal processes and acting as the limiting distribution of the forward recurrence time in a renewal process. Equilibrium distribution has wide applications in insurance, financial investment, reliability, stochastic processes, repairable systems and many areas of applied probability such as renewal risk model, tail distributions of ladder heights of random walk, limiting distributions of waiting time and busy model of queueing model etc [cf. Embrechts et. al. (1997), Rolski et. al. (1999) and Asmussen (2000)].

Consider a random variable $\tilde{X}$ with probability density function

$$\tilde{f}(x) = \frac{1}{E(X)} \bar{F}(x),\ x \in S,$$

the distribution function

$$\tilde{F}(x) = \frac{1}{E(X)} \int_0^x \bar{F}(u)du, \ x \in S,$$

and the survival function

$$\bar{\tilde{F}}(x) = \frac{1}{E(X)} \int_x^\infty \bar{F}(u)du, \ x \in S.$$

Then, $\tilde{X}$ is called the equilibrium random variable of the original random variable X, and its distribution as equilibrium distribution of original random variable X.

In this chapter, we have discussed the reliability properties of mean inactivity time under weighing and reliability properties of series and parallel systems under equilibrium distribution.

3.2 Reliability Properties of Mean Inactivity Time under Weighing

Jain et. al. (1989), Nanda & Jatin (1999), Misra et. al. (2008), Barlow et. al (1981), and Bartoszewicz et. al (2006) have studied the reliability properties of weighted distributions in relation to corresponding reliability measures of parent distributions. Here, we further derive some results on preservation of mean inactivity time ordering by weighted distributions.

Let w_1, w_2 be two functions where $w_i : \mathbb{R} \to \mathbb{R}^+$, $i = 1, 2$ such that $0 < E[w_1(X)] < \infty$, $0 < E[w_2(Y)] < \infty$ and $w_1 = E[w_1(X)]$,$w_2 = E[w_2(Y)]$.

Let X_{w_1} and Y_{w_2} be the weighted versions of $X$and Y with weight functions $w_1(\cdot)$ and $w_2(\cdot)$ respectively. Then X_{w_1} and Y_{w_2} have probability density functions given by

$$f_{w_1}(x) = \frac{w_1(x)f(x)}{w_1}, \quad x \in \mathbb{R}$$

and

$$g_{w_2}(x) = \frac{w_2(x)g(x)}{w_2}, \quad x \in \mathbb{R}$$

respectively.

Let $F_1(x)\,(G_2(x)),\ x \in \mathbb{R}$ be the distribution function of $X_{w_1}(Y_{w_2})$ and let

$$\overline{F}_1(x) = 1 - F_1(x)\,(\overline{G}_2(x) = 1 - G_2(x)), \quad x \in \mathbb{R}$$

be the survival function of $X_{w_1}(Y_{w_2})$.

Let us consider a random variable X with absolutely continuous distribution function

$$F(x) = P(X \leq x),\ x \in \mathbb{R},$$

survival (reliability) function

$$\overline{F}(x) = 1 - F(x),\ x \in \mathbb{R}$$

and the probability density function $f(x)$, $x \in \mathbb{R}$, to provide definitions of notions of reliability classes, statistical dependence, stochastic orders etc.

Let the reverse failure rate (rfr) of a random variable X is given by

$$r_X(x) = \frac{f(x)}{F(x)},\ x > 0.$$

The residual life of random variable X with fixed age/time $t(t > 0)$ is

$$X_t \ = \ (X - t|\, X > t),$$

and inactivity time of random variable X with fixed age/time $s(s > 0)$ is

$$X_{(s)} \ = \ (s - X|\, X \leq \ s).$$

The conditional distribution of $X - t$ given $X > t$ and $s - X$ given $X \leq s$ are the distributions of X_t and $X_{(s)}$ respectively. The mean residual life function and mean inactivity time function when the random variable X has finite mean are defined as

$$m_X(t) = E(X_t) = \frac{\int\limits_t^{\infty} \overline{F}(u)du}{\overline{F}(t)}$$

and

$$\mu_X(s) = E(X_{(s)}) = \frac{\int\limits_{-\infty}^{s} F(u)du}{F(s)}$$

respectively.

Next, we present the following definitions which are standard in literature [cf. Shaked, M. & Shanthikumar (2007) and Ahmad & Kayid (2005)]:

Definition 3.2.1:

(a) Let $\Omega = (a, b) \subseteq \mathbb{R}$, where $-\infty \leq a < b \leq \infty$ and $h : \Omega \rightarrow \mathbb{R}^+$. The function $h(\cdot)$ is said to be log-concave (log-convex) on Ω if, $\forall x, y \in \Omega$ and $\forall \alpha \in (0, 1)$,

$$h(\alpha x + (1 - \alpha)y) \geq (\leq) (h(x))^{\alpha} (h(y))^{1-\alpha} \quad .$$

(b) Random Variable X is said to have decreasing (increasing) reversed failure rate (DRFR (IRFR)) if $F(\cdot)$ is log-concave (log-convex) on $[0, \infty)$, or equivalently if the reversed failure rate function $r_X(\cdot)$ is decreasing (increasing) on $[0, \infty)$.

(c) Random variable X is said to have decreasing (increasing) mean residual life (DMRL (IMRL)) if $\int\limits_x^{\infty} \overline{F}(t)\, dt$ is log-concave (log-convex) on $[0, \infty)$.

(d) Random Variable X is said to have increasing (decreasing) mean inactivity time (IMIT (DMIT)) if $\int\limits_0^x F(u)du$ is log-concave (log-convex) on $[0, \infty)$ or equivalently if the mean inactivity time function $\mu_X(\cdot)$ is decreasing (increasing) on $[0, \infty)$.

(e) Random variable X is said to be smaller than Y in the down shifted likelihood ratio order (written as $X \leq_{lr\downarrow} Y$) if $\frac{g(t+x)}{f(t)}$ is increasing in $t \geq 0$ for all $x \geq 0$, where f and g denote the density functions of X and Y respectively.

(f) Random Variable X is said to be smaller than random variable Yin the reversed failure rate ordering (written as $X \leq_{rfr} Y$) if

$$G(s)F(t) \leq G(t)F(s),$$

whenever $-\infty < s < t < \infty$, or equivalently, if $r_X(x) \leq r_Y(x)$, $\forall\, x \in (0, \infty)$.

(g) Random variable X is said to be smaller then random variable Y in the mean inactivity time ordering ($X \leq_{mit} Y$) if

$$\left[\int_{-\infty}^{s} G(u)du\right]\left[\int_{-\infty}^{t} F(u)du\right] \leq \left[\int_{-\infty}^{s} F(u)du\right]\left[\int_{-\infty}^{t} G(u)du\right]$$

whenever $-\infty < s < t < \infty$, or equivalently, $\mu_X(t) \geq \mu_Y(t)$, $\forall\, t \in (0, \infty)$.

3.2.1 Ageing Properties and Stochastic Dependence

Let X and Y be non-negative random variables.

Lemma 3.2.1.1 (Misra et. al. (2008), Theorem 2.3(a)).

If X has DRFR, $w_1(\cdot)$ is decreasing on $[0, \infty)$ and log-concave on $[0, \infty)$ then X_{w_1} has DRFR.

Lemma 3.2.1.2 (Misra et. al. (2008), Theorem 3.2(c)).

If $X \leq_{rfr} Y$, $w_1(t)$ is decreasing on $[0, \infty)$ and $w_2(t)/w_1(t)$ is increasing on $[0, \infty)$, then $X_{w_1} \leq_{rfr} Y_{w_2}$.

Here we obtain the conditions on weight function $w(\cdot)$ under which such preservation of property of IMIT under weighing is possible.

Theorem 3.2.1.1:

X_{w_1} has IMIT if X has IMIT and $A_1(\cdot)$ is decreasing and log-convex on $[0, \infty)$ where $A_1(x) = E[\, w_1(x) \mid X \leq x\,]$.

Proof:

Let Z_1 and Z_2 be random variables having probability density function

$$f_{Z_1}(x) = \frac{F(x)}{\int\limits_0^\infty F(u)du},$$

and

$$f_{Z_2}(x) = \frac{F_{w_1}(x)}{\int\limits_0^\infty F_{w_1}(u)du},$$

respectively.

Z_1 has weighted version Z_2 with weight function $A_1(\cdot)$. The random variable Z_1 has DRFR since X has IMIT. Under the premise of the contention, using Lemma 3.2.1.1, it follows that random variable Z_2 has DRFR which follows that X_{w_1} has IMIT.

Consider

$$F_1(x) = \frac{A_1(x)F(x)}{w_1};$$

where

$$A_1(x) = E[w_1(X)|X \leq x].$$

From above, it is clear that X_{w_1} has DRFR (IRFR), if X has DRFR (IRFR).

The following theorem provides conditions on the weight function $w_1(\cdot)$ and the mean inactivity time function $\mu_X(t)$, under which a random variable X having IMIT, yield a weighted version which is DRFR (and hence IMIT).

Theorem 3.2.1.2:

X_{w_1} has DRFR if X has IMIT, $w_1(\cdot)$ is decreasing and log-concave on $[0, \infty)$ and the mean inactivity time function $\mu_X(t)$ is log-convex on $[0, \infty)$.

Proof:

In view of Lemma 3.2.1.1, it is enough to show that $X \leq_{rfr} Z_1$ where Z_1 has probability density function $f(t-\theta)$, i.e., X has DRFR and mean inactivity time function $\mu_{Z_1}(t) = \mu_X(t-\theta)$.

Consider

$$\begin{aligned} r_{Z_1}(t) - r_X(t) &= \tfrac{1-\mu'_{Z_1}(t)}{\mu_{Z_1}(t)} - \tfrac{1-\mu'_X(t)}{\mu_X(t)} \\ &= \left[\tfrac{\mu'_X(t)}{\mu_X(t)} - \tfrac{\mu'_X(t-\theta)}{\mu_X(t-\theta)}\right] + \left[\tfrac{1}{\mu_X(t-\theta)} - \tfrac{1}{\mu_X(t)}\right] \\ &\geq 0, \end{aligned}$$

since $\mu_X(t)$ is log-convex on $[0, \infty)$ and X has IMIT. Therefore, $X \leq_{rfr} Z_1$ and hence X has DRFR. Now contention follows using Lemma 3.2.1.1.

Theorem 3.2.1.3:

$X_{w_1} \leq_{mit} Y_{w_2}$ *if* $X \leq_{mit} Y$, $A_1(\cdot)$ *is decreasing and* $A_2(\cdot)/A_1(\cdot)$ *is increasing on* $[0, \infty)$, *where* $A_1(x) = E[w_1(X) \mid X \leq x]$ *and* $A_2(x) = E[w_2(Y) \mid Y \leq x]$.

Proof:

We have

$$F_{w_1}(x) = \frac{A_1(x)\,F(x)}{w_1}$$

and

$$G_{w_2}(x) = \frac{A_2(x)\,G(x)}{w_2}.$$

Let X^* and Y^* be random variables with probability density functions given by

$$f_{X^*}(x) = \frac{F(x)}{\int\limits_0^\infty F(u)\,du}$$

and

$$f_{Y^*}(x) = \frac{G(x)}{\int\limits_0^\infty G(u)\,du}$$

respectively.

Now $X \leq_{mit} Y$ implies that $X^* \leq_{rfr} Y^*$. Let $X^*_{A_1}$ and $Y^*_{A_2}$ be weighted version of X^* and Y^* with weight functions $A_1(\cdot)$ and $A_2(\cdot)$ respectively. Hence by Lemma 3.2.1.2,

$$X^* \leq_{rfr} Y^*.$$

Also, $X^* \leq_{rfr} Y^*$, if and only if $X_{w_1} \leq_{mit} Y_{w_2}$.

Proposition:

$A_1(\cdot)$ is increasing (decreasing) if $w_1(\cdot)$ is increasing (decreasing).

Proof:

Consider

$$F(x)\, w(x) \;-\; \int_0^x w(t)\, f(t)\, dt \;\geq (\leq)\, 0$$

if and only if $A_1'(x) \;\geq (\leq)\; 0$.

It may be noted that $w_1(\cdot)$ is increasing (decreasing) implies that $A_1(\cdot)$ is increasing (decreasing). Hence, the result follows by using the above argument.

Corollary:

If $X \leq_{mit} Y$, $w_1(\cdot)$ is decreasing and $w_2(\cdot)$ is increasing on $[0, \infty)$, then $X_{w_1} \leq_{mit} Y_{w_2}$.

Example:

If $f(x) = e^{-x}$ and $w_1 = x^{\alpha_1 - 1}$ where $\alpha_1 > 1$, $w_2 = x^{\alpha_2 - 1}$ where $\alpha_2 < 1$ then $X_{w_1} \leq_{mit} Y_{w_2}$.

Theorem 3.2.1.4:

If $f(\cdot)$ is log-convex and $w(\cdot)$is increasing on $[0, \infty)$, then $X \leq_{lr\downarrow} X_w$.

Proof:

For fixed $a > 0$, consider

$$\frac{f_w(x+a)}{f(x)} = \frac{w(x+a)\,f(x+a)}{E[\,w(X+a)]\,f(x)}$$

$$= \frac{1}{E[\,w(X+a)]} \cdot w(x+a) \cdot \frac{f(x+a)}{f(x)},$$

which is an increasing function, since $f(\cdot)$ is log-convex and $w(\cdot)$ is increasing.

Hence $X \ \leq_{lr\downarrow} X_w$.

Theorem 3.2.1.5:

If $X \ \leq_{lr\downarrow} Y$ and $w(\cdot)$ is log-convex, then $X_w \ \leq_{lr\downarrow} Y_w$.

Proof:

For fixed $a > 0$, consider

$$\frac{g_w(x+a)}{f_w(x)} = \frac{w(x+a)\,g(x+a)}{E[\,w(Y+a)]} \cdot \frac{E[w(x)]}{w(x)\,f(x)}$$

$$= \frac{E[w(x)]}{E[\,w(Y+a)]} \cdot \frac{w(x+a)}{w(x)} \cdot \frac{g(x+a)}{f(x)} \quad ,$$

which is an increasing function, since $w(\cdot)$ is log-convex and $X \ \leq_{lr\downarrow} Y$.

Hence $X_w \ \leq_{lr\downarrow} Y_w$.

3.3 Reliability Properties of Series and Parallel Systems under Equilibrium Distribution

Let X and Y be two statistically independent random variables with an absolutely continuous distribution function $F(\cdot)$ and $G(\cdot)$, survival function $\overline{F}(\cdot) = 1 - F(\cdot)$

and $\overline{G}(\cdot) = 1 - G(\cdot)$, probability density function $f(\cdot)$ and $g(\cdot)$, the reversed hazard function $\lambda_F(\cdot)$ and $\lambda_G(\cdot)$ and the eta function $\eta_F(\cdot)$ and $\eta_G(\cdot)$, respectively.

Here $\lambda_F(x) = \frac{f(x)}{F(x)}$ and $\eta_F(x) = -\frac{f'(x)}{f(x)},\ x \in \mathbb{R}$, where $\mathbb{R} = (-\infty, \infty)$.
Suppose that

$$\{x \in \mathbb{R} : f(x) > 0\} = (0, \infty) = S \text{ (say)},$$
$$\{x \in \mathbb{R} : g(x) > 0\} = S.$$

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ denote the n independently and identically distributed (i.i.d.) copies of random variables X and Y, respectively. A series (parallel) system comprising of these n i.i.d components functions if and only if all (at least one) of its component function(s). Clearly, $\min\limits_{1\le i\le n} X_i$ and $\max\limits_{1\le i\le n} X_i$ are respectively the lifetime of series and parallel systems having the components $X_1, X_2, \cdots, X_n$, where $\min\limits_{1\le i\le n} X_i$ $\left(\max\limits_{1\le i\le n} X_i\right)$ denote minimum (maximum) of $X_1, X_2, \ldots, X_n$.

The residual life of X with time $t \ge 0$ is given by

$$X_t = (X - t \,|\, X > t),\ t \ge 0,$$

and inactivity time of X at time $t \ge 0$ is given by

$$X_{(t)} = (t - X \,|\, X \le t),\ t \ge 0.$$

Next, we include below some definitions of stochastic orders which are standard in the literature [cf. Muller & Stoyan (2002) and Shaked & Shanthikumar (2007)].

Definition 3.3.1:

The random variable X is said to be smaller than random variable Y in the

(a) likelihood ratio (lr) ordering ($X \le_{lr} Y$) if

$$\frac{g(x)}{f(x)} \text{ increases in } x \in S;$$

(b) hazard rate (hr) ordering ($X \leq_{hr} Y$) if

$$\frac{\bar{G}(x)}{\bar{F}(x)} \text{ increases in } x \in S;$$

(c) reversed hazard rate (rh) ordering ($X \leq_{rh} Y$) if

$$\frac{G(x)}{F(x)} \text{ increases in } x \in S;$$

(d) usual stochastic (st) ordering ($X \leq_{st} Y$) if

$$\overline{F}(x) \leq \overline{G}(x), \text{ for all } x \in \mathbb{R};$$

(e) mean residual life (mrl) ordering ($X \leq_{mrl} Y$) if

$$E(X_t) \leq E(Y_t), \quad \text{for all } t.$$

(f) harmonic mean residual life (hmrl) ordering

($X \leq_{hmrl} Y$) if

$$\left(\frac{1}{x}\int_0^x \frac{1}{E(X_t)}dt\right)^{-1} \leq \left(\frac{1}{x}\int_0^x \frac{1}{E(Y_t)}dt\right)^{-1}, \ \forall x \geq 0.$$

Various researchers provide the characterization of stochastic orders in terms of ordering of equilibrium distributions.

Whitt (1985) proved that

$$X \leq_{hr(mrl,hmrl)} Y \Leftrightarrow \tilde{X} \leq_{lr(hr,st)} \tilde{Y}.$$

Bon & Illayk (2005) proved that if X_1 and X_2 are independent DMRL random variables, then

$$\min(\tilde{X}_1, \tilde{X}_2) \leq_{lr} \widetilde{\min(X_1, X_2)}.$$

Li and Xu (2008) proves that $X \leq_{rh} Y \Rightarrow \tilde{X} \leq_{st} \tilde{Y}$; and shows that reverse implication may not be true. Additionally, it has also been proved that if $\tilde{X} \leq_{rh} \tilde{Y}$, then

$$\widetilde{\min_{1\leq i\leq n} X_i} \leq_{rh} \widetilde{\min_{1\leq i\leq n} Y_i}.$$

In subsection 3.3.1, we establish some reliability properties of series and parallel systems under the equilibrium distribution. The likelihood ratio ordering and the log-concavity properties of series and parallel system (s) having independently and identically distributed (i.i.d.) components under the condition of equilibrium have been studied.

3.3.1 Results on Reliability Properties

The following result provides the preservation of the likelihood ratio order for the formation of series system under equilibrium:

Theorem 3.3.1.1:

Let $X_1, X_2, \ldots, X_n$; $Y_1, Y_2, \ldots, Y_n$ be i.i.d. copies of X and Y respectively. If $\tilde{X} \leq_{lr} \tilde{Y}$ then

$$\widetilde{\min_{1\leq i\leq n}} X_i \leq_{lr} \widetilde{\min_{1\leq i\leq n}} Y_i.$$

Proof:

It is sufficient to show the result for $n = 2$, as the result for any n will follow similarly. The random variable $\widetilde{\min(X_1, X_2)}$ has the survival function

$$\begin{aligned}\bar{H}_{1,S}(x) &= P\left(\widetilde{\min(X_1, X_2)} > x\right) \\ &= \frac{1}{E\left(\widetilde{\min(X_1, X_2)}\right)} \int_x^\infty \bar{F}^2(u)du,\ x \in S,\end{aligned}$$

and the probability density function

$$h_{1,S}(x) = \frac{\bar{F}^2(x)}{E\left(\widetilde{\min(X_1, X_2)}\right)},\ x \in S. \tag{3.3.1}$$

Similarly, the random variable $\widetilde{\min(Y_1, Y_2)}$ has the survival function

$$\bar{H}_{2,S}(x) = \frac{1}{E\left(\widetilde{\min(Y_1, Y_2)}\right)} \int_x^\infty \bar{G}^2(u)du,\ x \in S,$$

and the probability density function

$$h_{2,S}(x) = \frac{\bar{G}^2(x)}{E\left(\widetilde{\min(Y_1, Y_2)}\right)}, \ x \in S.$$

For fixed $x > 0$, consider

$$\frac{h_2(x)}{h_1(x)} = \frac{E\left(\widetilde{\min(X_1, X_2)}\right)}{E\left(\widetilde{\min(Y_1, Y_2)}\right)} \frac{\bar{G}^2(x)}{\bar{F}^2(x)},$$

which is clearly increasing in x, if $X \leq_{hr} Y$.
Now the result follows by observing that $X \leq_{hr} Y$ if and only if $\tilde{X} \leq_{lr} \tilde{Y}$.

The following result provides the conditions for which the parallel system have likelihood ratio order under equilibrium:

Theorem 3.3.1.2:

Let $X_1, X_2, \ldots, X_n$; $Y_1, Y_2, \ldots, Y_n$ be i.i.d. copies of X and Y respectively. If $X \leq_{rfr} Y$ and $g(x)G(x) \leq f(x)F(x), \ x \in \mathbb{R}$, then

$$\widetilde{\max_{1\leq i\leq n}} X_i \leq_{lr} \widetilde{\max_{1\leq i\leq n}} Y_i.$$

Proof:

It is sufficient to show the result for $n = 2$, as the result for any n will follow similarly. The random variable $\widetilde{\max(X_1, X_2)}$ has the survival function

$$\begin{aligned}\bar{H}_{1,P}(x) &= P\left(\widetilde{\max(X_1, X_2)} > x\right) \\ &= \frac{1}{E\left(\widetilde{\max(X_1, X_2)}\right)} \int_x^{\infty} \left(1 - F^2(u)\right) du, \ x \in S,\end{aligned}$$

and the probability density function

$$h_{1,P}(x) = \frac{1 - F^2(x)}{E\left(\widetilde{\max(X_1, X_2)}\right)}, \ x \in S.$$

Similarly, the random variable $\widetilde{\max(Y_1, Y_2)}$ has the survival function

$$\begin{aligned}\bar{H}_{2,P}(x) &= P\left(\widetilde{\max(Y_1, Y_2)} > x\right) \\ &= \frac{1}{E\left(\widetilde{\max(Y_1, Y_2)}\right)} \int_x^\infty \left(1 - G^2(u)\right) du, \ x \in S,\end{aligned}$$

and the probability density function

$$h_{2,P}(x) = \frac{1 - G^2(x)}{E\left(\widetilde{\max(Y_1, Y_2)}\right)}, \ x \in S.$$

For fixed $x > 0$, consider

$$\frac{h_{2,P}(x)}{h_{1,P}(x)} = C\psi(x),$$

where

$$C = \frac{E\left(\widetilde{\max(X_1, X_2)}\right)}{E\left(\widetilde{\max(Y_1, Y_2)}\right)} \text{ and } \psi(x) = \frac{1 - G^2(x)}{1 - F^2(x)}.$$

It is easy to verify that for any fixed $x \in \mathbb{R}$,

$$\begin{aligned}\psi'(x) &= \frac{2}{\left(1 - F^2(x)\right)^2} \left(\left(1 - G^2(x)\right) f(x)F(x) \right. \\ &\qquad \left. - \left(1 - F^2(x)\right) g(x)G(x)\right) \\ &= \frac{2}{\left(1 - F^2(x)\right)^2} \left(f(x)F(x) - g(x)G(x) \right. \\ &\qquad \left. + G^2(x)F^2(x) \left(\lambda_G(x) - \lambda_F(x)\right)\right) \\ &\geq 0,\end{aligned}$$

since $X \leq_{rfr} Y$ and $g(x)G(x) \leq f(x)F(x), \ x \in \mathbb{R}$.

Hence the result follows.

In order to study the log-concavity of series and parallel systems under equilibrium, here we present some of the ageing notions:

Definition 3.3.1.1:

The random variable X is said be

(a) log-concave if $\ln(f(\cdot))$ is concave on S;

(b) increasing hazard rate (IHR) if $\bar{F}(\cdot)$ is log-concave on S.

It is well known that

$$X \text{ is log-concave on } S \Rightarrow X \text{ is IHR on } S.$$

In the following theorem we provide the conditions under which a series system with i.i.d. components have log-concave life-time.

Theorem 3.3.1.3:

Let $X_1, X_2, \ldots, X_n$ be i.i.d. copies of X. Then X is IHR if and only if $\widetilde{\min_{1\leq i\leq n} X_i}$ is log-concave.

Proof:

It is sufficient to show the result for $n = 2$, as the result for any n will follow similarly. The random variable $\widetilde{\min(X_1, X_2)}$ has the probability density function (3.3.1).

Therefore,

$$\ln(h_{1,S}(x)) = 2\ln\left(\bar{F}(x)\right) - \ln\left(E\left(\widetilde{\min(X_1, X_2)}\right)\right)$$

is concave in S if and only if $\ln\left(\bar{F}(x)\right)$ is concave in S. Hence the result.

In the following theorem we provide the conditions under which a parallel system with i.i.d. components have log-concave life-time.

Theorem 3.3.1.4:

Let $X_1, X_2, \ldots, X_n$ be i.i.d. copies of X. If $\eta_F(\cdot) \leq 0$ then, $\widetilde{\max_{1\leq i\leq n} X_i}$ is log-concave.

Proof:

The random variable $\widetilde{\max_{1\leq i\leq n} X_i}$ has the probability density function

$$h_P(x) = \frac{1 - F^n(x)}{E\left(\widetilde{\max_{1\leq i\leq n} X_i}\right)}, \; x \in S.$$

Consider

$$\begin{aligned}\psi(x) &= \ln\left(h_P(x)\right) \\ &= \ln\left(1 - F^n(x)\right) - \ln\left(E\left(\widetilde{\max_{1\leq i\leq n} X_i}\right)\right).\end{aligned}$$

Then

$$\psi'(x) = -\frac{nf(x)F^{n-1}(x)}{1 - F^n(x)},$$

and

$$\begin{aligned}\psi''(x) = -&\frac{1}{\left(1 - F^n(x)\right)^2}\Big(n\left(1 - F^n(x)\right)\big((n-1)f^2(x) \\ &F^{n-2}(x) - \eta_F(x)f(x)F^{n-1}(x)\big) - \left(nf(x)F^{n-1}(x)\right)^2\Big) \\ &\leq 0, \text{ as } \eta_F(x) \leq 0.\end{aligned}$$

Hence $\widetilde{\max_{1\leq i\leq n} X_i}$ is log-concave.

3.4 Conclusions

In context of reliability and life testing problems, reliability properties of mean inactivity time under weighting and reliability properties of series and parallel systems having independently and identically distributed (i.i.d.) components under the equilibrium distribution have been studied. The conditions of stochastic comparison of weighted distributions in terms of mean inactivity time and shifted likelihood ratio order has been obtained. Further, it has been found that the likelihood ratio order is preserved between equilibrium random variable under the formation of series system. Some conditions are provided under which a parallel system under equilibrium distribution have likelihood ratio order. If the lifetimes of i.i.d. components have IHR then a series system composed of these i.i.d. components have log-concave life-time. It has also been established that if the eta function of i.i.d. components is negative then a parallel system composed of these i.i.d. components have log-concave life-time.

Chapter 4

Preservation Properties of Moment Generating Function & Laplace Transform ordering of Residual Life and Inactivity Time

4.1 Introduction

Stochastic comparison of probability distributions plays a fundamental role in the probability theory, the decision theory and the related disciplines. It also finds various applications in the field of reliability theory, survival analysis, actuarial sciences etc. The theory of stochastic orders provides various tools for the stochastic comparison of probability distributions. For a detailed study on the theory of stochastic orders, one may refer Muller & Stoyan (2002) and Shaked & Shanthikumar (2007). Some of these orders are moment generating function (or exponential) order and laplace transform order and their residual life and inactivity time (or reversed residual life) [cf. Ahmed & Kayid (2004), Elbatal (2007), Kayid (2011) and Kayid & Alamoudi (2013)].

Throughout the chapter, we use the terms increasing and decreasing in place of non-decreasing and non-increasing, respectively. Let X be a non-negative random variable with an absolutely continuous distribution function $F_X(\cdot)$, the survival function $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$, the probability density function $f_X(\cdot)$, the moment generating function $\psi_X(\cdot)$ and the Laplace-Stieltjes transform $L_X(\cdot)$, respectively; here $\psi_X(s) = E\left(e^{sX}\right),\ s > 0$, and $L_X(s) = E\left(e^{-sX}\right),\ s > 0$.

Let the residual life and inactivity time (or reversed residual life) of X with time/age $t \in (0, l_X)$, such that $l_X = \sup\{t : F_X(t) < 1\}$, be

$$X_t = (X - t | X > t) \text{ and } X_{(t)} = (t - X | X \leq t),$$

respectively.

We include below some definitions of stochastic orders which are standard in the literature (cf. Marshall & Olkin (1979), Barlow & Proschan (1981), Belzunce et al (1999), Muller & Stoyan (2002), Ahmed & Kayid (2004), Elbatal (2007), Shaked & Shanthikumar (2007), Wang & Ma (2009), Kayid (2011) and Kayid & Alamoudi (2013)).

Definition 4.1.1:

The random variable X is said to be smaller than random variable Y in the

(a) Moment generating function (mg) order (or exponential (exp) order) (written as $X \leq_{mg} Y$) if $\psi_X(s) \leq \psi_Y(s),\ \forall\ s > 0$;

(b) Laplace transforms (Lt) order (written as $X \leq_{Lt} Y$) if $L_X(s) \geq L_Y(s),\ \forall\ s > 0$;

(c) Moment generating function of residual life (mg-rl) order (written as $X \leq_{mg-rl} Y$) if $X_t \leq_{mg} Y_t,\ \forall\ t$, or equivalently $\frac{\int_t^\infty e^{su}\bar{F}_X(u)du}{\int_t^\infty e^{su}\bar{F}_Y(u)du}$ is decreasing in $t \in (0, l_X) \cap (0, l_Y),\ \forall\ s > 0$;

(d) Moment generating function of inactivity time (mg-it) order (written as $X \leq_{mg-it} Y$) if $X_{(t)} \geq_{mg} Y_{(t)}$, $\forall\ t$, or equivalently $\frac{\int_0^t e^{-su} F_X(u)du}{\int_0^t e^{-su} F_Y(u)du}$ is decreasing in $t \in (0, l_X) \cap (0, l_Y)$, $\forall\ s > 0$;

(e) Laplace transforms of residual life order (Lt-rl) order (written as $X \leq_{Lt-rl} Y$) if $X_t \leq_{Lt} Y_t$, $\forall\ t$, or equivalently $\frac{\int_t^\infty e^{-su} \bar{F}_X(u)du}{\int_t^\infty e^{-su} \bar{F}_Y(u)du}$ is decreasing in $t \in (0, l_X) \cap (0, l_Y)$, $\forall\ s > 0$;

(f) Laplace transforms of inactivity time (Lt-it) order (written as $X \leq_{Lt-it} Y$) if $X_{(t)} \geq_{Lt} Y_{(t)}$, $\forall\ t$, or equivalently $\frac{\int_0^t e^{su} F_X(u)du}{\int_0^t e^{su} F_Y(u)du}$ is decreasing in $t \in (0, l_X) \cap (0, l_Y)$, $\forall\ s > 0$.

Definition 4.1.2:

A function $f(x)$ is said to be a Polya function of order 2 (PF_2) in $-\infty < x < \infty$, if

(a) $f(x) \geq 0$ for $-\infty < x < \infty$, and

(b) $\begin{vmatrix} f(x_1 - y_1) & f(x_1 - y_2) \\ f(x_2 - y_1) & f(x_2 - y_2) \end{vmatrix} \geq 0$ for all $-\infty < x_1 < x_2 < \infty$ and $-\infty < y_1 < y_2 < \infty$, or equivalently, $\log(g(x))$ is concave on $(-\infty, \infty)$.

Belzunce et al (1999), Ahmed & Kayid (2004) and Elbatal (2007) studied several preservation properties of the laplace transform ordering of residual lives/inactivity times under the reliability operations of convolutions, mixtures and weak convergence. Further, Kayid (2011) and Kayid & Alamoudi (2013) established the preservation properties of the moment generating function ordering of residual lives/inactivity times under the reliability operations of convolutions and mixtures.

In this chapter, we study the concept of laplace transform ordering of residual life/inactivity time and the moment generating function ordering of residual life/inactivity time. The preservation properties of these ordering have been studied under the reliability operations of convolution. These results are in addition to the existing of Ahmed & Kayid (2004) and Kayid (2011).

4.2 Preservation Properties

In reliability theory, studying preservation properties of an stochastic order under the reliability operations such as convolution, mixture, transformations etc. is of much importance.

4.2.1 MGF Ordering of Residual life and Inactivity Time

Let X_1, X_2 and Y be non-negative random variables, such that Y is independent of X_1 as well as X_2, such that Y has the density function g. Then the following theorem provides the conditions for preservation of *mg-rl* order under convolution.

Theorem 4.2.1:

If $X_1 \leq_{mg-rl} X_2$ and g is log-concave then

$$X_1 - Y \leq_{mg-rl} X_2 - Y.$$

Proof:

It may be noted that for fixed t and $i = 1, 2$

$$\begin{aligned}\int_t^{\infty} e^{su} P(X_i - Y \geq u)\, du &= e^{st} \int_0^{\infty} e^{sx} P(X_i - Y \geq x + t)\, dx \\ &= e^{st} \int_0^{\infty} e^{sx} \left(\int_t^{\infty} P(X_i \geq x + u)\, g(u - t)\, du \right) dx.\end{aligned}$$

Now, in view of definition 4.1.1(c), it is enough to show that for all $0 \leq t_1 \leq t_2$ and $x > 0$,

$$\frac{\int_0^\infty \int_{t_1}^\infty e^{sx} P(X_1 \geq x+u)\, g(u-t_1)\, du\, dx}{\int_0^\infty \int_{t_1}^\infty e^{sx} P(X_2 \geq x+u)\, g(u-t_1)\, du\, dx} \geq \frac{\int_0^\infty \int_{t_2}^\infty e^{sx} P(X_1 \geq x+u)\, g(u-t_2)\, du\, dx}{\int_0^\infty \int_{t_2}^\infty e^{sx} P(X_2 \geq x+u)\, g(u-t_2)\, du\, dx}.$$

Since $Y$is non-negative, therefore, $g(u-t) = 0$ when $u < t$. Hence the above inequality is equivalent to

$$\frac{\int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1}(x+u)\, g(u-t_1)\, du\, dx}{\int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2}(x+u)\, g(u-t_1)\, du\, dx} \geq \frac{\int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1}(x+u)\, g(u-t_2)\, du\, dx}{\int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2}(x+u)\, g(u-t_2)\, du\, dx},$$

or equivalently,

$$\begin{vmatrix} \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2}(x+u)\, g(u-t_2)\, du\, dx & \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1}(x+u)\, g(u-t_2)\, du\, dx \\ \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_2}(x+u)\, g(u-t_1)\, du\, dx & \int_0^\infty \int_{-\infty}^\infty e^{sx} \bar{F}_{X_1}(x+u)\, g(u-t_1)\, du\, dx \end{vmatrix} \geq 0. \tag{4.2.1}$$

Applying the basic composition formula (cf. Karlin (1968), p-17), the left side of equation (4.2.4) is

$$\iint_{u_1<u_2} \begin{vmatrix} g(u_1-t_2) & g(u_2-t_2) \\ g(u_1-t_1) & g(u_2-t_1) \end{vmatrix} \begin{vmatrix} \int_0^\infty e^{sx} \bar{F}_{X_2}(x+u_1)\, dx & \int_0^\infty e^{sx} \bar{F}_{X_1}(x+u_1)\, dx \\ \int_0^\infty e^{sx} \bar{F}_{X_2}(x+u_2)\, dx & \int_0^\infty e^{sx} \bar{F}_{X_1}(x+u_2)\, dx \end{vmatrix} du_1\, du_2.$$

It may be noted that since $g(\cdot)$ is log-concave, the first determinant is non-positive and the second determinant is non-positive as $X_1 \leq_{mg-rl} X_2$. Hence, the result follows.

Let X_1, X_2 and Y be non-negative random variables, such that Y is independent of X_1 as well as X_2, such that Y has the density function g. Then the following theorem provides the conditions for preservation of *mg-it* order under convolution.

Theorem 4.2.2:

If $X_1 \leq_{mg-it} X_2$ and g is log-concave then

$$X_1 - Y \leq_{mg-it} X_2 - Y.$$

Proof:

It may be noted that for fixed t and $i = 1, 2$

$$\begin{aligned}\int_{-\infty}^{t} e^{-su}\, P\left(X_i - Y \leq u\right) du &= e^{-st}\int_{0}^{\infty} e^{sy}\, P\left(X_i - Y \leq\ t - y\right)\, dy \\ &= e^{-st}\int_{0}^{\infty} e^{sy}\left(\int_{t}^{\infty} P\left(X_i \leq\ u - y\right)\, g(u-t)\, du\right) dy.\end{aligned}$$

Now, in view of definition 4.1.1(d), it is enough to show that for all $0 \leq t_1 \leq t_2$ and $y > 0$,

$$\frac{\int_0^\infty\int_{t_1}^\infty e^{sy}\, P\left(X_1 \leq\ u - y\right)\, g(u - t_1)\, du\, dy}{\int_0^\infty\int_{t_1}^\infty e^{sy}\, P\left(X_2 \leq\ u - y\right)\, g(u - t_1)\, du\, dy} \geq \frac{\int_0^\infty\int_{t_2}^\infty e^{sy}\, P\left(X_1 \leq\ u - y\right)\, g(u - t_2)\, du\, dy}{\int_0^\infty\int_{t_2}^\infty e^{sy}\, P\left(X_2 \leq\ u - y\right)\, g(u - t_2)\, du\, dy}.$$

Since Y is non-negative, therefore, $g(u-t) = 0$ when $u < t$. Hence the above inequality is equivalent to

$$\frac{\int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_1}\left(u - y\right)\, g(u - t_1)\, du\, dy}{\int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_2}\left(u - y\right)\, g(u - t_1)\, du\, dy} \geq \frac{\int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_1}\left(u - y\right)\, g(u - t_2)\, du\, dy}{\int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_2}\left(u - y\right)\, g(u - t_2)\, du\, dy},$$

or equivalently,

$$\begin{vmatrix} \int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_2}\left(u - y\right)\, g(u - t_2)\, du\, dy & \int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_1}\left(u - y\right)\, g(u - t_2)\, du\, dy \\ \int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_2}\left(u - y\right)\, g(u - t_1)\, du\, dy & \int_0^\infty\int_{-\infty}^\infty e^{sy}\, F_{X_1}\left(u - y\right)\, g(u - t_1)\, du\, dy \end{vmatrix} \geq 0. \tag{4.2.2}$$

Applying the basic composition formula (cf. Karlin (1968), p-17), the left side of equation (4.2.2) is

$$\int\int_{u_1<u_2} \begin{vmatrix} g(u_1-t_2) & g(u_2-t_2) \\ g(u_1-t_1) & g(u_2-t_1) \end{vmatrix} \begin{vmatrix} \int_0^\infty e^{sy}\, F_{X_2}\,(u_1-y)\,dy & \int_0^\infty e^{sy}\, F_{X_1}\,(u_1-y)\,dy \\ \int_0^\infty e^{sy}\, F_{X_2}\,(u_2-y)\,dy & \int_0^\infty e^{sy}\, F_{X_1}\,(u_2-y)\,dy \end{vmatrix} du_1 du_2.$$

It may be noted that since $g\,(\cdot)$ is log-concave, the first determinant is non-positive and the second determinant is non-positive as $X_1 \leq_{mg-it} X_2$. Hence, the result follows.

4.2.2 Laplace Transform Ordering of Residual Life Time and Inactivity Time

Let X_1, X_2 and Y be non-negative random variables, such that Y is independent of X_1 as well as X_2, such that Y has the density function g. Then the following theorem provides the conditions for preservation of *Lt-rl* order under convolution.

Theorem 4.2.3:

If $X_1 \leq_{Lt-rl} X_2$ and g is log-concave then

$$X_1 - Y \leq_{Lt-rl} X_2 - Y.$$

Proof:

It may be noted that for fixed t and $i = 1, 2$

$$\begin{aligned} \int_t^\infty e^{-su}\, P\,(X_i - Y \geq u)\, du &= e^{-st} \int_0^\infty e^{-sx}\, P\,(X_i - Y \geq x + t)\; dx \\ &= e^{-st} \int_0^\infty e^{-sx} \left(\int_t^\infty P\,(X_i \geq\; x + u)\; g(u-t)\, du \right) dx. \end{aligned}$$

Using definition 4.1.1(e), it is enough to show that for all $0 \le t_1 \le t_2$ and $x > 0$,

$$\frac{\int_0^\infty \int_{t_1}^\infty e^{-sx} P(X_1 \ge x+u)\, g(u-t_1)\, du\, dx}{\int_0^\infty \int_{t_1}^\infty e^{-sx} P(X_2 \ge x+u)\, g(u-t_1)\, du\, dx} \ge \frac{\int_0^\infty \int_{t_2}^\infty e^{-sx} P(X_1 \ge x+u)\, g(u-t_2)\, du\, dx}{\int_0^\infty \int_{t_2}^\infty e^{-sx} P(X_2 \ge x+u)\, g(u-t_2)\, du\, dx}.$$

Since Y is non-negative, therefore, $g(u-t) = 0$ when $u < t$, hence the above inequality is equivalent to

$$\frac{\int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_1}(x+u)\, g(u-t_1)\, du\, dx}{\int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_2}(x+u)\, g(u-t_1)\, du\, dx} \ge \frac{\int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_1}(x+u)\, g(u-t_2)\, du\, dx}{\int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_2}(x+u)\, g(u-t_2)\, du\, dx},$$

or equivalently,

$$\begin{vmatrix} \int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_2}(x+u)\, g(u-t_2)\, du\, dx & \int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_1}(x+u)\, g(u-t_2)\, du\, dx \\ \int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_2}(x+u)\, g(u-t_1)\, du\, dx & \int_0^\infty \int_{-\infty}^\infty e^{-sx} \bar{F}_{X_1}(x+u)\, g(u-t_1)\, du\, dx \end{vmatrix} \ge 0. \tag{4.2.3}$$

Applying the basic composition formula (cf. Karlin (1968), p-17), the left side of equation (4.2.3) is

$$\iint_{u_1<u_2} \begin{vmatrix} g(u_1-t_2) & g(u_2-t_2) \\ g(u_1-t_1) & g(u_2-t_1) \end{vmatrix} \begin{vmatrix} \int_0^\infty e^{-sx} \bar{F}_{X_2}(x+u_1)\, dx & \int_0^\infty e^{-sx} \bar{F}_{X_1}(x+u_1)\, dx \\ \int_0^\infty e^{-sx} \bar{F}_{X_2}(x+u_2)\, dx & \int_0^\infty e^{-sx} \bar{F}_{X_1}(x+u_2)\, dx \end{vmatrix} du_1\, du_2.$$

It may be noted that since $g(\cdot)$ is log-concave, the first determinant is non-positive and the second determinant is non-positive as $X_1 \le_{Lt-rl} X_2$. Hence, the result follows.

Let X_1, X_2 and Y be non-negative random variables, such that Y is independent of X_1 as well as X_2, such that Y has the density function g. Then the following theorem provides the conditions for preservation of *Lt-it* order under convolution.

Theorem 4.2.4:

If $X_1 \leq_{Lt-it} X_2$ and g is log-concave then

$$X_1 - Y \leq_{Lt-it} X_2 - Y.$$

Proof:

It may be noted that for fixed t and $i = 1, 2$

$$\begin{aligned}\int_{-\infty}^{t} e^{su} P(X_i - Y \leq u)\, du &= e^{st}\int_0^{\infty} e^{-sx} P(X_i - Y \leq t - x)\, dx \\ &= e^{st}\int_0^{\infty} e^{-sx} \left(\int_t^{\infty} P(X_i \leq u - x)\, g(u-t)\, du\right) dx.\end{aligned}$$

Now, in view of definition 4.1.1(f), it is enough to show that for all $0 \leq t_1 \leq t_2$ and $x > 0$,

$$\frac{\int_0^{\infty}\int_{t_1}^{\infty} e^{-sx} P(X_1 \leq u - x)\, g(u - t_1)\, du\, dx}{\int_0^{\infty}\int_{t_1}^{\infty} e^{-sx} P(X_2 \leq u - x)\, g(u - t_1)\, du\, dx} \geq \frac{\int_0^{\infty}\int_{t_2}^{\infty} e^{-sx} P(X_1 \leq u - x)\, g(u - t_2)\, du\, dx}{\int_0^{\infty}\int_{t_2}^{\infty} e^{-sx} P(X_2 \leq u - x)\, g(u - t_2)\, du\, dx}.$$

Since Y is non-negative, therefore, $g(u - t) = 0$ when $u < t$. Hence the above inequality is equivalent to

$$\frac{\int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_1}(u - x)\, g(u - t_1)\, du\, dx}{\int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_2}(u - x)\, g(u - t_1)\, du\, dx} \geq \frac{\int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_1}(u - x)\, g(u - t_2)\, du\, dx}{\int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_2}(u - x)\, g(u - t_2)\, du\, dx},$$

or equivalently,

$$\begin{vmatrix} \int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_2}(u - x)\, g(u - t_2)\, du\, dx & \int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_1}(u - x)\, g(u - t_2)\, du\, dx \\ \int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_2}(u - x)\, g(u - t_1)\, du\, dx & \int_0^{\infty}\int_{-\infty}^{\infty} e^{-sx} F_{X_1}(u - x)\, g(u - t_1)\, du\, dx \end{vmatrix} \geq 0. \tag{4.2.4}$$

Applying the basic composition formula (cf. Karlin (1968), p-17), the left side of equation (4.2.4) is

$$\int\int_{u_1<u_2} \begin{vmatrix} g(u_1-t_2) & g(u_2-t_2) \\ g(u_1-t_1) & g(u_2-t_1) \end{vmatrix} \begin{vmatrix} \int_0^\infty e^{-sx}\, F_{X_2}(u_1-x)\, dx & \int_0^\infty e^{-sx}\, F_{X_1}(u_1-x)\, dx \\ \int_0^\infty e^{-sx}\, F_{X_2}(u_2-x)\, dx & \int_0^\infty e^{-sx}\, F_{X_1}(u_2-x)\, dx \end{vmatrix} du_1\, du_2.$$

It may be noted that since $g(\cdot)$ is log-concave, the first determinant is non-positive and the second determinant is non-positive as $X_1 \leq_{Lt-it} X_2$. Hence, the result follows.

4.3 Conclusions

In the context of reliability, we have studied some preservation properties of the Laplace transform ordering and the moment generating function ordering of residual life and inactivity time under the reliability operation of convolution with their proofs. These results are in addition to the existing results available in the literature.

Chapter 5

Sorting of Decision Making Units in Data Envelopment Analysis with Intuitionistic Fuzzy Weighted Entropy

5.1 Introduction

In the fuzzy environment, Data Envelopment Analysis (DEA) was first introduced by Sengupta in 2005 as a linear programming based technique used for measuring and evaluating the relative performance of activities in organizations e.g. hospital, bank etc., where the presence of multiple inputs generate multiple outputs. This makes the comparison complex and difficult. DEA is a useful management tool to the assessment and evaluation of decision making units. DEA defines the relative efficiency of decision making unit and has clear advantages over competing approaches. DEA involves identification of units and uses this information to construct efficiency frontiers over the data of available organization units. Noura & Saljooghi (2009) computed the fuzzy efficiency scores of decision making units (DMUs) and maximum entropy as a special class weighting function to rank DMUs. Decision making process is very important for functions such as investments, new product development, delivery personnel selections, allocation of resources and many others. Atanassov [(1986),(1989)] introduced the concept of intuitionistic fuzzy set (IFS) which is generalization of the theory of fuzzy set. Li (2005) proposed multicriteria decision making methods with IFS using various linear programming approaches to generate optimal weights.

It may be recalled that the intuitionistic fuzzy set is characterized by two functions - the degree of membership function and a non membership function. It may be noted that the sum of membership function and non membership function must be smaller than or equal to one. The theory of IFS is well suited in dealing with imprecise or uncertain decision information, image edge detection, uncertainty, incompleteness and vagueness in decision making. It has been used to build soft decision making models that can accommodate imprecise information and analyze the extent of agreement in a group of experts. Feasibility and effectiveness of IFSs are illustrated in its applications of decision making by many researchers [cf. Szmidt & Kacprzyk (2001), Atanassov et.al (2005) , Liu & Wang (2007), Xu & Yager (2008) and Jian-Zhang & Qiang Zhang (2011)]. Next, we present some basic definitions which are well known in literature.

Definition 5.1.1: Atanassov's intuitionistic fuzzy set (IFS) over a finite non empty fixed set X, is a set $\widetilde{A} = \{< x, \mu_{\tilde{A}}(x), \gamma_{\tilde{A}}(x) > \mid x \in X\}$ which assigns to each element $x \in X$ to the set $\tilde{A}$, which is subset of X having the degree of membership $\mu_{\tilde{A}}(x) : X \to [0,1]$ and degree of non-membership $\gamma_{\tilde{A}}(x) : X \to [0,1]$, satisfying $0 \leq \mu_{\tilde{A}}(x) + \gamma_{\tilde{A}}(x) \leq 1$, for all $x \in X$. For each IFS in X, a hesitation margin $\pi_{\tilde{A}}(x)$, which is the intuitionistic fuzzy index of element x in the IFS $\widetilde{A}$, defined by $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$, denotes a measure of non-determinacy.

Definition 5.1.2:

Let $\widetilde{a_i} = (\mu_i, \gamma_i), \quad i = 1, 2, \ldots.., n$, be a collection of intuitionistic fuzzy values, the intuitionistic fuzzy weighted averaging operator is defined as

$$IFWA_w(\widetilde{a_1}, \widetilde{a_2}, \ldots., \widetilde{a_n}) = \sum_{i=1}^{n} w_i \, \widetilde{a_i} = \left(1 - \prod_{i=1}^{n} (1 - \mu_i)^{w_i}, \prod_{i=1}^{n} \gamma_i^{w_i}\right);$$

where w_i is the weight of $\widetilde{a_i}$, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 5.1.3:

Let $\widetilde{a_i} = (\mu_i, \gamma_i), \quad i = 1, 2, \ldots.., n$, be a collection of intuitionistic fuzzy values, the intuitionistic fuzzy weighted geometric operator is defined as

$$IFWG_w(\widetilde{a_1}, \widetilde{a_2}, \ldots., \widetilde{a_n}) = \sum_{i=1}^{n} \widetilde{a_i}^{w_i} = \left(\prod_{i=1}^{n} \mu_i^{w_i}, \; 1 - \prod_{i=1}^{n} (1 - \gamma_i)^{w_i}\right);$$

where w_i is the weight of $\widetilde{a_i}$, $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$.

Definition 5.1.4:

Let $\widetilde{a} = (\mu, \gamma)$ be an intuitionistic fuzzy value, the score of $\widetilde{a}$ is defined by $s(\widetilde{a}) = \mu - \gamma$, s is called score function. The degree of accuracy of $\widetilde{a}$ is defined by $p(\widetilde{a}) = \mu + \gamma$, p is called accuracy function.

Let $\widetilde{a}_1 = (\mu_1, \gamma_1)$, $\widetilde{a}_2 = (\mu_2, \gamma_2)$ be two intuitionistic fuzzy values,

- If $s(\widetilde{a}_1) < s(\widetilde{a}_2)$, then $\widetilde{a}_1 < \widetilde{a}_2$;
- If $s(\widetilde{a}_1) = s(\widetilde{a}_2)$, then

  (i) $p(\widetilde{a_1}) < p(\widetilde{a_2}) \Rightarrow \widetilde{a_1} < \widetilde{a_2}$;

  (ii) $p(\widetilde{a_1}) = p(\widetilde{a_2}) \Rightarrow \widetilde{a_1} = \widetilde{a_2}$.

In section 5.2, we have presented and studied the fuzzy CCR Data Envelopment Analysis Model. A brief discussion on Intuitionistic Fuzzy Entropy and weighted entropy in subsections 5.2.1 and 5.2.2, respectively, has been given. Further, we have proposed a new algorithm for decision making units in context of intuitionistic fuzzy weighted entropy in order to rank decision making units in data envelopment analysis in section 5.3. In section 5.4, we provide illustrative examples to show the validity of the proposed algorithm. Finally, we conclude the chapter in section 5.5.

5.2 The Fuzzy CCR DEA Model

Charnes, Cooper & Rhodes (1978) (CCR) first introduced the DEA into the operations research literature. DEA is a nonparametric method of measuring the efficiency of decision-making unit(DMU). The original CCR model was applicable only to technologies characterized by constant returns to scale globally. Let us consider the following fuzzy CCR DEA model which consists of n decision making units and each requires varying amounts of m different fuzzy inputs to produce s different fuzzy outputs [cf. Guo and Tanaka (2001)]:

$$\max \quad E_0 = \sum_{r=1}^{s} u_r \widetilde{O_{ro}}$$

such that

$$\begin{aligned} \sum_{i=1}^{m} v_i \widetilde{I_{io}} &= \widetilde{1}; \\ \sum_{r=1}^{s} u_r \widetilde{O_{rj}} - \sum_{i=1}^{m} v_i \widetilde{I_{ij}} &\leq 0; \quad j = 1, \ldots., n, \\ u_r,\ v_i &\geq 0; \quad r = 1, 2, \ldots, s \quad \text{and} \quad i = 1, 2, \ldots, m, \end{aligned}$$

where $\widetilde{I_{io}}$; $i = 1, 2, \ldots, m$ and $\widetilde{O_{ro}}$; $r = 1, 2, \ldots, s$, are input and output values for DMU_o, the decision making unit under consideration.

The α-cuts of $\widetilde{I_{ij}}$ and $\widetilde{O_{rj}}$ are defined as

$$\begin{aligned} (\widetilde{I_{ij}})_\alpha &= \big(x \in X\, |\mu_{I_{ij}(x)} \geq \alpha\big) &= [I_{ij}^l, I_{ij}^u] \\ \text{and} \quad (\widetilde{O_{rj}})_\alpha &= \big(x \in X\, |\mu_{O_{rj}(x)} \geq \alpha\big) &= [O_{rj}^l, O_{rj}^u]. \end{aligned}$$

On applying the α-level of fuzzy data envelopment analysis, the following model is formed:

$$\max \quad E_0 = \sum_{r=1}^{s} u_r \left[O_{ro}^l,\, O_{ro}^u\right]$$

such that

$$\begin{aligned} \sum_{i=1}^{m} v_i \big[I_{io}^l,\, I_{io}^u\big] &= \widetilde{1}; \\ \sum_{r=1}^{s} u_r \big[O_{rj}^l,\, O_{rj}^u\big] - \sum_{i=1}^{m} v_i \big[I_{ij}^l,\, I_{ij}^u\big] &\leq 0; \quad j = 1, \dots, n, \\ u_r,\ v_i &\geq 0;\ r = 1, 2, \dots, s \quad \text{and} \quad i = 1, 2, \dots, m. \end{aligned}$$

For measuring the lower and the upper bounds of the best relative efficiency of each decision making units with interval input and output data, the following DEA model is achieved:

$$\max \quad (E_0)_\alpha^u = \sum_{r=1}^{s} u_r \,(O_{ro})_\alpha^u$$

such that

$$\begin{aligned} \sum_{i=1}^{m} v_i (I_{io})_\alpha^l &= \widetilde{1} \\ \sum_{r=1}^{s} u_r (O_{ro})_\alpha^u - \sum_{i=1}^{m} v_i \,(I_{io})_\alpha^l &\leq 0; \\ \sum_{r=1}^{s} u_r (O_{rj})_\alpha^l - \sum_{i=1}^{m} v_i \,(I_{ij})_\alpha^u &\leq 0; \quad j = 1, \dots, n, \quad j \neq 0; \\ u_r,\ v_i &\geq 0; \quad r = 1, 2, \dots, s \quad \text{and} \quad i = 1, 2, \dots, m. \end{aligned}$$

Also,

$$\max(E_0)^l_\alpha = \sum_{r=1}^{s} u_r\,(O_{ro})^l_\alpha$$

such that

$$\sum_{i=1}^{m} v_i (I_{io})^u_\alpha = \widetilde{1};$$

$$\sum_{r=1}^{s} u_r (O_{ro})^l_\alpha - \sum_{i=1}^{m} v_i\,(I_{io})^u_\alpha \leq 0;$$

$$\sum_{r=1}^{s} u_r (O_{rj})^u_\alpha - \sum_{i=1}^{m} v_i\,(I_{ij})^l_\alpha \leq 0; \qquad j = 1, \ldots, n, \quad j \neq 0,$$

$$u_r,\ v_i \geq 0; \quad r = 1, 2, \ldots, s \quad \text{and} \quad i = 1, 2, \ldots m.$$

It may be noted that for every α, $E^l_\alpha \leq E^u_\alpha$ and if $\alpha_1 \leq \alpha_2$, then $\left[E^l_{\alpha_2}, E^u_{\alpha_2}\right] \subseteq \left[E^l_{\alpha_1}, E^u_{\alpha_1}\right]$.

5.2.1 Intuitionistic Fuzzy Entropy Measure

Let us consider that Atanassov's intuitionistic fuzzy set *(IFS)*, $\widetilde{A}$, over a finite non empty fixed set $X = \{x_1, x_2, \ldots, x_n\}$. The concept of the intuitionistic fuzzy entropy measure for IFSs has been characterized and discussed by Szmidt & Kacprzyk [(2001), (2002)] and a set of following four properties, which an intuitionistic fuzzy entropy should satisfy, was introduced:

- **(IFS1) :** $H(\widetilde{A}) = 0$ iff $\widetilde{A}$ is a crisp set, i.e. $\mu_{\widetilde{A}}(x_i) = 0$ and $\gamma_{\widetilde{A}}(x_i) = 1$ or $\mu_{\widetilde{A}}(x_i) = 1$ and $\gamma_{\widetilde{A}}(x_i) = 0$ for all $x_i \in X$.

- **(IFS2) :** $H(\widetilde{A}) = 1$ iff $\mu_{\widetilde{A}}(x_i) = \gamma_{\widetilde{A}}(x_i)$ for all $x_i \in X$.

- **(IFS3) :** $H(\widetilde{A}) \leq H(\widetilde{B})$ if $\widetilde{A}$ is less fuzzy then $\widetilde{B}$, i.e . $\mu_{\widetilde{A}}(x_i) \leq \mu_{\widetilde{B}}(x_i)$ and $\gamma_{\widetilde{A}}(x_i) \geq \gamma_{\widetilde{B}}(x_i)$ for $\mu_{\widetilde{B}}(x_i) \leq \gamma_{\widetilde{B}}(x_i)$ or $\mu_{\widetilde{A}}(x_i) \geq \mu_{\widetilde{B}}(x_i)$ and $\gamma_{\widetilde{A}}(x_i) \leq \gamma_{\widetilde{B}}(x_i)$ for $\mu_{\widetilde{B}}(x_i) \geq \gamma_{\widetilde{B}}(x_i)$ for all $x_i \in X$.

- **(IFS4) :** $H(\widetilde{A}) = H(\overline{\widetilde{A}})$, where $\overline{\widetilde{A}}$ is complement of $\widetilde{A}$.

It may be noted that the above four axiomatic requirements, i.e, sharpness, maximality, resolution and symmetry of intuitionistic fuzzy entropy are widely accepted and have become a criterion for defining any new intuitionistic fuzzy entropy.

Corresponding to IFS $\widetilde{A}$ with n elements (intuitionistic fuzzy values) $a_i = (\mu_i, \gamma_i)$, $i = 1, 2, \ldots, n$, Szmidt & Kacprzyk (2001) introduced the following entropy measure of IFS $\widetilde{A}$:

$$H_{sk} = \frac{1}{n} \sum_{i=1}^{n} \frac{\max count(a_i \wedge \overline{\widetilde{a}}_i)}{\max count(a_i \vee \overline{\widetilde{a}}_i)};$$

where $\max count(\widetilde{A}) = \sum_{i=1}^{n} (\mu_{\widetilde{A}}(x_i) + \pi_{\widetilde{A}}(x_i))$, $\widetilde{A} \in F(X)$. Here $F(X)$ is set of all the IFSs on X.

Also, corresponding to De Luca −Termini (1972) entropy, we have the following measure of IFS $\widetilde{A}$ of n elements (intuitionistic fuzzy values) $a_i = (\mu_i, \gamma_i)$, $i = 1, 2, \ldots\ldots, n$:

$$E_{LT}(\widetilde{A}) = -\frac{1}{n \ln 2} \sum_{i=1}^{n} \left[\mu_i \ln\left(\frac{\mu_i}{\mu_i + \gamma_i}\right) + \gamma_i \ln\left(\frac{\gamma_i}{\mu_i + \gamma_i}\right) - \pi_i \ln 2 \right].$$

The concept of De Luca −Termini entropy for IFSs has been properly derived Vlachos & Sergiadis (2007) from Intuitionistic fuzzy cross-entropy of the IFSs.

5.2.2 Intuitionistic Fuzzy Weighted Entropy Measure

The concept of Intuitionistic Fuzzy Weighting function can be seen as the decision function representing the attitude of decision maker for many real life problems such as investments, new product development, delivery personnel selections, allocation of resources and especially in multicriteria decision making and many others.

Let ϕ be a real valued function defined as

$$\phi : \varepsilon \rightarrow [0,1], \text{ where } \varepsilon = \{(\alpha,\beta) : \alpha,\beta \in [0,1], \alpha+\beta \leq 1\},$$

be the set of all intuitionistic fuzzy values.

Consider two intuitionistic fuzzy values such as $\widetilde{p} = (\mu_{\widetilde{p}}, \gamma_{\widetilde{p}})$, $\widetilde{q} = (\mu_{\widetilde{q}}, \gamma_{\widetilde{q}}) \in \varepsilon$. ϕ is an entropy measure of IFSs, characterised as the intuitionistic fuzzy weighted entropy, if following four properties are satisfied:

- **(IFWE1) :** $\phi(\widetilde{p}) = 0$ iff $\mu_{\widetilde{p}} = 0$ and $\gamma_{\widetilde{p}} = 1$ (or $\mu_{\widetilde{p}} = 1$ and $\gamma_{\widetilde{p}} = 0$).
- **(IFWE2) :**$\phi(\widetilde{p}) = 1$ iff $\mu_{\widetilde{p}} = \gamma_{\widetilde{p}}$.
- **(IFWE3) :** $\phi(\widetilde{p}) \leq \phi(\widetilde{q})$, if $\widetilde{p}$ is less than $\widetilde{q}$, i.e., $\mu_{\widetilde{p}} \leq \mu_{\widetilde{q}}$ and $\gamma_{\widetilde{p}} \geq \gamma_{\widetilde{q}}$ for $\mu_{\widetilde{q}} \leq \gamma_{\widetilde{q}}$ (or $\mu_{\widetilde{p}} \geq \mu_{\widetilde{q}}$ and $\gamma_{\widetilde{p}} \leq \gamma_{\widetilde{q}}$ for $\mu_{\widetilde{q}} \geq \gamma_{\widetilde{q}}$).
- **(IFWE4) :** $\phi(\widetilde{p}) \leq \phi(\overline{\widetilde{p}})$.

Above four axiomatic requirements, i.e., sharpness, maximality, resolution and symmetry of intuitionistic fuzzy weighted entropy are widely accepted and have become a criterion for defining any new intuitionistic fuzzy weighted entropy.

Let ϕ be a function defined as $\phi : \varepsilon \rightarrow [0,1]$, and $\widetilde{A} = \{\widetilde{a}_1, \widetilde{a}_2, \ldots.., \widetilde{a}_n\}$ where $\widetilde{a}_i = (\mu_i, \gamma_i)$, $i = 1, 2, \ldots.., n$, we have

$$\phi(\widetilde{a}_i) = \pi_i - \frac{1}{\ln 2} \cdot \sum_{i=1}^{n} \left[\mu_i \ln\left(\frac{\mu_i}{\mu_i+\gamma_i}\right) + \gamma_i \ln\left(\frac{\gamma_i}{\mu_i+\gamma_i}\right)\right]; \qquad (5.2.1)$$

$\phi(\widetilde{a}_i)$ fulfils the requirement for intuitionistic fuzzy value entropy measure.

Hence, we get $E_{LT} = \frac{1}{n} \sum_{i=1}^{n} \phi(\widetilde{a}_i)$.

From above equation we get the weighted De Luca-Termini entropy for IFSs

$$E_{WLT}(\widetilde{A}) = \sum_{i=1}^{n} w_i \, \phi(\widetilde{a}_i);$$

where $w_i \in (0,1]$, $i = 1, 2, \ldots.., n$ and $\sum_{i=1}^{n} w_i = 1$ i.e. $w_1 = \cdots = w_n = \frac{1}{n}$.

5.3 Algorithm for Sorting of DMUs

As multicriteria decision making problems are defined on set of alternatives, so in this section we will discuss how to utilize the efficiency of DMUs to identify the best alternative according to some criteria. The procedure for intuitionistic fuzzy multicriteria decision making (IFMCDM) based on efficiency of DMUs and intuitionistic fuzzy weighted entropy consists of following steps:

Step 1: Take multiple inputs and multiple outputs. Estimate the efficiency of DMUs by using Fuzzy DEA model.

Step2: Convert efficiency of DMUs to decision matrix by considering mean of efficiency interval as degree of membership of the alternatives y_j $(j = 1, 2, \ldots.., m)$ according to the criterion x_i $(i = 1, 2, \ldots.., n)$, and is denoted by intuitionistic fuzzy valued decision matrix $\widetilde{M} = [\widetilde{m}_{ij}]_{n\times m}$, where $\widetilde{m}_{ij} = (\mu_{ij}, \gamma_{ij})$. Here μ_{ij}, γ_{ij} are the degree of membership and non-membership of the alternatives.

Step 3: Make use of the principle of minimum entropy value to get the weight vector, which is defined as

$$\min E_w = \sum_{j=1}^{m} E_w\left(\widetilde{A}_j\right) = \sum_{j=1}^{m}\sum_{i=1}^{n} w_i \phi(\widetilde{m}_{ij});$$

such that

$$\begin{cases} K_w, \\ w_1 + w_2 + \ldots. + w_n = 1, \\ w_i \geq \eta\,(i = 1, 2, \ldots., n), \end{cases}$$

where K_w is the set of known information about the weight vector, $\widetilde{A}_j$ is the estimation given by decision maker and η is a small positive real number.

After calculating minimum value of E_w, we calculate optimal weight vector, which is given by

$$w^* = \arg\min E_w.$$

Step 4: Amassed the estimation of alternatives by intuitionistic fuzzy weighted averaging operator ($IFWA_w$) or intuitionistic fuzzy weighted geometric operator ($IFWG_w$).

Step 5: Final and most important step is to rank the alternatives y_j ($j = 1, 2, \ldots.., m$) and select the best one in accordance with the comparison method which is given by the definition 5.1.4.

5.4 Illustrative Example

Let us consider an example related to a software company, searching the best supplier for one of its most important software used in assembling of Laptops.

Table 5.1: Data Table consisting of two fuzzy inputs and two fuzzy outputs

Decision Making Units	Supplier A	Supplier B	Supplier C	Supplier D	Supplier E
I/P-1	4, 3.5, 4.5	2.9, 2.9, 2.9	4.9, 4.4, 5.4	4.1, 3.4, 4.8	6.5, 5.9, 7.1
I/P-2	2.1, 1.9, 2.3	1.5, 1.4, 1.6	2.6, 2.2, 3.0	2.3, 2.2, 2.4	4.2, 3.6, 4.6
O/P-1	2.6, 2.4, 2.8	2.2, 2.2, 2.2	3.2, 2.7, 3.7	2.9, 2.5, 2.3	5.1, 4.4, 5.8
O/P-2	4.1, 3.8, 4.4	3.5, 3.3, 3.7	5.1, 4.3, 5.9	5.7, 5.5, 5.9	7.4, 6.5, 8.3

Efficiency of DMUs is calculated by using DEA model. For $\alpha = 0$, we have

$$\max \quad E_0^u = u_1 O_{10}^u + u_2 O_{20}^u$$

such that

$$v_1 (I_{1o})^l + v_2 (I_{2o})^l = \tilde{1};$$
$$u_1 O_{1o}^u + u_2 O_{2o}^u - v_1 I_{1o}^l - v_2 I_{2o}^l \leq 0;$$

and

$$
\begin{aligned}
u_1 O_{11}^l + u_2 O_{21}^l - v_1\, I_{11}^u - v_2\, I_{21}^u &\leq 0;\\
u_1 O_{12}^l + u_2 O_{22}^l - v_1\, I_{12}^u - v_2\, I_{22}^u &\leq 0;\\
u_1 O_{13}^l + u_2 O_{23}^l - v_1\, I_{13}^u - v_2\, I_{23}^u &\leq 0;\\
u_1 O_{14}^l + u_2 O_{24}^l - v_1\, I_{14}^u - v_2\, I_{24}^u &\leq 0;\\
u_1 O_{15}^l + u_2 O_{25}^l - v_1\, I_{15}^u - v_2\, I_{25}^u &\leq 0.
\end{aligned}
$$

On substituting all values from Table 5.1, we get upper bound of efficiency when $\alpha = 0$. In the similar manner we get the other efficiencies of DMUs as follows in Table 5.2:

Table 5.2: Efficiency of DMUs

Decision Making Units	Supplier A	Supplier B	Supplier C	Supplier D	Supplier E
$\alpha = 0$	0.654, 1	0.836, 1	0.571, 1	0.855, 1	0.638, 1
$\alpha = 0.25$	0.702, 1	0.908, 1	0.642, 1	0.943, 1	0.735, 1
$\alpha = 0.50$	0.758, 0.963	0.99, 1	0.716, 1	1, 1	0.845, 1
$\alpha = 0.75$	0.807, 0.904	1, 1	0.791, 0.932	1, 1	0.969, 1
$\alpha = 1.00$	0.855, 0.855	1,1	0.861, 0.861	1,1	1,1

Coversion of Efficiency DMUs to Decision Matrix Table:

Looking at the efficiency interval, we consider mean of efficiency interval as degree of membership of the alternatives y_j $(A, B, C, D$ and $E)$, satisfying the criterion $x_i\,(\alpha = 0, 0.25, 0.50, 0.75, 1)$. The intuitionistic fuzzy index $\pi_{ij} = 1 - \mu_{ij} - \gamma_{ij}$ shows the decision maker's hesitation of the alternatives y_j with respect to criterion x_i and is zero whenever alternatives $y_j = 1$. Therefore, the decision matrix $\widetilde{M}$ obtained from efficiency of DMUs is given by

$$
\widetilde{M} = \begin{pmatrix}
(.827, .173) & (.918, .082) & (.785, .215) & (.927, .073) & (.819, .181)\\
(.851, .149) & (.954, .046) & (.821, .179) & (.971, .029) & (.867, .133)\\
(.860, .103) & (.995, .005) & (.858, .142) & (1, 0) & (.922, .078)\\
(.855, .049) & (1, 0) & (.861, .071) & (1, 0) & (.984, .016)\\
(.855, 0) & 1, 0) & (.861, 0) & (1, 0) & (1, 0)
\end{pmatrix}
$$

Let K_w the set of known information about the weight vector given by:

$$K_w = \left\{ \begin{array}{l} w_1 \leq 0.3,\, 0.1 \leq w_2 \leq 0.2,\, 0.2 \leq w_3 \leq 0.5,\, 0.1 \leq w_4 \leq 0.3,\; w_5 \leq 0.4, \\ w_3 - w_2 \geq\; w_5 - w_4,\, w_4 \geq w_1,\, w_3 - w_1 \leq 0.1 \end{array} \right\}$$

By using (5.2.1), we get the De Luca - Termini entropy of the intuitionistic fuzzy values as under:

$$\begin{pmatrix} 0.6645 & 0.4092 & 0.7509 & 0.3770 & 0.6882 \\ 0.6073 & 0.2691 & 0.6779 & 0.1893 & 0.5656 \\ 0.5095 & 0.0454 & 0.5894 & 0 & 0.3951 \\ 0.3708 & 0 & 0.4301 & 0 & 0.1183 \\ 0.1450 & 0 & 0.1390 & 0 & 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned} E_w = \sum_{j=1}^{5} E_w\left(\widetilde{A}_j\right) &= \sum_{j=1}^{5}\sum_{i=1}^{5} w_i \phi(\widetilde{m}_{ij}) \\ &= 2.8898\, w_1 + 2.3092\, w_2 + 1.5394 w_3 + 0.9192\, w_4 + 0.2840\, w_5. \end{aligned}$$

Hence, we have the following linear programming problem:

$$\min E_w = 2.8898\, w_1 + 2.3092\, w_2 + 1.5394 w_3 + 0.9192\, w_4 + 0.2840\, w_5$$

subject to

$$\left\{ \begin{array}{l} w_1 \leq 0.3, 0.1 \leq w_2 \leq 0.2,\, 0.2 \leq w_3 \leq 0.5,\, 0.1 \leq w_4 \leq 0.3,\; w_5 \leq 0.4, \\ -w_2 + w_3 + \; w_4 - w_5 \geq 0,\, -w_1 + w_4 \geq 0,\, -w_1 + w_3 \leq 0.1, \\ w_1 + w_2 + \, ... + \, w_n = 1, \\ w_i \geq 0.001\, (i = 1, 2, 3, 4, 5). \end{array} \right.$$

Its optimal solution is $w_1 = 0.1$, $w_2 = 0.1$, $w_3 = 0.2$, $w_4 = 0.25$, $w_5 = 0.35$.

Now apply either $IFWA_w$ or $IFWG_w$ operator (Definition 5.1.2 and Definition 5.1.3). Here we have applied $IFWG_w$ operator to get

$$\widetilde{a}_1 = (0.8527,\, 0.0671), \quad \widetilde{a}_2 = (0.9858,\, 0.0142), \quad \widetilde{a}_3 = (0.8484,\, 0.0888),$$

$$\widetilde{a}_4 = (0.9895, 0.0105), \quad \widetilde{a}_5 = (0.9469, 0.0530).$$

By applying Definition 5.1.4, we calculate score function $s(a_j)\,(\,j = 1, 2, 3, 4, 5\,)$,

$$s(a_1) = 0.7856, \quad s(a_2) = 0.9716, \quad s(a_3) = 0.7596,$$

$$s(a_4) = 0.9790, \quad s(a_5) = 0.8939.$$

Therefore, we can say that alternative D is best choice and the optimal ordering is $y_4 > y_2 > y_5 > y_1 > y_3$, i.e,

$$D > B > E > A > C.$$

5.5 Conclusions

Under the new algorithm proposed, the sorting of decision making units in data envelopment analysis has been accomplished and an optimal ranking order has been found out with the help of intuitionistic fuzzy weighted entropy according to minimum entropy model. The efficiency of the proposed methodology may be applied in regard of information measure for pattern recognition, medical diagnosis, and image segmentation.

Chapter 6

Intuitionistic Trapezoidal and Triangular Fuzzy Multiple Criteria Decision Making

6.1 Introduction

In order to select a product, the role of human behaviors which is influenced by some interrelating factors is an important factor in a consumer decision making process. The external characteristics such as price, brand, capability etc. are also concerned in making a choice. The concept of multiple criteria decision making (MCDM) involves a committee of decision makers assessing various alternatives versus selected criteria and has been extensively applied in real life decision situations such as public administration, engineering, society, management science, economics, military research, professional journals and conferences of diversified disciplines [cf. Wang et. al. (2006), Yang et. al. (2009) and Cavallaro (2010)]. MCDM is a suitable method for the evaluation and selection of most appropriate alternative and selecting their performance based on quantitative criteria (economical) as well as qualitative criteria (market reputation, relationship closeness etc.). In many realistic cases, the values of certain alternatives is usually difficult to judge accurately; instead, importance of criteria is usually expressed through linguistic judgement such as 'good', 'poor', 'excellent' and so on.

In decision making problems, particularly in the case of sales analysis, new product marketing, financial services, etc. there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object. Therefore, in various engineering applications, intuitionistic fuzzy sets techniques have been more popular than fuzzy sets techniques in recent years. Feasibility and effectiveness of IFSs are illustrated in its applications of decision making by many researchers such as Szmidt and Kacprzyk (2001), Atanassov and Pasi (2005), Liu and Wang (2007), Bottani and Rizzi (2008) and Cavallaro (2010). In this chapter, we have applied the concept of intuitionistic trapezoidal fuzzy number (ITFN) and triangular intuitionistic fuzzy numbers (TIFNs) to the study of multiple criteria decision making (MCDM) problem for finding the best alternative where the linguistic variables for the criteria are intuitively pre-defined in the form of ITFNs and TIFNs.

The concept of intuitionistic trapezoidal fuzzy numbers (ITFNs) was introduced by Wang (2008) and it may be noted that intuitionistic trapezoidal fuzzy numbers (ITFNs) express more flexible and abundant information than trapezoidal fuzzy numbers.

Definition 6.1.1

Intuitionistic trapezoidal fuzzy number (ITFN) $\tilde{\chi} = \{(a, b, c, d)\,;\, \mu_{\tilde{\chi}},\, \gamma_{\tilde{\chi}}\}$ is a special intuitionistic fuzzy set, whose membership function and non-membership function have been defined as follows:

$$\mu_{\tilde{\chi}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}\mu_{\tilde{\chi}} & \text{if } a \le x \le b, \\ \mu_{\tilde{\chi}} & \text{if } b \le x \le c, \\ \frac{(d-x)}{(d-c)}\mu_{\tilde{\chi}} & \text{if } c < x \le d, \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_{\tilde{\chi}}(x) = \begin{cases} \frac{(b-x)+\gamma_{\tilde{\chi}}(x-a)}{(b-a)} & \text{if } a_1 \le x \le b, \\ \gamma_{\tilde{\chi}} & \text{if } b \le x \le c, \\ \frac{(x-c)+\gamma_{\tilde{\chi}}(d_1-x)}{(d_1-c)}\mu_{\tilde{\chi}} & \text{if } c < x \le d, \\ 0 & \text{otherwise.} \end{cases}$$

where $0 \le \mu_{\tilde{\chi}} \le 1$ and $0 \le \gamma_{\tilde{\chi}} \le 1$. Also, $\mu_{\tilde{\chi}} + \gamma_{\tilde{\chi}} \le 1$ for all $a, b, c, d \in R$. The values $\mu_{\tilde{\chi}}$ and $\gamma_{\tilde{\chi}}$ represent the maximum membership degree and minimum non-membership degree, respectively.

Shu & Cheng (2006) defined triangular intuitionistic fuzzy numbers (TIFNs) which have a greater capability to handle more ample and flexible information than triangular fuzzy numbers.

Definition 6.1.2

Triangular intuitionistic fuzzy number $\tilde{\chi} = < (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ (TIFN) is a special intuitionistic fuzzy set, whose membership function and non-membership function have been defined as follows:

$$\mu_{\tilde{\chi}}(x) = \begin{cases} u_{\tilde{\chi}}(x-\underline{t})/(t-\underline{t}) & \text{if } \underline{t} \le x < t \\ u_{\tilde{\chi}} & \text{if } x = t \\ u_{\tilde{\chi}}(\bar{t}-x)/(\bar{t}-t) & \text{if } t < x \le \bar{t} \\ 0 & \text{if } x < \underline{t} \text{ or } x > \bar{t} \end{cases}$$

and

$$\nu_{\tilde{\chi}}(x) = \begin{cases} [t-x+w_{\tilde{\chi}}(x-\underline{t})]/(t-\underline{t}) & \text{if } \underline{t} \le x < t \\ w_{\tilde{\chi}} & \text{if } x = t \\ [x-t+w_{\tilde{\chi}}(\bar{t}-x)]/(\bar{t}-t) & \text{if } t < x \le \bar{t} \\ 1 & \text{if } x < \underline{t} \text{ or } x > \bar{t} \end{cases}$$

respectively, where the values $u_{\tilde{\chi}}$ and $w_{\tilde{\chi}}$ represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy
$0 \leq u_{\tilde{\chi}} \leq 1, \quad 0 \leq w_{\tilde{\chi}} \leq 1, \quad 0 \leq u_{\tilde{\chi}} + w_{\tilde{\chi}} \leq 1.$

Let $\pi_{\tilde{\chi}}(x) = 1 - \mu_{\tilde{\chi}}(x) - \gamma_{\tilde{\chi}}(x)$, which is called as intuitionistic fuzzy index of an element x in $\tilde{\chi}$. It is the degree of indeterminacy membership of the element x in $\tilde{\chi}$. The TIFN $\tilde{\chi} =< (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ is called as a positive TIFN, denoted by $\tilde{\chi} > 0$, if $t \geq 0$ and one of the three values $\underline{t}$, t and $\bar{t}$ is not equal to zero. Similarly, if $\bar{t} \leq 0$ and one of the three values $\underline{t}$, t and $\bar{t}$ is not equal to zero, then the TIFN $\tilde{\chi} =< (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ is called as a negative TIFN, denoted by $\tilde{\chi} < 0$.

6.2 Intuitionistic Trapezoidal Fuzzy MCDM

The concept of intuitionistic trapezoidal fuzzy numbers (ITFNs) was introduced by Wang (2008) and it may be noted that intuitionistic trapezoidal fuzzy numbers (ITFNs) express more flexible and abundant information than trapezoidal fuzzy numbers. In this section of the chapter, we have implemented the concept of intuitionistic trapezoidal fuzzy numbers (ITFNs) to the study of multiple criteria decision making (MCDM) problem for evaluating the best vendor/supplier whose information take the form of ITFNs. We propose a new algorithm for ITF-MCDM problem where the weights of the involved attributes are supposed to be completely unknown. These weights have been calculated on the basis of the decision maker's qualitative opinion to the attributes with the help of pre-defined linguistic variables and an entropy measure. Finally, the ranking of the vendors/suppliers has been determined by calculating the hamming distance between the ideal alternative and all the available alternatives.

6.2.1 Preliminaries

Here, we describe the basic aspects of intuitionistic trapezoidal fuzzy numbers (ITFNs), which is well known in literature.

Definition 6.2.1.1

Let $\tilde{\chi}_1 = \{(a_1, b_1, c_1, d_1); \mu_{\tilde{\chi}_1}, \gamma_{\tilde{\chi}_1}\}$ and $\tilde{\chi}_2 = \{(a_2, b_2, c_2, d_2); \mu_{\tilde{\chi}_2}, \gamma_{\tilde{\chi}_2}\}$ be two trapezoidal intuitionistic fuzzy numbers and δ is a real number. Some basic arithmetical operations (addition, multiplication etc.) are defined as follows:

$$\begin{aligned}
\tilde{\chi}_1 \oplus \tilde{\chi}_2 = \{&(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2);\\
&\mu_{\tilde{\chi}_1} + \mu_{\tilde{\chi}_2} - \mu_{\tilde{\chi}_1}\mu_{\tilde{\chi}_2},\ \gamma_{\tilde{\chi}_1}.\gamma_{\tilde{\chi}_2}\}\\
\tilde{\chi}_1 \odot \tilde{\chi}_2 = \{&(a_1.\,a_2, b_1.\,b_2, c_1.\,c_2, d_1.\,d_2);\ \mu_{\tilde{\chi}_1}.\,\mu_{\tilde{\chi}_2},\\
&\gamma_{\tilde{\chi}_1} + \gamma_{\tilde{\chi}_2} - \gamma_{\tilde{\chi}_1}.\gamma_{\tilde{\chi}_2}\}\\
\delta\tilde{\chi}_1 = &\left\{(\delta a_1, \delta b_1, \delta c_1, \delta d_1);\ 1 - (1 - \mu_{\tilde{\chi}_1})^{\delta},\ \gamma_{\tilde{\chi}_1}^{\delta}\right\}
\end{aligned}$$

Definition 6.2.1.2

Intuitionistic trapezoidal fuzzy ideal solution is defined as

$$\tilde{I}^+ = \left\{(a^+, b^+, c^+, d^+);\ \mu^+, \gamma^+\right\} = \{(1, 1, 1, 1);\ 1, 0\}.$$

Definition 6.2.1.3 (Normalized Hamming Distance)

Let $\tilde{\chi}_1 = \{(a_1, b_1, c_1, d_1); \mu_{\tilde{\chi}_1}, \gamma_{\tilde{\chi}_1}\}$ and
$\tilde{\chi}_2 = \{(a_2, b_2, c_2, d_2); \mu_{\tilde{\chi}_2}, \gamma_{\tilde{\chi}_2}\}$
be two intuitionistic trapezoidal fuzzy number. The normalized hamming distance between $\tilde{\chi}_1$ and $\tilde{\chi}_2$ is defined as

$$d(\tilde{\chi}_1, \tilde{\chi}_2) = \frac{1}{8}\left\{\begin{array}{l}
|(1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})a_1 - (1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})a_2|\\
+|(1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})b_1 - (1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})b_2|\\
+|(1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})c_1 - (1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})c_2|\\
+|(1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})d_1 - (1 + \mu_{\tilde{\chi}_1} - \gamma_{\tilde{\chi}_1})d_2|
\end{array}\right\}$$

6.2.2 Method for Evaluating Weights of Attributes with ITFNs

A multiple criteria decision making problem includes a discrete set of m possible alternatives $A = \{A_1, A_2, \ldots, A_m\}$, which is based on a set of n evaluation criterions $C = \{C_1, C_2, \ldots, C_n\}$.

The intuitionistic trapezoidal fuzzy decision matrix is expressed as

$$\tilde{D} = [\tilde{r}_{ij}]_{m\times n} = \{[a_{ij}, b_{ij}, c_{ij}, d_{ij}]; \mu_{ij}, \gamma_{ij}\}_{m\times n};$$

where $\tilde{r}_{ij}$ is the rating of i^{th} alternative meeting the j^{th} criteria which is jointly provided by the decision makers, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Weight measure plays an important role in multiple criteria decision making problems and have a direct relationship with the distance measure between two fuzzy numbers. In order to deal with decision information with intuitionistic trapezoidal fuzzy numbers, we use the normalized hamming distance between intuitionistic trapezoidal fuzzy numbers as given in definition 6.2.1.3.

Let w_j represents the weight vector of j^{th} attribute and let the information about these weights is unknown. However, the weights of the attributes have been provided by the decision maker's qualitative opinion. For the sake of intuitionistic formulation of the qualitative opinions, we define the following table 6.1:

Table 6.1: Linguistic Variables and ITFNS

Sr. No.	**Linguistic variables**	**ITFNs**
1	Very Poor (VP)	{[0.2,0.3,0.4,0.5] ; 0.7,0.1}
2	Poor (P)	{[0.3,0.4,0.5,0.6] ; 0.8,0.1}
3	Satisfactory (SF)	{[0.4,0.5,0.6,0.7] ; 0.2,0.7}
4	Good (G)	{[0.5,0.6,0.7,0.8] ; 0.5,0.4}
5	Very Good (VG)	{[0.6,0.7,0.8,0.9] ; 0.7,0.3}

If there are p persons in a decision making committee, who qualitatively define the weights of the n criterions, then the effective weight of each criteria in the form of intuitionistic trapezoidal fuzzy number can be evaluated as:

$$\tilde{w}_j = \frac{1}{p}\,(\tilde{w}_j^1 + \tilde{w}_j^2 + \ldots + \tilde{w}_j^p).$$

If $d(\tilde{w}_j, \tilde{I}^+)$ is distance between the weight ITFN $\tilde{w}_j$ and the intuitionistic trapezoidal fuzzy ideal solution $\tilde{I}^+$, then the distance vector is given by

$$N = \Big[d(\tilde{w}_1, \tilde{I}^+), d(\tilde{w}_2, \tilde{I}^+), \ldots, d(\tilde{w}_n, \tilde{I}^+)\Big].$$

Further, the normalized distance vector on vector N' is given by

$$N' = [\varepsilon_j] = \left[\frac{d(\tilde{w}_j, \tilde{I}^+)}{(\max)_j\, d(\tilde{w}_j, \tilde{I}^+)}\right], \quad j \;=\; 1, 2, ..., n.$$

The entropy measure of the j^{th} criteria (C_j) for m available alternatives can be obtained from:

$$e_j \;=\; -\frac{1}{\ln(m)}\left[\frac{\varepsilon_j}{\sum\limits_{j=1}^{n}\varepsilon_j}\;\ln\left(\frac{\varepsilon_j}{\sum\limits_{j=1}^{n}\varepsilon_j}\right)\right].$$

Finally, the crisp value of weight for j^{th} criterion, which is based on the above entropy measure, can be calculated as follows:

$$w_j = \frac{1 - e_j}{n - \sum_{k=1}^{n} e_k}; \qquad\qquad j = 1, 2, \ldots, n.$$

6.2.3 Algorithm for Intuitionistic Trapezoidal Fuzzy Multiple Criteria Decision Making

The ranking procedure for a discrete set of m possible alternatives based on a set of n evaluation criteria in case of intuitionistic trapezoidal fuzzy multi criteria decision making (ITF-MCDM) problem is given below:

Input A discrete set of m possible alternatives $A = \{A_1, A_2, \ldots, A_m\}$, a set of n evaluation criteria $C = \{C_1, C_2, \ldots, C_n\}$ and weights of criteria in terms of qualitative opinions of decision makers.

Step 1 : If there are p persons in a decision making committee, then construct the decision matrix D by calculating the rating of each alternative meeting the criteria as follows:

$$\tilde{r}_{ij} = \frac{1}{p}\,(\tilde{r}_{ij}^{1} + \tilde{r}_{ij}^{2} + \ldots + \tilde{r}_{ij}^{p}).$$

Step 2 : Since the information about the weights of attributes is unknown, we find the attribute weights using the entropy method as discussed in section 6.2.2.

Step 3 : Make use of definition 6.2.1.3 and the obtained weight vector in step 2 to compute the distances $d(A_i, \tilde{I}^{+})$ for each i as follows:

$$d(A_i, \tilde{I}^{+}) = \sum_{j=1}^{n} w_j\, d(\tilde{I}^{+}, \tilde{r}_{ij})$$

Step 4 : Finally, the ranking of the alternatives is performed using the values of the distances $d(A_i, \tilde{I}^{+})$ where $i = 1, 2, \ldots, m$. The basic idea of ranking the alternatives used is - smaller the value of $d(A_i, \tilde{I}^{+})$ better the performance/closeness of an alternative to intuitionistic trapezoidal fuzzy ideal solution.

6.2.4 Illustrative Example

Let us consider an example concerning with a communication system, searching for the best global supplier (GS_1, GS_2, GS_3) for one of its most critical parts (e.g. chip) used in assembling process by three decision makers (DM_1, DM_2, DM_3). Criteria used for evaluating global suppliers are (C_1) economical, (C_2) functioning, (C_3) on time performance, (C_4) quality, (C_5) risk factors.

Decision makers use the linguistic variables such as very poor, poor, satisfactory, good, very good to describe the weights of the criteria and rating of

Table 6.2: Linguistic Variables for Weight of Criteria

Criteria/Decisions	DM_1	DM_2	DM_3
C_1	VG	SF	SF
C_2	G	P	G
C_3	VG	G	SF
C_4	G	G	VG
C_5	SF	VP	VP

Table 6.3: Rating of Alternatives in Different Criterion

Criteria	**Supplier**	DM_1	DM_2	DM_3
C_1	GS_1	G	VG	VG
	GS_2	SF	SF	G
	GS_3	G	P	SF
C_2	GS_1	P	G	G
	GS_2	G	G	SF
	GS_3	VG	SF	VG
C_3	GS_1	G	SF	G
	GS_2	SF	G	G
	GS_3	SF	SF	G
C_4	GS_1	G	SF	SF
	GS_2	VG	G	VG
	GS_3	SF	G	SF
C_5	GS_1	P	SF	P
	GS_2	G	SF	G
	GS_3	G	VG	SF

the alternatives qualitatively. The linguistic variables for weights of criteria is provided in the above table 6.2.

The rating of the alternatives with respect to various criterion as given by the decision makers is provided in the above table 6.3.

In order to solve the problem, we first evaluate the weight of each criterion with the help of pre-defined linguistic variables in the form of ITFNs and tabulate them in the following table 6.4:

Table 6.4: Conversion of Linguistic Variables into Summated Weight ITFN

Criteria	**Weight**
C_1	{[0.47,0.57,0.67,0.77] ; 0.27,0.05}
C_2	{[0.43,0.53,0.63,0.73] ; 0.32,0.01}
C_3	{[0.5,0.6,0.7,0.8] ; 0.29,0.03}
C_4	{[0.53,0.63,0.73,0.83] ; 0.31,0.02}
C_5	{[0.27,0.37,0.47,0.57] ; 0.31,0.002}

Based on the normalized decision matrix given in table 6.5, the attribute weights can be calculated by using the entropy method with ITFNs:

Table 6.5: Decision Matrix for ITFNs

	GS_1	GS_2	GS_3
C_1	[.57,.67,.77,.87]; .19,.01	[.43,.53,.63,.73]; .23,.06	[.40,.50,.60,.70]; .31,.01
C_2	[.43,.53,.63,.73]; .32,.01	[.47,.57,.67,.87]; .27,.11	[.53,.63,.73,.83]; .31,.02
C_3	[.47,.57,.67,.87]; .27,.11	[.47,.57,.67,.87]; .27,.11	[.43,.53,.63,.73]; .23,.06
C_4	[.43,.53,.63,.73]; .23,.06	[.57,.67,.77,.87]; .19,.01	[.43,.53,.63,.73]; .23,.06
C_5	[.33,.43,.53,.63]; .32,.00	[.47,.57,.67,.87]; .27,.11	[.50,.60,.70,.80]; .29,.03

The computed weights are as follows:
$w_1 = 0.19987$; $w_2 = 0.2000$; $w_3 = 0.2015$; $w_4 = 0.2032$; and $w_5 = 0.1954$.
It may be noted that

$$\sum_i w_i = 0.9999 \approx 1.$$

By using the weight vector, we get the distances

$$d(GS_1, \tilde{I}^+) = 0.63576$$

$$d(GS_2, \tilde{I}^+) = 0.59069$$

$$d(GS_3, \tilde{I}^+) = 0.6240$$

Finally, based on the idea of ranking given in step 4 of the algorithm, we conclude that desirable order of selecting a global supplier is

$$GS_2 \quad > \quad GS_3 \quad > \quad GS_1.$$

6.3 Triangular Intuitionistic Fuzzy MCDM

Shu & Cheng (2006) defined triangular intuitionistic fuzzy numbers (TIFNs) which have a greater capability to handle more ample and flexible information than triangular fuzzy numbers. The main aim of this section of the chapter is to study triangular intuitionistic fuzzy multiple criteria decision making (TIF-MCDM) problem for finding the best alternative where the linguistic variables for the criteria are intuitively pre-defined in the form of TIFNs. Here, in addition to the decision maker's qualitative opinions, we also consider the management's opinions to the criteria for the ranking of the available alternatives. In view of this, the weight of each criterion has been calculated with the help of parametric entropy as well as 'useful' parametric entropy under $\alpha-$cut/$(\alpha, \beta)-$cut based distance measures for different possible values of parameters. Further, this methodology in a multi-criteria decision making problem related to a pre-defined survey structure for the purchase of a car has been implemented. Also, an algorithm for Triangular Intuitionistic Fuzzy Multi-criteria Decision Making (TIF-MCDM) problem where the ranking of the available alternatives by calculating the various distances between the ideal alternative and all the available alternatives has been provided. An illustrative example to rank the alternatives in view of different opinions has also been provided.

6.3.1 Preliminaries

Here, we present the basics of triangular intuitionistic fuzzy numbers (TIFNs), which is well known in literature.

Definition 6.3.1.1

Let $\tilde{\chi} =< (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ and $\tilde{\lambda} =< (\underline{s}, s, \bar{s}); u_{\tilde{\lambda}}, w_{\tilde{\lambda}} >$ be two TIFNs and δ is a real number. Some arithmetical operations (addition, multiplication etc.) are defined by Wang and Zhang (2009) as follows:

$$\tilde{\chi} \oplus \tilde{\lambda} = < (\underline{t} + \underline{s},\ t + s,\ \bar{t} + \bar{s});\ u_{\tilde{\chi}} + u_{\tilde{\lambda}} - u_{\tilde{\chi}} u_{\tilde{\lambda}},\ w_{\tilde{\chi}} w_{\tilde{\lambda}} >;$$

$$\delta\tilde{\chi} = \begin{cases} < (\delta\underline{t}, \delta t, \delta\bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} > & \text{if } \delta > 0 \ , \\ < (\delta\underline{t}, \delta t, \delta\bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} > & \text{if } \delta < 0 \ ; \end{cases}$$

and

$$\tilde{\chi} \odot \tilde{\lambda} = \begin{cases} < (\underline{t}\,\underline{s}, ts, \bar{t}\,\bar{s});\ u_{\tilde{\chi}} \wedge u_{\tilde{\lambda}},\ w_{\tilde{\chi}} \vee w_{\tilde{\lambda}} > & \text{if } \tilde{\chi} > 0 \text{ and } \tilde{\lambda} > 0, \\ < (\underline{t}\,\bar{s}, ts, \bar{t}\,\underline{s});\ u_{\tilde{\chi}} \wedge u_{\tilde{\lambda}},\ w_{\tilde{\chi}} \vee w_{\tilde{\lambda}} > & \text{if } \tilde{\chi} < 0 \text{ and } \tilde{\lambda} > 0, \\ < (\bar{t}\,\bar{s}, ts, \underline{t}\,\underline{s});\ u_{\tilde{\chi}} \wedge u_{\tilde{\lambda}},\ w_{\tilde{\chi}} \vee w_{\tilde{\lambda}} > & \text{if } \tilde{\chi} < 0 \text{ and } \tilde{\lambda} < 0; \end{cases}$$

where the symbols "$\wedge$" and " $\vee$" are the min and max operators, respectively.

Obviously, if $u_{\tilde{\chi}} = 1$ and $w_{\tilde{\chi}} = 0$, i.e. $\tilde{\chi} = < (\underline{t}, t, \bar{t}); 1, 0 >$ and $\tilde{\lambda} = < (\underline{s}, s, \bar{s}); 1, 0 >$ are Triangular Fuzzy Numbers (TFNs), then above equations degenerate to the arithmetic operations of the TFNs. Hence arithmetic operations of TIFNs are the generalization of those of the TFNs.

Definition 6.3.1.2

(i) Triangular intuitionistic fuzzy positive ideal solution is

$$\tilde{I}^{+} = < (\underline{t}^{+}, t^{+}, \bar{t}^{+}); u^{+}, w^{+} > = < (1, 1, 1); 1, 0 > .$$

(ii) Triangular intuitionistic fuzzy negative ideal solution is

$$\tilde{I}^{-} = < (\underline{t}^{-}, t^{-}, \bar{t}^{-}); u^{-}, w^{-} > = < (0, 0, 0); 0, 1 > .$$

Definition 6.3.1.3

An (α, β)-cut set of $\tilde{\chi} = < (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ is a crisp subset of $\mathbb{R}$, which is defined [Refer Li et. al. (2010) and Guha et. al. (2010)] as $\tilde{\chi}^{\alpha}_{\beta} = \{x \mid \mu_{\tilde{\chi}}(x) \geq \alpha, \nu_{\tilde{\chi}}(x) \leq \beta\}$; where $0 \leq \alpha \leq u_{\tilde{\chi}}$, $w_{\tilde{\chi}} \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1$.

Definition 6.3.1.4

A α-cut set of $\tilde{\chi} =< (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ is a crisp subset of $\mathbb{R}$, which is defined as $\tilde{\chi}^{\alpha} = \{x \,|\, \mu_{\tilde{\chi}}(x) \geq \alpha\}$.

From the definitions of TIFNs and α-cut set, it follows that $\tilde{\chi}^{\alpha}$ is a closed interval, denoted by $\tilde{\chi}^{\alpha} = [\,L^{\alpha}(\tilde{\chi}), R^{\alpha}(\tilde{\chi})]$, which can be calculated as

$$[L^{\alpha}(\tilde{\chi}), R^{\alpha}(\tilde{\chi})] = \left[\underline{t} + \frac{\alpha(t-\underline{t})}{u_{\tilde{\chi}}}\,,\, \bar{t} - \frac{\alpha(\bar{t}-t)}{u_{\tilde{\chi}}}\right].$$

Definition 6.3.1.5

A β-cut set of $\tilde{\chi} =< (\underline{t}, t, \bar{t}); u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ is a crisp subset of $\mathbb{R}$, which is defined as $\tilde{\chi}_{\beta} = \{x \,|\, \nu_{\tilde{\chi}}(x) \leq \beta\}$.

Using the definitions of TIFNs and β-cut set, it follows that $\tilde{\chi}_{\beta}$ is a closed interval, denoted by $\tilde{\chi}_{\beta} = [\,L_{\beta}(\tilde{\chi}), R_{\beta}(\tilde{\chi})]$, which can be calculated as

$$[L_{\beta}(\tilde{\chi}), R_{\beta}(\tilde{\chi})] = \left[\frac{[(1-\beta)t + (\beta - w_{\tilde{\chi}})\underline{t}]}{1-w_{\tilde{\chi}}}, \frac{[(1-\beta)t + (\beta - w_{\tilde{\chi}})\bar{t}]}{1-w_{\tilde{\chi}}}\right].$$

Definition 6.3.1.6

Let $\tilde{\chi} =< \tilde{t}\,; u_{\tilde{\chi}}, w_{\tilde{\chi}} > n$ and $\tilde{\lambda} =< \tilde{s}\,; u_{\tilde{\lambda}}, w_{\tilde{\lambda}} >$ be two arbitrary triangular intuitionistic fuzzy numbers where $\tilde{t}$ and $\tilde{s}$ are two triangular fuzzy numbers with α-cut representations, $\tilde{t}_{\alpha} = [t^{L}(\alpha),\, t^{R}(\alpha)]$ and $\tilde{s}_{\alpha} = [s^{L}(\alpha),\, s^{R}(\alpha)]$. The distance between $\tilde{\chi}$ and $\tilde{\lambda}$ is defined by Chen & Li (2011) is as follows :

$$d(\tilde{\chi}, \tilde{\lambda}) = \sqrt{\int_0^1 \left[(t^{L}(\alpha) - s^{L}(\alpha))^2 + (t^{R}(\alpha) - s^{R}(\alpha))^2\right] d\alpha \quad + } \\ \sqrt{\tfrac{1}{2}\left[\left(u_{\tilde{\chi}} - u_{\tilde{\lambda}}\right)^2 + \left(w_{\tilde{\chi}} - w_{\tilde{\lambda}}\right)^2 + \left(u_{\tilde{\chi}} + w_{\tilde{\chi}} - u_{\tilde{\lambda}} - w_{\tilde{\lambda}}\right)^2\right]}\,.$$

Definition 6.3.1.7

Let $\tilde{\chi} =< \tilde{t}\,; u_{\tilde{\chi}}, w_{\tilde{\chi}} >$ and $\tilde{\lambda} =< \tilde{s}\,; u_{\tilde{\lambda}}, w_{\tilde{\lambda}} >$ be two arbitrary triangular intuitionistic fuzzy numbers where $\tilde{t}$ and $\tilde{s}$ are two triangular fuzzy numbers with

α-cut representations, $\tilde{t}_\alpha = [t_u^L(\alpha), t_u^R(\alpha)]$ and $\tilde{s}_\alpha = [s_u^L(\alpha), s_u^R(\alpha)]$ and β-cut representations, $\tilde{t}_\beta = [t_{1-w}^L(\beta), t_{1-w}^R(\beta)]$and $\tilde{s}_\beta = [s_{1-w}^L(\beta), s_{1-w}^R(\beta)]$. The distance between $\tilde{\chi}$ and $\tilde{\lambda}$ defined by Grzegorzewski (2003) is as follows :

$$d(\tilde{\chi}, \tilde{\lambda}) = \frac{1}{4}\left(\begin{array}{l} \int_0^1 \left|t_u^L(\alpha) - s_u^L(\alpha)\right|^z d\alpha + \int_0^1 \left|t_u^R(\alpha) - s_u^R(\alpha)\right|^z d\alpha + \\ \qquad + \int_0^1 \left|t_{1-w}^L(\alpha) - s_{1-w}^L(\alpha)\right|^z d\alpha + \int_0^1 \left|t_{1-w}^R(\alpha) - s_{1-w}^R(\alpha)\right|^z d\alpha \end{array} \right)^{\frac{1}{z}}$$

where $1 \le z \le \infty$.

6.3.2 Evaluating Weights of Criteria

In this section, we explain the methodology to use the entropy method for evaluating weights of attributes with triangular intuitionistic fuzzy numbers and utility distribution related information.

Let us consider a multiple criteria decision making problem where a discrete set of m possible alternatives $A = \{A_1, A_2, ..., A_m\}$, which is based on a set of n evaluation criteria $C = \{C_1, C_2, ..., C_n\}$. We represent the triangular intuitionistic fuzzy multiple criteria decision matrix is as follows:

$$\tilde{D} = [\tilde{r}_{ij}]_{m\times n} = \{[t_{ij}, t_{ij}, \bar{t}_{ij}];\, u_{ij}, w_{ij}\}_{m\times n};$$

where $\tilde{r}_{ij}$ is the rating of the i^{th} alternative $(i = 1, 2, \ldots, m)$ meeting the j^{th} criteria $(j = 1, 2, \ldots, n)$ which is jointly provided by the decision makers. It may be noted that weight measure has a direct relationship with the distance measure between two fuzzy numbers. In order to deal with decision information with triangular intuitionistic fuzzy numbers, we use the distance between triangular intuitionistic fuzzy numbers as given in definitions 6.3.1.6 & 6.3.1.7.

Let $\tilde{w}_j$ represents the weight vector of j^{th} criteria, where the weights of the criteria have been provided by the decision maker's qualitative opinion. For the sake of formulation of the qualitative opinion, we intuitively define the following Table 6.6:

Table 6.6: Linguistic Variables $\sim$ TIFNs

Sr. No.	**Linguistic Variables**	**TIFNs**
1	Very Poor (VP)	$<(0.2308, 0.3, 0.4286); 0.8, 0.1>$
2	Poor (P)	$<(0.3, 0.4286, 0.75); 0.8, 0.1>$
3	Satisfactory (SF)	$<(0.55, 0.7, 0.85); 0.6443, 0.252>$
4	Good (G)	$<(0.7, 0.8667, 0.9667); 0.7846, 0.1587>$
5	Very Good (VG)	$<(0.8, 1, 1); 0.8413, 0.126>$

If there are p persons in a decision making committee, who qualitatively define the weights of the n criteria, then the effective weight of each criteria in the from of triangular intuitionistic fuzzy number can be evaluated as:

$$\tilde{w}_j = \frac{1}{p}\left(\tilde{w}_j^1 + \tilde{w}_j^2 + \ldots\ldots\ldots\ldots + \tilde{w}_j^p\right).$$

If $d(\tilde{w}_j, \tilde{I}^+)$ is distance between the weights $\tilde{w}_j$ (TIFN) and the triangular intuitionistic fuzzy ideal solution $\tilde{I}^+$, then the distance vector is given by

$$N = \left[d(\tilde{w}_1, \tilde{I}^+), d(\tilde{w}_2, \tilde{I}^+), \ldots, d(\tilde{w}_n, \tilde{I}^+)\right].$$

Further, the normalized distance vector on vector N' is given by

$$N' = [\varepsilon_j] = \left[\frac{d(\tilde{w}_j, \tilde{I}^+)}{\max d(\tilde{w}_j, \tilde{I}^+); j = 1, 2, 3, \ldots, n}\right], j = 1, 2, \ldots, n.$$

For a discrete random variable with probability distribution $P = (p_1, p_2, \ldots, p_n)$ associated with an experiment, Renyi (1961) defined the parametric probabilistic entropy measure as

$$e^\eta = \frac{1}{1-\eta} \log \sum_{k=1}^{n} (p_k)^\eta; \ 0 < \eta < 1.$$

The entropy measure of the j^{th} criteria (C_j) can be obtained from Renyi's (1961) entropy in an analogous way as follows:

$$e_j^\eta = \frac{1}{1-\eta} \log \sum_{k=j} \left(\frac{\varepsilon_k}{\sum_{k=1}^{n} \varepsilon_k}\right)^\eta$$

Finally, the crisp value of weight for j^{th} criterion in view of positive ideal solution ($\tilde{I}^{+}$), which is based on the above entropy measure, can be calculated as follows:

$$w_j^+ \;=\; \frac{1-e_j^\eta}{n-\sum_{k=1}^{n} e_k^\eta}; \quad j \;=\; 1,2,......,n.$$

Similarly, the crisp value of the weight for j^{th} criterion in view of negative ideal solution ($\tilde{I}^{-}$) can be calculated. Thereafter, both the calculated weights are being used in the proposed algorithm in section 6.3.4.

It may be noted that a general probabilistic entropy does not take into account the effectiveness or importance of the events, while in some practical situations of probabilistic nature these subjective considerations also play an important role. Here, we assume that the MCDM process is not completely dependent on the deputed decision makers, but also there is some other management authority whose opinion/importance for different criteria have been taken into consideration.

Belis & Guiasu (1968) considered a qualitative aspect of information called 'useful' information by implementing a utility distribution given by $U = (u_1, u_2, u_3, \ldots, u_n)$, where $u_i > 0$, for each i and is utility or importance of an event x_i whose probability of occurrence is p_i. Also, it is assumed that u_i is independent of p_i. It has also been suggested that the occurrence of an event removes two types of uncertainty - the quantitative type related to its probability of occurrence and the qualitative type related to its utility (importance) for fulfillment of some goal set by the experimenter. In view of this, they proposed the following 'useful' information measure as

$$H(U;P) \;=\; -\sum u_i p_i \log p_i.$$

In case $u_i = 1 \forall i$, the above equation reduces to $H(P) = -\sum p_i \log p_i$, which is well known Shannon's Entropy (1948).

We propose to take the following generalized 'useful' information measure of the j^{th} criteria (C_j) as

$$e_j^{\eta}(U) = \frac{1}{1-\eta} \log \sum_{k=j} u_k \left(\frac{\varepsilon_k}{\sum\limits_{k=1}^{n} \varepsilon_k} \right)^{\eta}$$

where $p_k = \frac{\varepsilon_k}{\sum\limits_{k=1}^{n} \varepsilon_k}$. If $u_k = 1, \forall k$ then $e_j^{\eta}(U)$ reduces to Renyi's entropy.

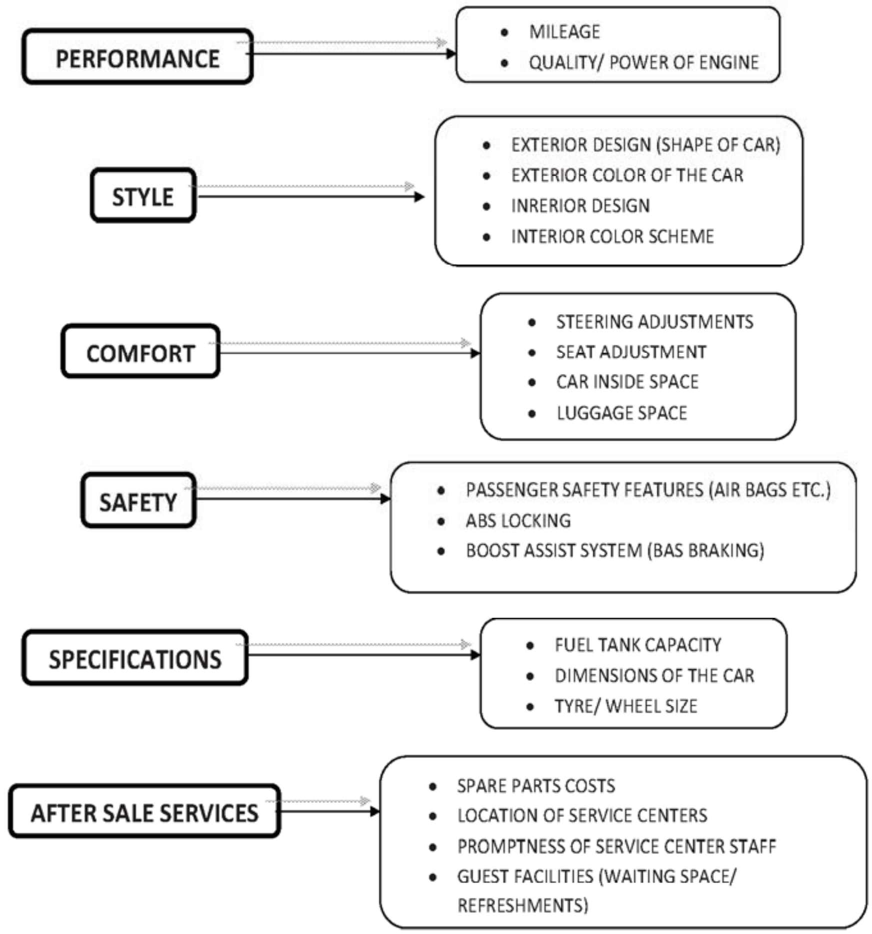
Next, the procedure for finding the weights for further calculations is similar as it is with the non-utility part of this section discussed earlier.

6.3.3 Survey Structure

In order to apply the methodology of evaluating the weights of different criteria in a multi-criteria decision making shown in section 6.3.2, a small survey has been conducted among a certain domain of intellectual people. The survey comprises of a short priority sheet for finding the ranking of various evaluation criteria under different category while purchasing a car.

The priority from a customer (decision maker) for a particular criterion has been taken in terms of linguistic variables - Very Good(VG), Good(G), Satisfactory (SF), Poor(P) and Very Poor(VP) are used in this research paper to determine the satisfaction level of the customer. For a particular criterion, the customers have been asked to indicate the degree of priority level on discrete scale of 1(VP) to 5 (VG).

The broad categories of the priority sheet have been chosen to be performance, style, comfort, safety, specifications and after sale services. Further, in the priority sheet, these categories have been sub-divided into twenty different evaluation criteria from top to bottom in the framework presented below:



On compiling all the data obtained through the brief survey conducted and applying the methodology of evaluating the weights of criteria as discussed in section 6.3.2, we present the following table 6.7 showing the ranking of the evaluation criteria.

From the table 6.7, we observe that the criterion ‘location of service centres’ scores the highest rank and the criterion ‘interior colour scheme’ was ranked as the least.

Table 6.7: Ranking of the Evaluation Criteria

Evaluation Criteria	**Weights (β = 0.1)**	**Ranking**	**Weights (β = 0.5)**	**Ranking**
C_1	0.05012	6	0.05035	6
C_2	0.05019	4	0.05058	4
C_3	0.05022	2	0.05065	2
C_4	0.04977	18	0.04931	18
C_5	0.05016	5	0.05048	5
C_6	0.04985	16	0.04953	16
C_7	0.04997	11	0.04993	11
C_8	0.04995	13	0.04985	13
C_9	0.05008	9	0.05023	9
C_{10}	0.05011	7	0.05033	7
C_{11}	0.05011	8	0.05033	8
C_{12}	0.04986	14	0.04959	14
C_{13}	0.04975	19	0.04926	19
C_{14}	0.04984	17	0.04952	17
C_{15}	0.05002	10	0.05006	10
C_{16}	0.04986	15	0.04959	15
C_{17}	0.04996	12	0.04989	12
C_{18}	0.05044	1	0.05133	1
C_{19}	0.05021	3	0.05062	3
C_{20}	0.04952	20	0.04856	20

6.3.4 Ranking Algorithm for Triangular Intuitionistic Fuzzy MCDM

The ranking procedure for a discrete set of m possible alternatives based on a set of n evaluation criteria in case of triangular intuitionistic fuzzy multiple criteria decision making (TIF-MCDM) problem is given below:

Input A discrete set of m possible alternatives $A = \{A_1, A_2, ..., A_m\}$, a set of n evaluation criteria $C = \{C_1, C_2, ..., C_n\}$ and computed weights of criteria based on qualitative opinions of decision makers.

Step 1 : If there are p persons in a decision making committee, then construct the decision matrix DM by calculating the rating of each alternative meeting the criteria as follows:

$$\tilde{r}_{ij} = \frac{1}{p}(\tilde{r}_{ij}^1 + \tilde{r}_{ij}^2 + \ldots + \tilde{r}_{ij}^p).$$

Step2: Since the information about the weights of attributes is unknown, we find the attribute weights using the entropy method as discussed in section 6.3.2.

Step3: Make use of definitions 6.3.1.6 & 6.3.1.7and the obtained weight vector to compute the distances $d(A_i, \tilde{I}^+)$ and $d(A_i, \tilde{I}^-)$ for each i as follows:
$d(A_i, \tilde{I}^+) = \sum_{j=1}^{n} w_j^+ \, d(\tilde{I}^+, \tilde{r}_{ij})$ and $d(A_i, \tilde{I}^-) = \sum_{j=1}^{n} w_j^- \, d(\tilde{I}^-, \tilde{r}_{ij})$.
Step 4: Calculate the closeness coefficient, $CC_i \quad (i = 1, 2, \ldots, m)$ of all alternatives and rank all alternatives, according to the closeness coefficient as follows:

$$CC_i = \frac{d(A_i, \tilde{I}^-)}{d(A_i, \tilde{I}^+) + d(A_i, \tilde{I}^-)}$$

Step5: Finally, the ranking of the alternatives is performed using the values of the closeness coefficient, CC_i where $i = 1, 2, \ldots, m$. The basic idea of ranking the alternatives used is – higher the value of CC_i better the performance/closeness of an alternative to the triangular intuitionistic fuzzy ideal solution.

6.3.5 Illustrative Example

The applicability and effectiveness of triangular intuitionistic fuzzy multiple criteria decision making (TIF-MCDM) model is illustrated by a numerical example.

Let us consider an example of company which desires to hire an administrator (ADMN). Let there be three short-listed candidates $ADMN_1$, $ADMN_2$ and $ADMN_3$ after preliminary screening. A group of three decision makers is formed, which will assess the three candidates based upon the criteria including stability(C_1), experience (C_2), functioning (C_3), personality (C_4) and self-confidence (C_5).

Decision makers use the linguistic variables such as very poor, poor, satisfactory, good, very good to describe the weights of criteria and rating of alternatives qualitatively.

The linguistic variables for weights of criteria have been provided in Table 6.8. The rating of the alternatives with respect to various criterion as given by the decision makers has been tabulated in Table 6.9. In order to solve the problem, we first evaluate the weights of each criterion with the help of pre-defined linguistic variables in the form of TIFNs and tabulate them in the following Table 6.10.

Table 6.8: Linguistic Variables for Weight of Criteria

Criteria/Decisions	DM_1	DM_2	DM_3
C_1	VG	SF	SF
C_2	G	P	G
C_3	VG	G	SF
C_4	G	G	VG
C_5	SF	VP	VP

Table 6.9: Rating of Alternatives in Different Criterion

Criteria	**Alternative**	DM_1	DM_2	DM_3
C_1	$ADMN_1$	G	VG	VG
	$ADMN_2$	SF	SF	G
	$ADMN_3$	G	P	SF
C_2	$ADMN_1$	P	G	G
	$ADMN_2$	G	G	SF
	$ADMN_3$	VG	SF	VG
C_3	$ADMN_1$	G	SF	G
	$ADMN_2$	SF	G	G
	$ADMN_3$	SF	SF	G
C_4	$ADMN_1$	G	SF	SF
	$ADMN_2$	VG	G	VG
	$ADMN_3$	SF	G	SF
C_5	$ADMN_1$	P	SF	P
	$ADMN_2$	G	SF	G
	$ADMN_3$	G	VG	SF

Table 6.10: Linguistic Variables $\sim$ Summated Weight (TIFN)

Criteria	**Weight**
C_1	$< (0.63, 0.80, 0.90); 0.327, 0.003 >$
C_2	$< (0.57, 0.72, 0.89); 0.330, 0.001 >$
C_3	$< (0.68, 0.86, 0.94); 0.329, 0.002 >$
C_4	$< (0.73, 0.91, 0.98); 0.331, 0.001 >$
C_5	$< (0.34, 0.43, 0.57); 0.329, 0.001 >$

Based on the normalized decision matrix given in Table 6.11, the criteria weights can be calculated by using the entropy method with TIFNs:

Table 6.11: Decision Matrix for TIFNs

	$ADMN_1$	$ADMN_2$	$ADMN_3$
C_1	$<(0.77, 0.96, 0.99); 0.33, 0.001>$	$<(0.60, 0.76, 0.89); 0.32, 0.003>$	$<(0.52, 0.67, 0.86); 0.32, 0.001>$
C_2	$<(0.57, 0.72, 0.89); 0.33, 0.001>$	$<(0.65, 0.81, 0.93); 0.33, 0.002>$	$<(0.72, 0.90, 0.95); 0.33, 0.001>$
C_3	$<(0.65, 0.81, 0.93); 0.33, 0.002>$	$<(0.65, 0.81, 0.93); 0.33, 0.002>$	$<(0.60, 0.76, 0.89); 0.32, 0.003>$
C_4	$<(0.60, 0.76, 0.89); 0.32, 0.003>$	$<(0.77, 0.96, 0.99); 0.33, 0.001>$	$<(0.60, 0.76, 0.89); 0.32, 0.003>$
C_5	$<(0.38, 0.52, 0.78); 0.32, 0.0001>$	$<(0.65, 0.81, 0.93); 0.33, 0.002>$	$<(0.68, 0.86, 0.94); 0.32, 0.002>$

The computed weights with triangular intuitionistic fuzzy positive ideal solution and triangular intuitionistic fuzzy negative ideal solution using definition 6.3.1.6 are as follows:

	w_1^+	w_1^-	w_2^+	w_2^-	w_3^+	w_3^-	w_4^+	w_4^-	w_5^+	w_5^-
$\beta = 0.1$	.2013	.1991	.1992	.2004	.2029	.1982	.2038	.1976	.1927	.2051
$\beta = 0.5$	.2054	.1959	.1968	.2013	.2118	.1924	.2154	.1900	.1705	.2204
$\beta = 0.9$	.2082	.1938	.1952	.2020	.2178	.1885	.2233	.1849	.1555	.2309

By using the weight vector, we get the distances, closeness coefficients and ranking order of selecting an administrator for different values of β as shown in the following Table 6.12:

Table 6.12: Ranking Results Obtained by TIF-MCDM using Chen and Li (2011) and Parametric Entropy

For β=0.1	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	1.09421	1.97896	0.6439	3
$ADMN_2$	1.08705	2.09335	0.6582	2
$ADMN_3$	1.02884	2.03923	0.6647	1
For β=0.5	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	1.0838	1.9709	0.6452	3
$ADMN_2$	1.0917	2.0906	0.6569	2
$ADMN_3$	1.0328	2.0412	0.6640	1
For β=0.9	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	1.0769	1.9664	0.6461	3
$ADMN_2$	1.0952	2.0895	0.6561	2
$ADMN_3$	1.0357	2.0437	0.6637	1

Further, the computed weights with triangular intuitionistic fuzzy positive ideal solution and triangular intuitionistic fuzzy negative ideal solution using definition 6.3.1.7 are as follows:

	w_1^+	w_1^-	w_2^+	w_2^-	w_3^+	w_3^-	w_4^+	w_4^-	w_5^+	w_5^-
$\beta = 0.1$	.2005	.1980	.1942	.2000	.2091	.1966	.2152	.1956	.1811	.2097
$\beta = 0.5$	.2018	.1920	.1776	.2001	.2349	.1864	.2583	.1822	.1274	.2392
$\beta = 0.9$	.2026	.1879	.1673	.2002	.2510	.1795	.2852	.1733	.0939	.2591

By using the weight vector, we get the distances, closeness coefficients and ranking order of selecting an administrator for different values of β as shown in the following Table 6.13.

Table 6.13: Ranking Results Obtained by TIF-MCDM using Grzegorzewski's (2003) Method and Parametric Entropy

For β=0.1	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	0.2461	0.7510	0.7532	3
$ADMN_2$	0.1737	0.8294	0.8268	1
$ADMN_3$	0.2099	0.7916	0.7904	2
For β=0.5	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	0.2311	0.7423	0.7626	3
$ADMN_2$	0.1683	0.8277	0.8310	1
$ADMN_3$	0.2185	0.7952	0.7845	2
For β=0.9	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	0.2217	0.7363	0.7685	3
$ADMN_2$	0.1652	0.8267	0.8335	1
$ADMN_3$	0.2238	0.7976	0.7809	2

The values of closeness coefficients in Table 6.13 is more distinguishable than in Table 6.12. Thus, we conclude that results obtained by Grzegorzewski's distance measure are better then the results obtained by Chen and Li distance measure.

As we discussed earlier that if the MCDM process is not completely dependent on the deputed decision makers, but also there is some other management authority whose opinion/importance for different criteria can be associated with the criteria.

For example, if we take $u_1 = 6$; $u_2 = 7$; $u_3 = 8$; $u_4 = 8$ and $u_5 = 9$ to be the utilities of criteria - stability, experience, functioning, personality and self confidence respectively on a 10-point scale, then the computed weights with triangular intuitionistic fuzzy positive ideal solution and triangular intuitionistic fuzzy negative ideal solution using definition 6.3.1.7 are presented.

	w_1^+	w_1^-	w_2^+	w_2^-	w_3^+	w_3^-	w_4^+	w_4^-	w_5^+	w_5^-
$\beta = 0.1$	.1510	.1547	.1912	.1847	.2024	.2165	.1954	.2176	.2599	.2265
$\beta = 0.5$	.1212	.1501	.2281	.1789	.1389	.2431	.0846	.2510	.4273	.1769
$\beta = 0.7$	.1240	.1602	.2462	.1808	.1116	.2485	.0398	.2585	.4784	.1521

By using the weight vector, we get the distances, closeness coefficients and ranking order of selecting an administrator for different values of β as shown in the following Table 6.14:

Table 6.14: Ranking Results Obtained by TIF-MCDM Using Grzegorzewski's (2003) Method and 'Useful' Parametric Entropy

For β=0.1	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	.2738	.7400	.7299	3
$ADMN_2$	.1736	.8349	.8279	1
$ADMN_3$	.1967	.7951	.8017	2
For β=0.5	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	.3238	.7526	.6992	3
$ADMN_2$	.1857	.8402	.8190	2
$ADMN_3$	.1696	.7897	.8232	1
For β=0.7	$d(ADMN_i, \tilde{I}^+)$	$d(ADMN_i, \tilde{I}^-)$	CC_i	Ranking
$ADMN_1$	.3377	.7607	.6926	3
$ADMN_2$	.1913	.8408	.8146	2
$ADMN_3$	.1615	.7865	.8297	1

6.4 Conclusions

The study of multiple criteria decision making (MCDM) problem for evaluating the best alternative has been done with the concept of intuitionistic trapezoidal fuzzy numbers (ITFNs). A new algorithm for ITF-MCDM problem has been proposed where the weights of the involved attributes are unknown. On the basis of the decision maker's qualitative opinion to the attributes with the help of pre-defined linguistic terms and an entropy measure, these weights have been calculated. Finally, the selection on the basis of ranking of the vendors/suppliers has been done by calculating the hamming distance between the ideal alternative and all the available alternatives. Triangular intuitionistic fuzzy multiple criteria decision making (TIF-MCDM) problem for finding the best alternative where the linguistic variables for the criteria are intuitively pre-defined in the form of TIFNs. We have implemented the concept of management's opinions to the criteria as a 'useful' information - utility distribution in addition to the decision maker's qualitative opinions for the ranking of the available alternatives. On the basis of this approach, the weight of each criterion has been well calculated with the help of parametric entropy as well as 'useful' parametric entropy under $\alpha-$cut/$(\alpha, \beta)-$cut based distance measures for different possible values of parameters. Also, the ranking of the evaluation criteria involved in the survey structure based on a questionnaire for the purchase of a car has been presented. A new ranking algorithm for Triangular Intuitionistic Fuzzy Multi-criteria Decision Making (TIF-MCDM) problem for the available alternatives by calculating the various distances between the ideal alternative and all the available alternatives has been described. Finally, an example to rank the alternatives in view of the different opinions has also been illustrated.

Chapter 7

Conclusions

The stochastic comparison of residual life and inactivity time of series and parallel systems had been studied in the literature when the random variables are independent and identically distributed. By assuming that X and Y are independent, but not necessarily identical distributed and letting $X \leq_{lr} Y$, $\eta_f \leq 0$ and $\eta_g \geq 0$, (or $Y \leq_{lr} X$, $\eta_f \geq 0$ and $\eta_g \leq 0$) we proved that the parallel system of used components, i.e., $\max(X_t, Y_t)$, is better than the used parallel system, i.e., $(\max(X, Y))_t$, in the sense of likelihood ratio order.

It has been found that, by assuming X and Y are independent, but not necessarily identical distributed and letting $X \leq_{lr} Y$, (or $Y \leq_{lr} X$), for any $t \geq 0$,

$$(\max(X, Y))_{(t)} \leq_{lr} \max(X_{(t)}, Y_{(t)});$$

and

$$\min(X_{(t)}, Y_{(t)}) \leq_{lr} (\min(X, Y))_{(t)}.$$

Also, various aging properties of used/inactive parallel/series systems and the parallel/series system of used/inactive components has been proved. These results are supported by well known distributions, such as Weibull and Gompertz distributions.

In context of reliability and life testing problems, we obtain reliability properties of mean inactivity time under weighting. The conditions of stochastic comparison of weighted distributions in terms of mean inactivity time and shifted likelihood ratio order has been obtained.

It has been found that the likelihood ratio order is preserved between equilibrium random variable under the formation of series system. Also some conditions are provided under which a parallel system under equilibrium distribution have likelihood ratio order.

If the life-times of i.i.d. components have increasing hazard rate (IHR) then a series system composed of these i.i.d. components have log-concave life-time. Further if the eta function of i.i.d. components is negative then a parallel system composed of these i.i.d. components have log-concave life-time.

We further studied some preservation properties of the Laplace transform ordering and the moment generating function ordering of residual life and inactivity time under the reliability operation of convolution with their proofs.

The sorting of decision making units in the data envelopment analysis has been accomplished and an optimal ranking order has been found out with the help of intuitionistic fuzzy weighted entropy according to minimum entropy model. An illustrative example has been provided in order to show the implementation of the proposed algorithm.

The study of multiple criteria decision making (MCDM) problem for evaluating the best alternative has been carried out with the concept of intuitionistic trapezoidal fuzzy numbers (ITFNs) and triangular intuitionistic fuzzy numbers (TIFNs). An example based on a short survey and questionnaire has been considered. Further, a new methodology to implement the management's opinions to the criteria as a 'useful' information in addition to the decision maker's qualitative opinions for the ranking of the available alternatives has been introduced. In order to show the implementation of the proposed algorithm, illustrative examples have been provided.

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