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Exact Soliton solutions for Nonlinear Evolutionary Partial Differential Equations

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CERTIFICATE

This is to certify that the thesis entitled "**Exact Soliton solutions for Nonlinear Evolutionary Partial Differential Equations**" submitted by **Abhinav Anand** to the Jaypee University of Information Technology, Waknaghat in fulfillment of the requirement for the award of the degree of **Bachelor of Technology in Bioinformatics** is a record of bona fide research work carried out by them under my supervision and guidance and no part of this work has been submitted for any other degree or diploma.

Signature of Supervisor ....

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As I conclude my project, I have many people to thank; for all the help, guidance and support they lent me, throughout the course of my endeavour.

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Signature of the student ....

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Date ...27...05..11.....

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List of Notations

We use the following notation:

• Derivatives are denoted using subscripts. $\partial_x = \frac{\partial}{\partial x}$ and $\frac{\partial u}{\partial x} = u_x$

Higher derivatives may be denoted by repeating the independent variable in the subscript, or using a coefficient. e.g., Second derivatives are denoted as $u_{2x} = u_{xx}$

• We will introduce an infinite number of independent variables, $x_1, x_2, x_3 ...$, and denote these using subscripts.

•We use the notation o to denote a composition (of operators or functions). For clarity, we will always specify a composition, meaning that the operator is yet to act on a function.

- GCCKdV equation means generalized complex coupled Kortweg-de Vries equation.

ABSTRACT

"The world around us has been inherently nonlinear."

In recent years, solitons and their nonlinear evolutionary equations (NLEEs) have attracted the attention of many biomathematicians and physicists. A soliton is a particular type of solitary wave, which is not destroyed when it collides with another wave of the same kind. Such behaviour is suggested by numerical simulation, but is it really possible that the soliton completely recovers its original shape after a collision? In detailed analysis of the results of such numerical simulations, some ripples can be observed after a collision, and it therefore seems that the original shape is not completely recovered. Therefore, in order to clarify whether or not solitons are destroyed through their collisions, it is necessary to find exact solutions of soliton equations.

A direct method has been investigated to find exact solutions of nonlinear partial differential equations, including soliton equations. A new binary operator, called the *D*-operator, is derived. General formulae, through which nonlinear partial differential equations are transformed into bilinear forms, are presented. By virtue of special properties of the *D*-operator, solving these bilinear forms by ordinary reductive perturbation methods leads to perturbation expansions that may sometimes be truncated as finite sums. Such a truncation yields an exact solution for the equation. Through the dependent variable transformations and symbolic computation, GCCKdV equations are bilinearized, based on which the one- and two-

soliton solutions are obtained. Through the interactions of two solitons, the regular elastic collisions are shown.

When the wave numbers are complex, three kinds of solitonic collisions are presented: (i) two solitons merge and separate from each other periodically; (ii) two solitons exhibit the attraction and repulsion nearly twice, and finally separate from each other after such type of interaction; (iii) two solitons are fluctuant in the central region of the collision.

Propagation features of solitons are investigated with the effects of the coefficients in the GCCKdV equations considered.

The work suggests that a variety of effective analytical methods can be developed considerably to find exact solutions for nonlinear PDEs.

CHAPTER 1

INTRODUCTION

1.1 Fundamentals of PDEs

Definition 1. A partial differential equation (PDE) is an equation containing partial derivatives of the dependent variable.

For example, the following are PDEs

$$u_t + cu_x = 0 \tag{1.1.1}$$

$$u_{xx} + u_{yy} = f(x, y) \tag{1.1.2}$$

$$a(x, y)u_{xx} + 2u_{xy} + 3x^2u_{yy} = 4e^x \tag{1.1.3}$$

$$u_x u_{xx} + (u_y)^2 = 0 \tag{1.1.4}$$

$$(u_{xx})^2 + u_{yy} + a(x, y)u_x + b(x, y)u = 0\ . \tag{1.1.5}$$

In general we may write a PDE as

$$F(x, y, \cdots, u, u_x, u_y, \cdots, u_{xx}, u_{xy}, \cdots) = 0 \tag{1.1.6}$$

where *x, y,* .... are the independent variables and *u* is the unknown function of these variables.

Of course, we are interested in solving the problem in a certain domain D. A solution is a function *u* satisfying (1.1.6). From these many solutions we will select the one satisfying certain conditions on the boundary of the domain D.

For example, the functions

$$\begin{aligned} u(x, t) &= e^{x-ct} \\ u(x, t) &= \cos(x - ct) \end{aligned}$$

are solutions of (1.1.1), as can be easily verified.

Definition 2. The order of a PDE is the order of the highest order derivative in the equation.

For example (1.1.1) is of first order and (1.1.2) - (1.1.5) are of second order.

Definition 3. A PDE is linear if it is linear in the unknown function and all its derivatives with coefficients depending only on the independent variables.

For example (1.1.1) - (1.1.3) are linear PDEs.

Definition 4. A PDE is nonlinear if it is not linear.

Definition 5. A PDE is quasilinear if it is linear in the highest order derivatives with coefficients depending on the independent variables, the unknown function and its derivatives of order lower than the order of the equation.

For example (1.1.4) is a quasilinear second order PDE, but (1.1.5) is not.

We shall primarily be concerned with linear second order PDEs which have the general form

$$A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=G(x,y). \quad (1.1.7)$$

Definition 6. A PDE is called homogeneous if the equation does not contain a term independent of the unknown function and its derivatives.

For example, in (1.1.7) if $G(x, y) \equiv 0$, the equation is homogenous. Otherwise, the PDE is called inhomogeneous.

Partial differential equations are more complicated than ordinary differential ones.For PDEs, selecting a particular solution satisfying the supplementary conditions may be as difficult as finding the general solution. This is because the general solution of a PDE involves an arbitrary function as can be seen in the next example. Also, for linear homogeneous ODEs of order n, a linear combination of n linearly independent solutions is the general solution. This is not true for PDEs, since one has an infinite number of linearly independent solutions.

1.2 Basics of Nonlinear Waves

The wave equation.

Consider the wave equation

$$u_{tt} - u_{xx} = 0,$$

with u, x, and t real. Typically pose an initial-value problem to find u given u(x, 0) = f(x) and u_t(x, 0) = g(x), and subject to suitable boundary conditions on a finite or infinite interval of x. General solution is given by d'Alembert's formula:

$$u(x,t) = F_+(x-t) + F_-(x+t),$$

with $F_\pm$ arbitrary.

Determined from initial data by

$$f(x) = F_+(x) + F_-(x), \qquad g(x) = -F'_+(x) + F'_-(x).$$

Thus,

$$F'_\pm(x) = \frac{1}{2}f'(x) \mp \frac{1}{2}g(x).$$

Integrate and use

$f(x) = F_+(x) + F_-(x)$:

$$F_\pm(x) = \frac{1}{2}f(x) \mp \frac{1}{2}\int_y^x g(\xi)\,d\xi,$$

where y is arbitrary (its effect cancels out of u(x, t)).

Notes:

• F_+(x − t) and F_-(x + t) represent travelling wave solutions of the wave equation. They propagate without change of form at constant speed. If $F_+$and F_- (represent isolated wave forms (say have compact support), The two wave forms will eventually separate from one another and thus a complicated initial condition can resolve asymptotically into a sum of simple waves.

• Of course for all time u(x, t) is a sum of simple waves. This reflects the superposition principle shared by all linear equations: whenever $u_1(x,t)$and $u_2(x,t)$ are two solutions, then so is the sum $u(x,t) = u_1(x,t) + u_2(x,t)$.

The linear Klein-Gordon equation. Dispersion.

A simple modification of the wave equation is the following equation

$$u_{tt} - u_{xx} + u = 0.$$

This is called a (linear) Klein-Gordon equation. It is also a linear equation, and so has a superposition principle. However, there are no localized traveling wave solutions.

Substitute

$u(x,t) = F(\xi)$ with $\xi = x - ct$ and by the chain rule obtain

$u_{xx} = F''(\xi)$, $u_{tt} = c^2 F''(\xi)$,

so under the substitution the equation becomes

$$F''(\xi) + \frac{1}{c^2 - 1} F(\xi) = 0,$$

so if $c^2 < 1$ the

$$F(\xi) = a_+ e^{\xi/\sqrt{1-c^2}} + a_- e^{-\xi/\sqrt{1-c^2}},$$

which is unbounded and not localized,

and if $c^2 > 1$ then

$$F(\xi) = a\cos(\xi/\sqrt{c^2-1}) + b\sin(\xi/\sqrt{c^2-1}),$$

which is bounded but periodic (not pulse-like).

The nature of these sinusoidal traveling wave solutions explains why this equation doesn't support localized traveling waves. Generally a wavetrain solution of a wave equation has the form

$$u(x,t) = Ae^{i(kx-\omega t)}$$

and A is the amplitude, k is the wavenumber, and ω is the frequency. These are not independent.

Upon substitution into the linear Klein-Gordon equation we see that this is a solution as long as

$$-\omega^2 + k^2 + 1 = 0.$$

This formula is called a dispersion relation.

Since the phase velocity of the wave train is $v_p = \omega / k$, we see that in this problem

$$v_p^2 = 1 + \frac{1}{k^2},$$

which depends nontrivially on k. Thus waves of different lengths travel with different speeds. This phenomenon is known as dispersion. Since by Fourier theory we can write the general solution as

$$u(x,t) = \int_{-\infty}^{\infty} \left[A_+(k) e^{i(kx - \omega_+(k)t)} + A_-(k) e^{i(kx - \omega_-(k)t)} \right] dk$$

a general solution is made up of waves traveling at different speeds with respect to each other. This ultimately leads to the distortion of any wave form that does not resemble one of the basic Fourier components.

It should also be noted that Fourier theory gives a solution algorithm for the linear Klein-Gordon equation, even though there is no d'Alembert formula. That is, the functions $A_\pm(k)$ are determined in terms of $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ by Fourier transforms.

A nonlinear Klein-Gordon equation. Solitary waves.

If we want to further modify the equation to bring coherent structures back into the picture, we have to add nonlinearity. Consider the equation

$$u_{tt} - u_{xx} + \frac{4}{\pi} \left(\sin(2\pi u) + \frac{1}{3} \sin(6\pi u) \right) = 0.$$

The nonlinear terms here are chosen to be odd and periodic but are otherwise arbitrary (they happen to be the first two terms of the Fourier sine series of 1 on $0 < u < 1/2$). This equation has interesting localized traveling waves.

With u(x, t) = F(ξ) and ξ = x − ct, we arrive at

$$F''(\xi) + \frac{4}{\pi(c^2-1)}\left(\sin(2\pi F) + \frac{1}{3}\sin(6\pi F)\right) = 0.$$

This is a nonlinear differential equation, but it is easy to analyze by looking at the phase portrait in the (F, F') plane. Multiplying by F' we find that

$$\frac{d}{d\xi}\left[\frac{1}{2}\left(\frac{dF}{d\xi}\right)^2 - \frac{4}{\pi(c^2-1)}\left(\frac{1}{2\pi}\cos(2\pi F) + \frac{1}{18\pi}\cos(6\pi F)\right)\right] = 0$$

so the solution curves in the phase plane are level curves of an energy. If $c^2 < 1$ we have saddle points at

$$F = n,\ n \in \mathbb{Z}$$

(and F' = 0) and center points halfway in-between.

If $c^2 > 1$ it's the other way around.

Three kinds of "motions" in ξ :

• Periodic motion. Libration of the pendulum.

• Unbounded (in F) motion. Rotation of the pendulum.

• Separatrices. Heteroclinic orbits connecting saddle points.

The latter correspond to localized solutions of the nonlinear Klein-Gordon equation! Suppose $c^2 < 1$. To find the localized traveling waves it is enough to look at the energy level through F = F' = 0:

$$\frac{1}{2}\left(\frac{dF}{d\xi}\right)^2 - \frac{4}{\pi(c^2-1)}\left(\frac{1}{2\pi}\cos(2\pi F) + \frac{1}{18\pi}\cos(6\pi F)\right) = 0.$$

This first-order equation may be solved directly for ξ in terms of F because

$$\frac{d\xi}{dF} = \pm\sqrt{\frac{1-c^2}{2}} \cdot \frac{1}{\sqrt{\int_0^F \Phi(y)\, dy}},$$

Where $\Phi(u) := \frac{4}{\pi}\left(\sin(2\pi u) + \frac{1}{3}\sin(6\pi u)\right)$

As ξ is positive for $0 < u < 1$, d ξ /dF is always nonzero, so by the implicit function theorem we get F = F(ξ).

(Every monotone function has an inverse). Therefore, F(ξ) has the shape of a "front". There are traveling wave fronts of all speeds with $c^2 < 1$. This is different from the linear wave equation which only admitted the speeds c = ±1. Note the relativistic interpretation of the fronts: the characteristic width of the traveling wave solution u(x, t) = F (x−ct) is proportional to $\sqrt{1-c^2}$. The faster it goes, the shorter it is. The scaling is exactly that of special relativity. Also, because the passage of the front increases the angle 2πu by 2π, there is a kind of twisting going on, which motivates the terminology of calling these traveling waves kinks. Localized traveling waves in nonlinear wave equations are called solitary waves to contrast them with solutions that are periodic in ξ (and therefore resemble a whole train of waves instead of just one). Once dispersion is in the picture, nonlinearity is essential to have solitary waves in a system. The propagation of the solitary wave should be thought of as a dynamical balance between dispersive "forces" that try to pull the wave apart, and nonlinear "forces" that try to compress it together.

We can think of a solitary wave in terms of a group of kids of different sizes all walking along together. If they are walking on the pavement, then some kids walk faster than others and eventually the group spreads out, which is like dispersion. Now put the same group of kids on a huge trampoline, and the ones who get out in front suffer the disadvantage of having to walk uphill while the ones who fall behind are given a boost by walking downhill. The more kids are present, the greater the effect is, which is the essential property of nonlinearity. Combining the effects of nonlinearity and dispersion, the group of kids walking on the trampoline just remains the same size — a solitary wave.

The fact that we are now looking at a nonlinear equation means that we can no longer count on a superposition principle. Sums of solutions are no longer solutions. Also, there is no longer an algorithm for solving initial-value problems. That's just how it is.

The sine-Gordon equation. Solitons.

Another example of a nonlinear Klein-Gordon equation is just

$$u_{tt} - u_{xx} + \sin(2\pi u) = 0 ,$$

which is known as the sine-Gordon equation. Exactly the same kind of reasoning as before, now using

$$\Phi(u) = \sin(2\pi u)$$

gives a family of solitary wave solutions (kinks) parametrized by velocities $c^2 < 1$. In this case, we can find the traveling waves explicitly:

$$F(\xi) = \frac{2}{\pi}\tan^{-1}\left(\exp\left(\pm\sqrt{\frac{2\pi}{1-c^2}}(\xi-\xi_0)\right)\right)$$

Now, we can again carry out similar numerical experiments to examine collisions of these solitary waves.

Notes:

• The interaction is now "clean". There is no radiation shed.

• There is a "phase shift": after the collision the kinks reemerge unscathed except that they are shifted somewhat from where they would have been had there been no interaction.

This strange behavior of the solitary waves in the sine-Gordon equation justifies our promoting them to have a new name: solitons. Since the dynamics allows any number of solitons to propagate and "pass through each other" (albeit with a phase shift) the velocities of the solitons are observable constants of the motion. Since an initial condition could in principle be rigged to contain an arbitrary number of solitons, there are evidently an arbitrary number of conserved quantities for this equation.

In the theory of mechanics a Hamiltonian system with a sufficient number of conserved quantities (in involution with respect to each other) makes the mechanical system integrable by quadratures.

The sine-Gordon equation is an example of an infinite-dimensional integrable system.

There is also evidently a kind of "nonlinear superposition principle" for the sine-Gordon equation.

In one form it is the following: writing the sine-Gordon equation in characteristic form r = x + t, s = x − t as

$u_{rs} = \sin(u)$.

and considering the relations relating two functions u and v:

$$\frac{1}{2}(u+v)_r = a\sin\left(\frac{u-v}{2}\right); \qquad \frac{1}{2}(u-v)_s = \frac{1}{a}\sin\left(\frac{u+v}{2}\right).$$

By cross-differentiation, it follows that both

$u_{rs} = \sin(u)$, $v_{rs} = \sin(v)$.

so if u is a solution and we determine v through the first-order equations above, we get another solution of the same equation!

1.3 A Brief History of Soliton Theory

In 1834, whilst conducting experiments to determine the most efficient design for canal boats, naval engineer, John Scott Russell had what he described as a 'first chance interview with a singular and beautiful phenomenon'. Russell was the first to record a sighting of a solitary wave, and was intrigued by its unchanging form as he chased it along the Union Canal in Edinburgh. In his Report on Waves, Russell's prose elegantly describes what he calls the 'wave of translation'.

'[The wave] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height.'

Following this discovery, Scott Russell built a small wave tank in his back garden and made further observations of the properties of the solitary wave.

He was convinced that the solitary wave was of fundamental importance, but prominent nineteenth and early twentieth century scientists thought otherwise.

In fact, they hotly debated the existence of the solitary wave Scott Russell had described, and the respected mathematician Stokes doubted that the wave could propagate without change in form.

The issue was not settled until after Scott Russell's death. In fact, soliton theory lay dormant until the appearance of the important paper by Korteweg and deVries in 1895, in which the KdV equation first appeared as a model for shallow water waves in weakly dispersive media. We will see in section 1.4, that one solution to KdV describes an invariant, hump like wave travelling at constant speed.

But it wasn't until Zabusky and Kruskal published a paper in 1965 , that the full potential of soliton theory began to emerge. Working numerically on the Fermi-Pasta-Ulam problem, in which a system of N−1 identical masses are connected in a 1-dimensional lattice with N connecting springs, Zabusky and Kruskal recovered the KdV equation. They found (numerically) that KdV solitary waves could be shown to remain shape invariant upon interaction, undergoing only a phase shift and interacting elastically. Since this behaviour is more reminiscent of particle collisions than of wave interactions, Zabusky and Kruskal coined the word soliton, whose suffix on highlights the particle like nature of a solitary wave.

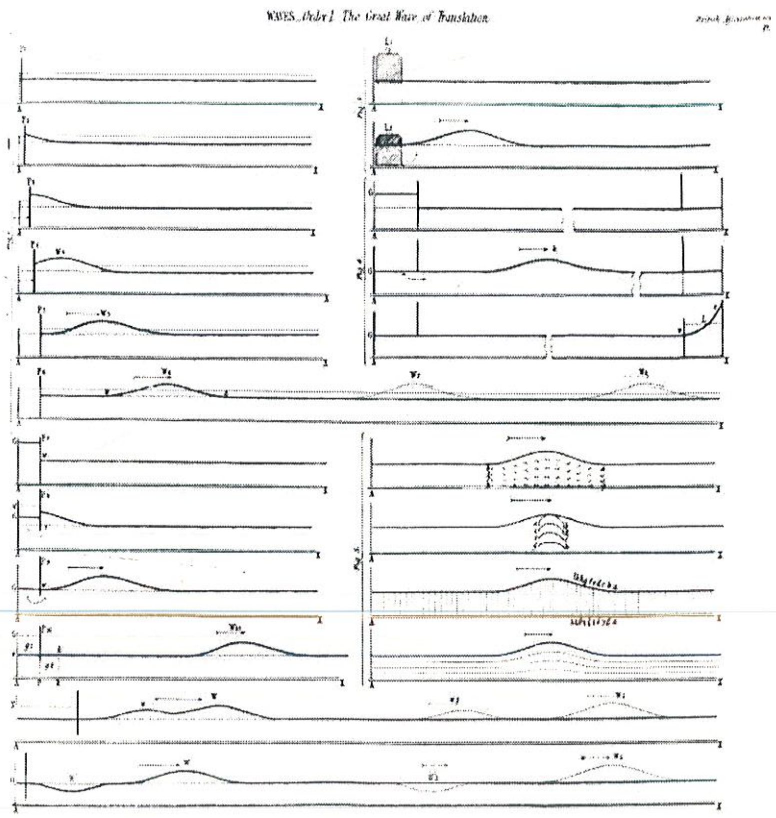

Figure 1.1: Russell's Waves of Translation

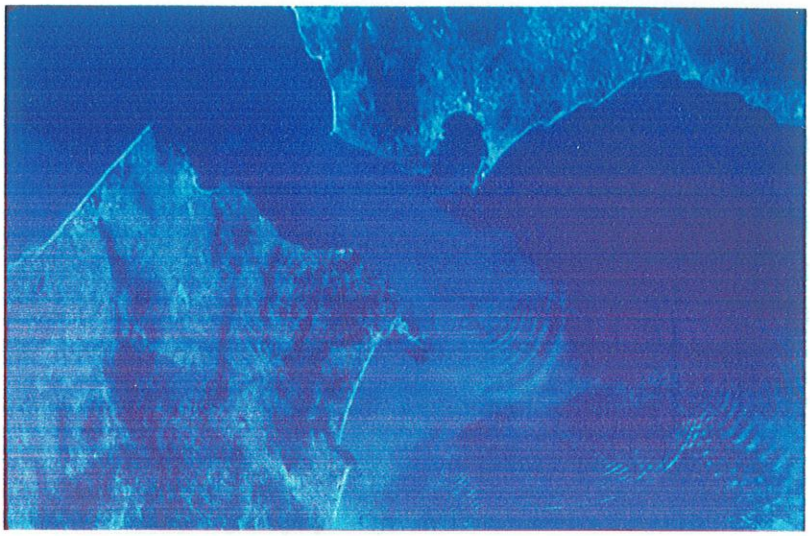

Figure 1.2: Solitons in the Strait of Gibraltar

The Zabusky and Kruskal discovery had revived the study of the solitary wave.

A seminal series of papers by Gardiner et al in 1967 paved the way for the inverse scattering transform to be developed. This technique finally showed that an analytical solution was possible for solitons

Zabusky and Kruskal had investigated the invariance properties of solitons numerically, but a slither of doubt had remained, as small oscillations appeared in the results. Without an analytical solution, they had been unable to tell if these oscillations were due to numerical error or instability, or if they were a property inherent in the solution itself. The inverse scattering transform finally put the question of soliton invariance firmly to rest, showing that solitons remain shape invariant as they travel, and interact elastically after a collision.

In 1972 soliton theory had its big break, when Zakharov and Shabat showed that inverse scattering could be generalised to other soliton equations, not just the KdV equation.

A direct method for solving soliton equations was invented by Hirota. Sato's theory followed in the late seventies and early eighties and was able to explain the underpinnings of Hirota's method and the properties of soliton equations from a unified viewpoint. In its full description, Sato's theory seems at once, arcane, beautiful and powerful.

There are various ways to formulate soliton theory, encompassing many areas of mathematics, from applied mathematics to complex analysis, group theory, geometry and field theory (in physics). Miwa, Date, Jimbo and Kashiwara have extended Sato's theory using the framework of bosonic and fermionic Fock spaces from quantum field theory. Today, soliton theory has multiple applications in physics and biology. Due to their invariance properties, solitons are of great potential use in light wave communication technology. Solitons can be observed in the ocean, often in the narrow Strait of Gibraltar as seen in Figure 1.2.

Solitons are also created in the atmosphere, forming cloud rolls called Morning Glory - and also in tidal bores, where they form solitary waves.

Figure 1.3: Morning Glory Cloud Solitons

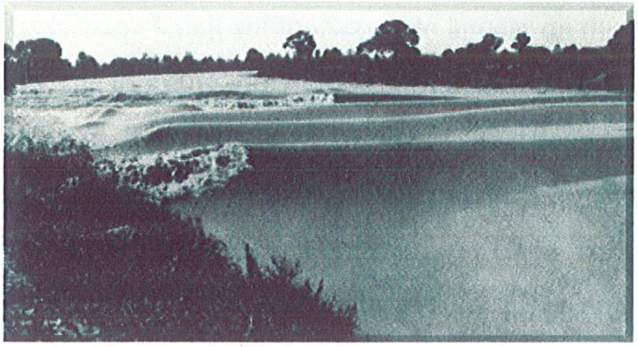

Figure 1.4: Bore Solitons

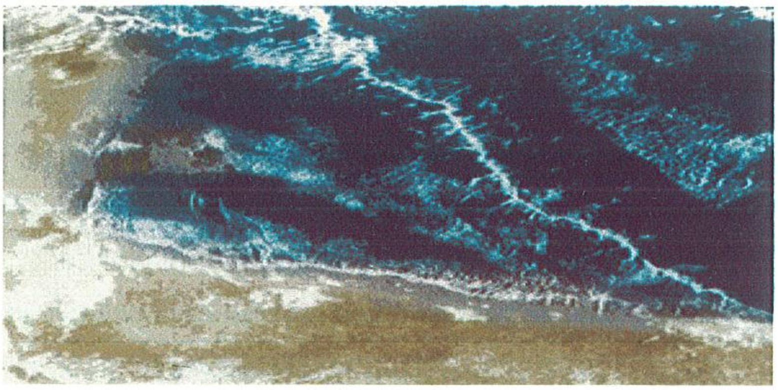

Figure 1.5: Morning Glory Satellite Image

The ocean solitons shown in Figure 1.2 & Figure 1.5 are triggered by inflowing Atlantic water accelerated by its passage through the narrow strait and across a sill at its entrance. If a body of water consists of an upper layer and a denser lower layer, the interface between the layers can undergo wave motion, causing an internal wave inside the body of water. At the interface between the fresher, lighter Atlantic water and more saline, denser Mediterranean water, internal wave sets are generated. Each soliton develops independently as the tidally driven, eastward flowing water is compressed, and upwelling results causing small soliton waves to appear on the surface.

The morning glory cloud soliton is unique to the Gulf of Carpenteria in Northern Australia. Like ocean solitons, the cloud rolls are generated by atmospheric layering caused by the interaction of nocturnal sea breezes over Cape York Peninsula. The cloud solitons have been observed to propagate for at least 300kms and can retain their shapes for several hours. Porter and Smyth have argued that the cloud rolls can be modelled by the forced Benjamin Ono equation.

Similarly, tidal bores (figure 1.4) are waves which may travel for miles down a narrow river or stream. In this case the wave maintains its narrow focus because the sides and bottom of the river provide just enough of a nonlinear compressing force to counter the normal dispersion of the wave.

In general solitons may propagate in many different media, ranging from infinitesimal, to meteorological and astrophysical. There are certain conditions in space in which quarks may form a soliton.

Boojums, compactons, fluxons, kinks, antikinks, and twists are other names for types of solitons predicted in everything from supercooled fluids to empty space.

1.4 What Exactly is a Soliton?

Instead of starting from a mathematical definition of a soliton, we list the observable properties of solitons and note that the solutions obtained analytically obey these.

We define a soliton as a wave that has the following properties:

- Soliton equations come in infinite hierarchies, with a high degree of symmetry, which are mostly integrable. This has been found by observation. Not only are solitons symmetrical in cross section about their point of maximum amplitude, but their governing equations have a great deal of intrinsic symmetry.

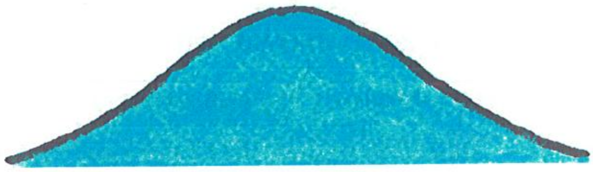

Figure 1.8: A Soliton in Cross-section

- Solitons scatter elastically, retaining their original speed and shape after a collision. The shape invariance property of a solitary wave occurs in systems displaying both nonlinearity and dispersion. Firstly, we will show that the speed of a soliton is proportional to its height. Secondly, we will show that for KdV type equations if more than one soliton is present in a given solution then these solitons must be of different heights and travel at different speeds.
- Solitons interact non-linearly. When solitons collide and overlap, the linear superposition principle does not apply, as solitons are governed by non-linear evolution equations. There actually exists a non-linear analog of the superposition principle, the Backlund transformation, which is essentially a transformation connecting two soliton solutions.

Solitons are a localised wave and it is smooth and continuous. Let the amplitude of a soliton be u. We assume that $u \in C^{\infty}$. A soliton vanishes sufficiently rapidly asymptotically, such that we may impose the following boundary condition.

$$u(x, t) \to 0 \text{ as } |x| \to \pm\infty \qquad (1.3.1)$$

We assume that 1.3.1 occurs sufficiently rapidly, by which we define the following.

Let u be a member of the Schwartz class, S(R). This means that u is infinitely differentiable and u and its derivatives decay faster than any power of $|x|^{-1}$ as $|x| \to \pm\infty$.

It is possible to solve KdV without this condition in terms of Abelian integrals, but hereafter we will only consider solutions which have the boundary conditions specified above, as we are interested only in the soliton solutions.

The KdV Equation and Other Miracles

As mentioned earlier, KdV is a celebrated soliton equation derived by Korteweg and de Vries in 1895 . It is derived using fluid dynamics to approximate a model for shallow water waves in weakly dispersive media. The KdV equation can be expressed as

$$u_t = u_{xxx} + uu_x \qquad (1.4.2)$$

This is a nonlinear evolution equation for u(x, t), where u is the amplitude of the wave. The u_{xxx} term is a dispersion term and the uu_x term is a non-linear term.

Scale Invariance

KdV is scale invariant, which means we can set the value of the coefficients of its three terms to arbitrary values by choosing appropriate scaling transforms for x, t and u. These coefficients have been set arbitrarily to one in 1.4.2.

Consider the arbitrary scaling transform from u to $\hat{u}$,

$$x \to a\hat{x} \qquad t \to b\hat{t} \qquad u \to c\hat{u} \tag{1.4.3}$$

where a, b, c are arbitrary constants. Using the chain rule we find

$$u_t = \frac{\partial}{\partial t}c\hat{u} = \frac{\partial}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t}c\hat{u} = \frac{c}{b}\frac{\partial \hat{u}}{\partial \hat{t}}$$

$$u_x = \frac{\partial}{\partial x}c\hat{u} = \frac{\partial}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial t}c\hat{u} = \frac{c}{a}\frac{\partial \hat{u}}{\partial \hat{x}}$$

$$u_{xxx} = \frac{\partial^2}{\partial x^2}\frac{c}{a}\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{c}{a^3}\frac{\partial \hat{u}}{\partial \hat{x}}$$

and substitution into 1.4.2 gives

$$\frac{c}{b}u_t = \frac{c}{a^3}u_{xxx} + \frac{c^2}{a}uu_x$$

$$\Rightarrow \frac{1}{b}u_t = \frac{1}{a^3}u_{xxx} + \frac{c}{a}uu_x$$

Thus, choosing arbitrary values of a, b and c, we may re-scale each of the coefficients separately to any arbitrary value.

Galilean Invariance

The KdV solution is invariant under a Galilean transformation to any inertial reference frame. The most general Galilean transformation is

$$\hat{x} \to x - v_c t \tag{1.4.4}$$

where v_c is the constant velocity of the new reference frame. Transforming 1.4.2 in this fashion, we find that the KdV equation is retained, but with u shifted by a constant, λ.

$$\hat{u} = u - \lambda$$

In their important series of paper in 1967, Gardiner et al showed that it is possible to re-construct the KdV variable u at any time, t> 0, given initial data that approaches a constant, μ, sufficiently rapidly as x → ±∞. By virtue of Galilean invariance, μ can be set to an arbitrary value, including zero.

Lemma :

Solutions of KdV that decay sufficiently rapidly are uniquely determined by initial data.

Proof :

Let us assume that a solution to KdV exists, and that the initial data takes the form of a soliton such that 1.4.1 is satisfied at all times, t.

Let us write KdV such that

$$u_t + u_{xxx} + 6uu_x = 0 \tag{1.4.5}$$

Let u and v be soliton solutions to KdV in 1.4.5. Let w be the difference between these solutions, such that w = u − v. We will assume that u and v satisfy the same initial data

$$u(x, 0) = v(x, 0) = f(x) \tag{1.4.6}$$

and we ask whether this initial data admits unique solutions under the evolution of KdV.

Substituting w into 1.4.5 gives

$$w_t + 6uw_x + 6wv_x + w_{xxx} = 0$$

Multiplying this equation by w and integrating over all x , we obtain

$$\int_{-\infty}^{\infty} (ww_t + 6wuw_x + 6w^2v_x + ww_{xxx}) = c_1$$

Where c_1 is an arbitrary constant of integration. Now, integrating by parts and using the chain rule and the asymptotic boundary condition in 1.4.1, the last term vanishes since

$$\begin{aligned}\int_{-\infty}^{\infty} w w_{xxx} dx &= [w_{xx} w]|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w_x w_{xx} dx \\ &= 0 - \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\frac{1}{2}(w_x)^2 dx \\ &= 0\end{aligned}$$

Then, integrating by parts on the second term (with the chain rule and asymptotic boundary condition), we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} w^2 dx + \int_{-\infty}^{\infty} (6v_x - 3u_x)\, w^2 dx = 0 \tag{1.4.7}$$

Where we have used the fact that u, v and w and their derivatives vanish as x ⟶ ±∞, to set $c_1 = 0$ for a non-trivial solution. After integration, this becomes an ODE for t. Since we are trying to show that w = 0, we don't need to perform the actual integration. Instead, we use an inequality. Let

$$\beta(t) = \frac{1}{2}\int_{-\infty}^{\infty} w^2 dx < \infty$$

and let M be

$$M = \sup |6v_x - 3u_x| < \infty$$

where the supremum is the least upper bound, such that no member of $|6v_x - 3u_x|$ exceeds it. Now, we can write the ode in 1.4.7 as an inequality without the need to integrate explicitly

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} w^2 dx \leq -M\int_{-\infty}^{\infty} w^2 dx$$

$$\beta(t) \leq 2c_2(x)e^{-Mt} \tag{1.4.8}$$

where c_2(x) is a constant of integration with respect to t.

From 1.4.6, we have w(x, 0) = 0 and thus, we have the initial condition β(0) = 0. Using this initial condition on 1.8, we find that c_2(x) = 0, such that β(t) ≤ 0. But, since w^2 is an even function, then β(t) ≥ 0.

This implies that β (t) =0 ∀ t ≥ 0 and w = 0, so u = v ∀ t ≥ 0. Therefore, only one unique solution is possible from an arbitrary initial soliton condition.

It is possible to find the simplest soliton solution for KdV by direct integration. Consider the KdV equation in 1.4.5

$$u_t = 6uu_x + u_{xxx}$$

Let us assume that the soliton solution takes the form of a travelling wave. We make the change of variables to a Galilean reference frame

$$u(x, t)= u(x + \lambda t)= u(z) \qquad (1.4.9)$$

where λ is the speed of the wave. Once again, since we are looking for a soliton solution, we assume that $u \in C^{\infty}$ and we have the asymptotic boundary condition given in 1.4.1 Substituting 1.4.9 into KdV in 1.4.2, and using the chain rule, we have

$$-\lambda u_z + 6\left(\frac{1}{2}u^2\right)_z + u_{zzz} = 0$$

Integrating this with respect to z gives

$$-\lambda u + 3u^2 + u_{zz} = c$$

where c is an arbitrary constant of integration. As x →∞, then z →∞, and thus from the asymptotic boundary condition, c = 0. We can integrate once more by multiplying by u_z, we can make use of the chain rule

$$\begin{aligned} -\lambda u u_z + 3u^2 u_z + u_{zz} u_z &= 0 \\ -\lambda\left(\frac{1}{2}u^2\right)_z + 3\left(\frac{1}{3}u^3\right)_z + \left(\frac{1}{2}u_z^2\right)_z &= 0 \\ \Longrightarrow \frac{-\lambda}{2}u^2 + u^3 + \frac{1}{2}u_z^2 &= 0 \end{aligned}$$

The constant of integration has vanished, once again, courtesy of the boundary condition.

We may now re-arrange this for u_z and perform one final integration by solving a first order ODE

$$\begin{aligned} u_z &= \sqrt{\lambda u^2 - 2u^3} \\ \int \frac{1}{\sqrt{\lambda u^2 - 2u^3}} du &= \int dz \\ &= z + \delta_0 \end{aligned} \qquad (1.4.10)$$

where δ_0 is a constant of integration.

Now the LHS of 1.4.10 can be integrated by substitution

$$LHS = \int \frac{du}{u\sqrt{\lambda - 2u}}$$

and making the substitution $v = \sqrt{(\lambda - 2u)}$ to remove the square root, we have

$$LHS = \int \frac{-v}{-\frac{1}{2}(v^2 - \lambda)v} dv = \int \frac{2}{v^2 - \lambda} dv \qquad (1.4.11)$$

And 1.4.10 becomes

$$\begin{aligned} 2\int \frac{\sqrt{\lambda}}{v^2 - \sqrt{\lambda}^2} dv &= \sqrt{\lambda}(z + \delta_0) \\ -\text{arctanh}\left(\frac{v}{\sqrt{\lambda}}\right) &= \frac{\sqrt{\lambda}}{2}(z + \delta_0) \\ \frac{v}{\sqrt{\lambda}} &= \tanh\left(-\frac{\sqrt{\lambda}}{2}(z + \delta_0)\right) \\ \sqrt{\frac{\lambda - 2u}{\lambda}} &= -\tanh\left(\frac{\sqrt{\lambda}}{2}(z + \delta_0)\right) \\ \lambda - 2u &= \lambda \tanh^2\left(\frac{\sqrt{\lambda}}{2}(z + \delta_0)\right) \\ u &= -\frac{\lambda}{2}\left[\tanh^2\left(\frac{1}{2}\sqrt{\lambda}(z + \delta_0)\right) - 1\right] \\ u &= \frac{\lambda}{2}\text{sech}^2\left(\frac{\sqrt{\lambda}}{2}(z + \delta_0)\right) \end{aligned}$$

$$u = \frac{\lambda}{2}\text{sech}^2\left(\frac{1}{2}\left\{\sqrt{\lambda}(x + \lambda t) + \eta_0\right\}\right) \qquad (1.4.11)$$

where η_0 is an arbitrary constant.

Letting p = $\sqrt{\lambda}$, we can remove the square root signs.

$$u = \frac{p^2}{2}\text{sech}^2\left(\frac{1}{2}\left\{px + p^3t + \eta_0\right\}\right) \quad (1.4.12)$$

Now, letting $\eta = \lambda(x + \lambda t) + \eta_0$ describe a Galilean reference frame, where η_0 sets the starting position of the soliton, we can express u as

$$u = \frac{p^2}{2}\text{sech}^2(\frac{\eta}{2}) \quad (1.4.13)$$

Notice that the amplitude of the wave in 1.4.12 is directly proportional to its velocity, and increases as the velocity increases. The soliton shape remains invariant as it travels at a constant speed, keeping a hump like shape given by $\text{sech}^2(\eta/2)$ as t increases .

Figure 1.3.4 shows a plot of this function, which can be compared to the shape of an actual soliton in 1.3.4.

Note that in 1.4.9 we have assumed that the wave travels in one direction, to the left. Conversely, if we assume that the wave travels to the right, such that u(x, t)= u(x − λt)= u(z), then this amounts to changing the sign of λ in the analysis above, and in this case, 1.4.11 becomes $\int \frac{2}{v^2+\lambda}dv$ and integrates to give arctan instead of arctanh. Thus, the solution becomes oscillatory.

However, if we wish to find a KdV soliton solution travelling to the right, we can change the sign of λ by changing the sign of the co-efficient of u_t. We can do this using the scale invariance property of KdV, and thus we are also able to obtain a soliton travelling to the right with the form of 1.4.13 for a KdV evolution equation. Thus, KdV admits solitary waves travelling to the left or to the right (but not in both directions at once) for a given KdV evolution equation.

Tools & Software used

1. STATISTICAL ANALYSIS SYSTEM (SAS 9.1)
2. MAPLE
3. MS-EXCEL 2007

CHAPTER 2

DIRECT METHOD IN SOLITON THEORY

2.1 The Idea of Hirota's Method

The basic idea of Hirota's method is to construct soliton solutions using exponentials. Eventually, we will motivate this idea by showing that a soliton cannot be constructed by a linear superposition of waves. Firstly, we develop some results from the well known wave theory. We then explore the avenues of dependent variable transforms and perturbation expansions. Armed with these all of these ideas, along with what we already know about the properties of solitons, we ask how a soliton solution might be possible. Finding a few small results and analytic approximations using our varied armoury of mathematical techniques, we can then make a few educated guesses. Eventually, Hirota's calculus begins to manifest in our algebra, and we are able to motivate the D operator.

The D operator describes a new calculus with unique properties. By virtue of these properties, the ordinary perturbation technique actually leads to an exact soliton solution. To illustrate all of this, we take the most famous of the soliton equations, the KdV equation, as our example.

2.2 Dependent Variable Transformations

2.2.1 *Transforming to Bilinear Form*

Hirota's method relies on a transformation to a bilinear equation. A bilinear function is a nonlinear function of degree two. A simple example is f = xy.

More generally, a quadratic form involving the real variables $x_1, x_2, x_3 \ldots$, and associated with the matrix A = a_{ij}, is given by

Q($x_1, x_2, x_3 \ldots\ldots$) = $a_{ij}\, x_i\, x_j$

The equations we will come across will be quadratic forms but, we use the term bilinear hereafter as this is done throughout the literature.

We begin by trying to find a dependent variable transform that might help to simplify the KdV equation. However, we are unable to categorise the non-linear equations that can be linearised or simplified using a transformation. All we can do is list a few examples.

2.2.2 *Linearising Differential Equations*

The Ricatti equation, Burgers' equation and the Liouville equation are three examples of nonlinear differential equations that can be linearised.

<u>The Ricatti equation</u>

$$\frac{\partial u}{\partial t} = a(t) + 2b(t)u + u^2 \quad \text{where } u = u(t) \tag{2.1}$$

can be linearised by a rational transform

$$u = \frac{g}{f} \tag{2.2}$$

Differentiating 2.2, we have

$$u_t = \frac{g_t f - g f_t}{f^2}$$

And substitution into the Ricatti equation in 2.1 yields

$$(g_t - a(t)f - b(t)g)\,f - (f_t + b(t)f + g)\,g = 0$$

Introducing an arbitrary function λ (t), we can separate this into two linear equations in two unknowns.

$$f_t + b(t)f + g = \lambda(t)f \qquad g_t - a(t)f - b(t)g = \lambda(t)g$$

\\

Burgers' equation

$$u_t = u_{xx} + 2uu_x$$

can be linearised with the Hopf-Cole transformation

$$u = (\log f)_x = \frac{f_x}{f} \qquad (2.3)$$

Firs tly we can integrate with respect to x by introducing a potential function, w such that u = w_x

$$w_t = w_{xx} + w_x^2 + c$$

where c is an arbitrary constant of integration. This can then be linearised via a logarithmic transform

$$w = \log f \qquad (2.4)$$

suc h that

$$\frac{f_t}{f} = \frac{f_{xx}f - f_x^2}{f^2} + \frac{f_x^2}{f^2} + c$$

simplifies to

$$f_t = f_{xx} + cf$$

2.3 Beyond Linear Wave Theory

2.3.1 Linear Non-dispersive Waves

Consider the linear wave equation for a unidirectional dispersionless wave (moving to the right)

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f(x,t) = 0 \qquad (2.5)$$

where v is a constant wave speed (or phase velocity of each wave). Assuming that this wave train is periodic, the most fundamental solution is a superposition of plane waves with various wave numbers, k

$$\begin{aligned} f(x,t) &= \exp\left[i(\omega t - kx)\right] \\ &= e^{-kx}\left(A\sin\omega t + B\cos\omega t\right) \end{aligned} \qquad (2.6)$$

2.6 is chosen because each of its superposed plane wave solutions satisfies equation 2.5 along with both of the the boundary conditions at $x \rightarrow +\infty$ and $x \rightarrow -\infty$. That is, in constructing a solution via linear superposition, we add plane waves which individually satisfy the governing equation and its boundary conditions over the whole domain. We note that the exponential solution $f(x, t) = \exp[\pm(px - \Omega t)]$ also satisfies equation 2.5, but diverges at one of the boundaries. Thus, it is rejected as a solution in linear wave theory. This is an important point to note.

The relationship between the angular frequency, ω, and the phase velocity for a particular solution is called the dispersion relation. This can be found by substituting the solution into the original PDE. Here we find that substituting 2.6 into
2.5 gives the dispersion relation

$$\omega = vk \qquad (2.7)$$

This is a linear relationship between angular frequency and phase velocity, which means that each of the superposed waves travels at the same speed and the wave solution is non-dispersive. The group velocity, $\partial\omega/\partial k$, for the superposed solution (or wave packet) is the rate of change of frequency with respect to wavenumber, giving a measure of how the wave packet disperses. Here the group velocity is given by

$$\frac{\partial \omega}{\partial k} = v$$

so the group velocity is the same as the phase velocity in a non-dispersive wave, and the wave packet does not spread out as it travels.

2.3.2 Linear Dispersive Waves

If we now add a dispersion term to the governing wave equation above, we have

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \delta\frac{\partial^3}{\partial x^3}\right) f(x,t) = 0 \qquad (2.8)$$

Substituting a plane wave solution (in the form of 2.6) into 2.8, we find

$$\begin{aligned} i\omega f - vikf + \delta(ik)^3 f &= 0 \\ if(\omega - vk - \delta i k^3) f &= 0 \end{aligned}$$

and for a non-trivial solution, a non-linear dispersion relation results

$$\omega = vk - \delta k^3 \qquad (2.9)$$

so that the group velocity is given by

$$\frac{\partial \omega}{\partial k} = v - 3\delta k^2$$

Thus the group velocity and the phase velocity are different, and the wave packet spreads, changing its shape as it travels.

2.3.3 Non-linear Non-dispersive Waves

We add a non-linear term to the dispersionless wave equation

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \alpha f^m \frac{\partial}{\partial x}\right) f(x,t) \;=\; 0 \qquad (2.10)$$

which can be re-written by setting $v(f) = v + \alpha f^m$

$$\left(\frac{\partial}{\partial t} + v(f)\frac{\partial}{\partial x}\right) f(x,t) \;=\; 0 \qquad (2.11)$$

Now the wave speed will depend on f and we can express a travelling wave using the ubiquitous D'Alembert's solution

$$f(x, t) = f(x - v(f)t)$$

If $v = v(f)$ and $\alpha > 0$, then the wave will travel faster as its amplitude, f, increases. This means that the top of the wave will move faster than the base of the wave, and the wave will steepen (and eventually break).

Thus, a non-linear non-dispersive wave will exhibit steepening and does not remain invariant like a soliton.

2.3.4 Nonlinear Dispersive Waves

If we now add both a non-linear term and a dispersion term to our governing wave equation, we have

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \delta\frac{\partial^3}{\partial x^3} + \alpha f^m \frac{\partial}{\partial x}\right) f(x,t) = 0 \qquad (2.12)$$

It is known that this equation has soliton solutions that travel with unchanging shape. Here we assume that soliton solutions exist.

We have seen that non-linearity or dispersion alone do not produce soliton solutions, and now we investigate how it might be possible for 2.3.4 to do so. To this end, it is necessary for the velocities at the top and the base of the wave to be the same. We consider a reference frame moving at constant velocity, introducing the change of variables $\eta = px - \Omega t$ where $v = \Omega/p$ and p is a free parameter.

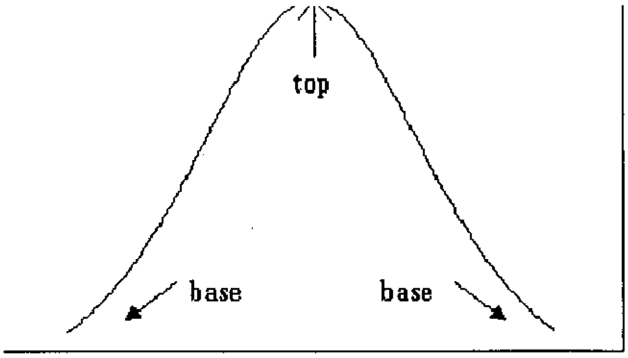

Figure 2.1: Soliton

From the form of the one soliton solution in 2.1, we know that it is symmetrical around the point of maximum amplitude and must be an even function in η.

Thus, at the top of the wave in the neighbourhood of the point of maximum amplitude, we can approximate f by a quadratic function in η. We will call this approximate solution f_{top}.

This means that the dispersion f_{xxx} will be zero in the neighborhood of the point of maximum amplitude. Thus f_{top} will satisfy the dispersionless governing equation given in 2.11 with v(f)= v + αA^m, where A is the maximum amplitude of the wave.

Meanwhile, close to the base of the wave, we may neglect the non-linear term because f is small here. We will find an approximate solution, f_{base} , in the neighborhood of the base of the soliton on each side. Now, f_{base} approximately satisfies the linear governing equation given in 2.8, where the group velocity and phase velocity are respectively

$$\begin{aligned} v_{gr} &= \frac{\partial\omega}{\partial k} = v - 3\delta k^2 \\ v_{ph} &= \frac{\omega}{k} = v - \delta k^2 \end{aligned} \qquad (2.13)$$

From this, we see that both the group and phase velocities are smaller than v for δ> 0, indicating that the top and bottom of the wave do not move at the same speed! This cannot be the soliton solution for 2.3.4.

But, we have claimed that the soliton solution does exist for 2.3.4. We must re-think our linear approximation at the base of the wave. In fact, the velocities at the base of the wave are incorrect for the soliton solution. It is certainly correct to approximate the governing equation as a linear equation in the neighbourhood of the base of the wave. The oversight arose in finding the solution to this governing equation, because we chose plane wave solutions (as in 2.6) according to the common understanding of linear theory. We rejected exponential solutions because they don't satisfy physical boundary conditions at $\eta \longrightarrow \pm\infty$ for a superposition of plane waves on the infinite domain.

But we must remember that here we are only interrogating the global solution by comparing approximations in different neighbourhoods -and the global solution is not made up by a superposition of linear waves on the infinite domain. It is nonlinear and is not subject to superposition. In fact, we may choose

$$f \approx e^{-\eta} \quad \text{as} \quad \eta \to +\infty \tag{2.14}$$

$$f \approx e^{+\eta} \quad \text{as} \quad \eta \to -\infty \tag{2.15}$$

and approximate the solution in certain neighbourhoods with localised pieces. We can't add up individual solutions over the entire domain by superposition to find the global solution, but we do know that the global solution must be asymptotic to these pieces in the associated neighbourhoods.

With this in mind, if we now include the case of exponential wave solutions at the base of the wave, (denoting them f_b(x, t))

$$f_b(x,t) = \exp\left[\pm(px - \Omega t)\right] \tag{2.16}$$

Now we obtain a single, non-linear dispersion relation from the linear equation in 2.8 and the solutions in 2.15.

$$\mp\Omega f_b \pm vpf_b + (\pm p)^3\delta f_b = 0$$

$$\Omega = vp + \delta p^3 \qquad (2.17)$$

The velocities at the top and base of the wave become

$$\begin{aligned} v_b &= \frac{\Omega}{p} = v + \delta p^2 && \text{at the base of the wave} \\ v_t &= v + \alpha A^m && \text{at the top of the wave} \end{aligned}$$

And it is now possible for these velocities to be equated at the top and base of the wave, such that, when $v_b = v_t$

$$\delta p^2 = \alpha A^m \qquad (2.18)$$

If this relation holds, then a soliton solution might still be possible, as we do not see the wave changing shape (at the top or the base) in this approximation.

2.4 Towards a Soliton Solution

In this section, we follow Hirota's idea to work towards a solution of the KdV equation. This will guide us to Hirota derivatives, and explain Hirota's motivation for inventing the direct method. The approach is partly algorithmic, and partly trial and error, but eventually the symmetry in KdV begins to emerge, producing certain patterns of derivatives. Hirota's method does not allow for a deep mathematical insight into solitons, but it does lead directly to a soliton solution.

So far, we have found that dispersive-nonlinear PDE's (such as that in 2.3.4) are candidates for soliton solutions. We found that the equation in 2.3.4 has one type of travelling wave solution that does not remain invariant in shape, and another type of travelling wave solution that looks promising on the invariance front, passing an asymptotic test requiring that the base and the top of the wave travel at the same speed. Hereafter, we will investigate the latter solution.

In order to approximate or learn more about this solution, we can compute a perturbation expansion, which is a power-series expansion in a small parameter. The discussion in section 2.3, provides an important clue for solving non-linear wave equations. We should not employ, as a first approximation, normal plane waves seen in linear theory -but rather, we should use exponential solutions.

As a first approximation, we might try to expand f in a power-series in ϵe^{η} such that

$$f(x,t) \approx \epsilon a_1 e^{\eta} + \epsilon^2 a_2 e^{2\eta} + \epsilon^3 a_3 e^{3\eta} + \dots \qquad (2.19)$$

where $\eta = px - \Omega t$ and ϵ is a small parameter.

In fact, with a few little miracles, we will see that such an expansion will actually provide an exact solution!

Following Hirota's train of thought that lead him to the direct method, let's try and guess a soliton solution to KdV using what we know already. Firstly, we know that 1.4.13 is an exact solution for a single soliton. Secondly, from section 2.3.4, we have seen that non-linear, dispersive waves admit an exponential candidate for a soliton solution. This solution also behaves asymptotically as an exponential (as in equation 2.15).

Connecting the above ideas and noting that the one soliton solution in 1.13 is a transcendental function, let's write it in terms of exponentials

$$u = \frac{p^2}{2} sech^2(\frac{\eta}{2})$$

By the definition of sech, we have,

$$sech(z) = \frac{2}{e^z + e^{-z}} = \frac{2e^z}{e^{2z}+1}$$

$$sech^2(z) = \frac{4e^{2z}}{(e^{2z}+1)^2}$$

Thus, we can express the one soliton solution in terms of exponentials

$$u_1 = \frac{p^2}{2} sech^2(\frac{\eta}{2}) = \frac{2p^2 e^\eta}{(e^\eta + 1)^2} \qquad (2.20)$$

Next, we use the idea of a dependent variable transform to exploit what we know about the one soliton solution in the form of 2.20. We try a rational dependent variable transform

$$u = \frac{g}{f} \qquad (2.21)$$

keeping in mind that

$$f = (e^\eta + 1)^2 \quad \text{and} \quad g = 2p^2 e^\eta \qquad (2.22)$$

give a known solution of KdV in the form of 2.21. Substituting 2.21 into the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (2.23)$$

where we have chosen the coefficients for algebraic convenience.

Using the quotient rule to find derivatives, we have

$$u_t = \frac{g_t f - g f_t}{f^2} \qquad u_x = \frac{g_x f - g f_x}{f^2}$$

$$u_{xxx} = \frac{g_{xxx}}{f} - \frac{3g_{xx}f_x + 3g_x f_{xx} + g f_{xxx}}{f^2} + 6\frac{g_x f_x^2 + g f_{xx} f_x}{f^3} - \frac{g f_x^3}{f^4}$$

and substituting these into the KdV equation, we find

$$\frac{g_t f - g f_t}{f^2} + 6\frac{g}{f}\left(\frac{g_x f - g f_x}{f^2}\right) + \frac{g_{xxx} f - 3g_{xx} f_x - 3g_x f_{xx} - g f_{xxx}}{f^2} \ldots$$

$$\ldots + 6\frac{f g_x f_x^2 + f g f_{xx} f_x - g f_x^3}{f^4} = 0 \qquad (2.24)$$

We will now try to decouple this equation into two equations in f and g, by setting part of it to zero.

If we set terms with denominator f^2 (the first and third terms) to zero

$$\frac{g_t f - g f_t + g_{xxx} f - 3g_{xx} f_x - 3g_x f_{xx} - g f_{xxx}}{f^2} = 0 \qquad (2.25)$$

then we find that f and g satisfy 2.25 if we change the sign of the $-3g_x f_{xx}$ term.

So, if we decouple 2.24 into two equations, 2.25 and the rest,

$$\frac{g_t f - g f_t + g_{xxx} f - 3g_{xx} f_x - 3g_x f_{xx} - g f_{xxx}}{f^2} = 0$$

$$6\frac{g}{f}\left(\frac{g_x f - g f_x}{f^2}\right) + 6\frac{f g_x f_x^2 + f g f_{xx} f_x - g f_x^3}{f^4} = 0 \tag{2.26}$$

and then add $\frac{6g_x f_{xx}}{f^2}$ to the first equation in 2.26 and minus it from the second, we find two decoupled equations which satisfy f and g in 2.22. The first equation in 2.26 becomes

$$\frac{g_t f - g f_t + g_{xxx} f - 3g_{xx} f_x + 3g_x f_{xx} - g f_{xxx}}{f^2} = 0 \tag{2.27}$$

and the second is

$$\begin{aligned}
6\frac{g}{f}\left(\frac{g_x f - g f_x}{f^2}\right) + 6\frac{f g_x f_x^2 + f g f_{xx} f_x - g f_x^3}{f^4} - \frac{6g_x f_{xx}}{f^2} &= 0 \\
\frac{6}{f^4}\left[g f (g_x f - g f_x) + f g_x f_x^2 + f g f_{xx} f_x - g f_x^3 - f^2 g_x f_{xx}\right] & \\
\frac{6}{f^4}(g_x f - g f_x)\left[g f - (f f_{xx} - f_x^2)\right] & \\
\frac{6}{f^4}(g_x f - g f_x)\left[g f - f f_{xx} + f_x^2\right] &
\end{aligned} \tag{2.28}$$

Note that since our decoupled equations, 2.27 and 2.28, are independently zero, they may be written as

$$\begin{aligned}
g_t f - g f_t + g_{xxx} f - 3g_{xx} f_x + 3g_x f_{xx} - g f_{xxx} &= 0 \\
g f - f f_{xx} + f_x^2 &= 0
\end{aligned} \tag{2.29}$$

which are bilinear equations in f and g. Thus, we have bilinearised the KdV equation using a rational transform and trial and error.

Note also that in the known one soliton solution,

$$u = \frac{2p^2 e^\eta}{(e^\eta + 1)^2}$$

the numerator is related to the derivative of the denominator, such that

$$\frac{2p^2 e^\eta}{(e^\eta + 1)} = 2p^2 \frac{\partial}{\partial \eta} log(e^\eta + 1)$$

$$\frac{\partial}{\partial \eta}\frac{2p^2 e^\eta}{(e^\eta + 1)} = \frac{2p^2 e^\eta}{(e^\eta + 1)^2} = 2p^2 \frac{\partial^2}{\partial \eta^2} log(e^\eta + 1) \tag{2.30}$$

Thus, we notice that

$$u = \frac{g}{f} \propto \frac{\partial^2}{\partial\eta^2} log(e^\eta + 1)$$
$$u \propto \frac{\partial^2}{\partial\eta^2} \log f^{\frac{1}{2}}$$
$$u \propto 2\frac{\partial^2}{\partial\eta^2} \log f \quad (2.31)$$

Now, since $\eta = px + \Omega t$ describes a Galilean reference frame and KdV is Galilean invariant, we can replace η with x in 2.31 for KdV, and we expect

$$u = 2\frac{\partial^2}{\partial x^2} \log f \quad (2.32)$$

to transform KdV in into bilinear form.

2.5 The D Operator (Hirota Derivatives)

In the above discussion, we noticed that in transforming KdV into a bilinear form, patterns of derivatives appeared with alternating minus signs. This lead Hirota to introduce the D-operator , or, Hirota Derivatives and develop the associated calculus.

2.5.1 Introducing a New Calculus

Given two functions of a single variable x, we can write

$$f(x+y)g(x-y) = \sum_{j=0}^{\infty} \frac{1}{j!}(D_x^j f \cdot g)y^j \quad (2.33)$$

where the operator (f, g) $\mapsto D_x^j$ f • g is defined as the Hirota derivative of order j with respect to x. Note that D_x^j f • g is to be thought of as a single entity, and thus the D operators themselves are not to be treated as individual operators.

If we write out Taylor expansions of f(x + y) and g(x − y), multiply these together on the LHS of 2.33 -and then compare coefficients with y^j on the RHS, then we have defined all D_x^j. The single variable Taylor expansion for f(x + h) can be expressed as

$$f(x+h) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d^j f(x)}{dx^j} h^j \tag{2.34}$$

Using 2.34, the LHS of 2.33 is expanded as

$$\begin{aligned} f(x+y)g(x-y) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(x)}{dx^n} y^n \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m g(x)}{dx^m} (-y)^m \\ &= \left(f + f_x y + \frac{f_{2x} y^2}{2} + \frac{f_{3x} y^3}{6} + \dots \right) \left(g - g_x y + \frac{g_{2x} y^2}{2} - \frac{g_{3x} y^3}{6} + \dots \right) \end{aligned}$$

By collecting combinations of products m and n terms which give terms in each j index (j $\in Z^+$), this can be written in the form

$$= \sum_{j=0}^{\infty} \frac{1}{j!} (C_j) y^j$$

where C_j are the coefficients of y_j.

We can write out the first few terms.

The y^0 coefficient is given by fg. The y^1 co-efficient is given by $-fg_x$ and $f_x g$. The coefficient of y^2 will be given by the combinations $\frac{1}{2} f_{xx} g$, $\frac{1}{2} f g_{xx}$, $-f_x g_x$ and so on...

So the LHS of 2.33 becomes

$$f(x+y)g(x-y) = fg\frac{y^0}{0!} + (f_x g - f g_x)\frac{y}{1!} + (f_{2x} g - 2 f_x g_x + f g_{2x})\frac{y^2}{2!} + \dots$$

and matching coefficients of $y^n/_{n!}$ with the RHS of 2.33, we can write down the first few single variable Hirota derivatives

$$\begin{aligned} D_x f \cdot g &= f_x g - f g_x \\ D_x^2 f \cdot g &= f_{xx} g - 2 f_x g_x + f g_{xx} \\ D_x^3 f \cdot g &= f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx} \\ D_x^4 f \cdot g &= f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{xxxx} \end{aligned} \tag{2.35}$$

For functions of many variables, f($x_1 + y_1, x_2 + y_2$,...) and g($x_1 - y_1, x_2 - y_2$...), we have

$$f(x_1 + y_1, x_2 + y_2 \dots) g(x_1 - y_1, x_2 - y_2 \dots) = e^{y_1 D_1 + y_2 D_2 + y_3 D_3 + \dots} f \cdot g \tag{2.36}$$

where D_1 is the Hirota derivative with respect to x_1. The many variable Taylor expansion of f $(x_1 + h_1, x_2 + h_2 \dots)$ can be expressed as

$$f(x_1+h_1,...,x_n+h_n)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_{k=1}^{n}h_k\frac{\partial}{\partial x_k}\right)^j f(x_1,x_2,...,x_n) \tag{2.37}$$

Writing out the LHS of 2.36 using 2.37 and comparing coefficients of $(y_k)^j$ with the RHS defines all of the Hirota Derivatives. For example, for the 2 variable case f = f(x, t) and g = g(x, t), we have

$$f(x+y,t+s)g(x-y,t-s)=e^{D_x y+D_t s}f\cdot g \tag{2.38}$$

Expanding out the LHS of 2.38 using 2.37, we have

$$\begin{aligned} f(x+y,t+s) &= \sum_{j=0}^{\infty}\frac{1}{j!}\left(y\frac{\partial}{\partial x}+s\frac{\partial}{\partial t}\right)^j f \\ &= f+(yf_x+sf_t)+\frac{1}{2}\left(y^2f_{xx}+2syf_{xt}+s^2f_{tt}\right)+... \end{aligned} \tag{2.39}$$

$$\begin{aligned} g(x-y,t-s) &= \sum_{j=0}^{\infty}\frac{1}{j!}\left(-y\frac{\partial}{\partial x}-s\frac{\partial}{\partial t}\right)^j g \\ &= g-(yg_x+sg_t)+\frac{1}{2}\left(y^2g_{xx}+2syg_{xt}+s^2g_{tt}\right)-... \end{aligned} \tag{2.40}$$

and the RHS of 2.38 can be expanded in a Taylor series also

$$\begin{aligned} e^{D_x y+D_t s}f\cdot g &= \sum_{n=0}^{\infty}\frac{1}{n!}(yD_x+sD_t)^n \\ &= \left(1+yD_x+sD_t+\frac{1}{2}y^2D_x^2+\frac{1}{2}s^2D_t^2+ysD_xD_t+...\right)f\cdot g \end{aligned} \tag{2.41}$$

To obtain a definition of $D_xD_tf\bullet g$, for example, we match coefficients of ys on the LHS and RHS of 2.38. The combinations of products from 2.39 and 2.40 that give ys coefficients on the LHS of 2.38 are $-f_xg_t-f_tg_x$ and $f_{xt}g+fg_{xt}$ .The co-efficient of ys on the RHS of 2.38 is D_xD_t from 2.41 . Hence we can define $D_xD_tf\bullet g$ and $D_xD_tf\bullet f$.

$$D_xD_tf\cdot g=-f_xg_t-f_tg_x+f_{xt}g+fg_{xt} \tag{2.42}$$

$$D_xD_tf\cdot f=-2f_xf_t+2f_{xt}f \tag{2.43}$$

Once again, we notice that Hirota derivatives are the same as normal derivatives of products (using the Leibnitz rule) except for the alternating minus signs.

In fact, 2.36 would define normal derivatives operating on products if there were no minus signs in the function g.

One way to write the Leibnitz rule for normal derivatives is

$$D_t^m D_x^n a(x,t) \cdot b(x,t) = \frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n} a(x+y,t+s)b(x-y,t-s)|_{y=0,s=0} \quad (2.45)$$

where m, n ∈ Z∗ , and we note that Hirota derivatives may also be defined as

$$D_x^n a(x) \cdot b(x) = \frac{\partial^n}{\partial y^n} a(x+y)b(x-y)|_{y=0} \quad (2.46)$$

where m, n ∈ Z∗ .

Writing out 2.45 for the case of a function of one variable, we can express the Hirota derivative as

$$D_x^n a(x) \cdot b(x) = \frac{\partial^n}{\partial y^n} a(x+y)b(x-y)|_{y=0} \quad (2.46)$$

Finally we note an important property of Hirota derivatives that normal derivatives do not have. Notice that

$$D_x^n f \cdot f = 0 \quad \text{for n odd} \quad (2.47)$$

It is easy to see this happening by direct computation. From 2.36, we have

$$\begin{aligned} D_x f \cdot f &= f_x f - f f_x = 0 \\ D_x^2 f \cdot f &= f_{xx} f - 2 f_x f_x + f f_{xx} \neq 0 \\ D_x^3 f \cdot f &= f_{xxx} f - 3 f_{xx} f_x + 3 f_x f_{xx} - f f_{xxx} = 0 \end{aligned}$$

... and so on. So from the pattern of derivatives in the Liebnitz rule and alternating minus signs in Hirota derivatives, we can see that every second Hirota derivative will be zero.

In fact, we can see from the definition in 2.33, that swapping the order of f and g in the Hirota derivative changes which terms the minus signs appear in. (A minus sign will appear where g is differentiated an odd number of times. Combine this with the pattern of derivatives in the Leibnitz rule). Thus, we have

$$D_x^n g \cdot f = (-1)^n D_x^n f \cdot g \tag{2.48}$$

and for g = f, we have $D_x^n f \bullet f = (-1)^n D_x^n f \bullet f$, which implies 2.47.

So, every second single variable Hirota derivative is zero, which implies that there might be more types of solutions to equations written in terms of Hirota derivatives than normal differential equations. This is a special property of soliton equations and we will see that it is this property that allows us to find an exact soliton solution in the case of Hirota's method.

2.5.2 Hirota Derivatives and the Exponential Identity

Writing the RHS of 2.33 in the form of an exponential, we have

$$\begin{aligned} a(x+y)b(x-y) &= \sum_{j=0}^{\infty} \frac{1}{j!}(D_x^j a \cdot b) y^j \\ &= e^{yD_x} a(x) \cdot b(x) \end{aligned} \tag{2.49}$$

This is sometimes called the exponential identity for Hirota derivatives and is nothing more than their definition. We can use 2.49 to prove the following useful logarithmic identity

$$2\cosh\left(y\frac{\partial}{\partial x}\log f\right) = \log\left[\cosh\left(yD_x\right) f \cdot f\right] \tag{2.50}$$

This identity will make life more pleasant when bilinearising the KdV equation with the logarithmic transform in 2.60.

Proof :

Firstly, notice that using 2.46 we can express the RHS of 2.49 as

$$e^{yD_x} a(x) \cdot b(x) = \exp\left(y\frac{\partial}{\partial \delta}\right) a(x+\delta) b(x-\delta)|_{\delta=0}$$

where δ is a dummy variable. Thus, we have

$$a(x+y)b(x-y) = \exp\left(y\frac{\partial}{\partial \delta}\right) a(x+\delta)b(x-\delta)|_{\delta=0} \tag{2.51}$$

Using 2.51 and setting f = f(x), a = log f and b = 1, we find that the RHS of 2.50 may be written as

$$
\begin{aligned}
RHS &= \left[\exp\left(y\frac{\partial}{\partial x}\right) + \exp\left(-y\frac{\partial}{\partial x}\right)\right]\log f \\
&= \log f(x+y) + \log f(x-y) \\
&= \log\left[f(x+y)f(x-y)\right]
\end{aligned}
$$

But, this is equal to the LHS of 2.50 since

$$
\begin{aligned}
LHS &= \log\left[\cosh\left(yD_x\right) f \cdot f\right] \\
&= \log\left[\frac{1}{2}\left(e^{yD_x} + e^{-yD_x}\right) f \cdot f\right]
\end{aligned}
$$

But using 2.49,

$$
\begin{aligned}
e^{yD_x} f \cdot f &= f(x+y)f(x-y) \\
e^{-yD_x} f \cdot f &= f(x-y)f(x+y)
\end{aligned}
$$

so LHS = log [f(x + y)f(x − y)] = RHS and the proof of 2.49 is complete.

2.5.3 The D_z operator

We wish to use Hirota derivatives to re-write the KdV equation in bilinear form using the logarithmic transform in 2.60. The KdV equation is a function of two variables, and so far, we have been proving results in one variable only. We have done this for simplicity, and now we take advantage of the linearity of differential operators (and thus the linearity of Hirota derivatives).

We define the D_z operator as

$$
D_z^n = (D_t + \epsilon D_x)^n \qquad (2.52)
$$

$$
\text{with } \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial x}
$$

D_z^n is a linear combination of (linear) differential operators such that

$$
(D_t + \epsilon D_x)^n a \cdot b = D_t a \cdot b + n\epsilon D_t^{n-1} D_x a \cdot b + \cdots + \epsilon^n D_x^n a \cdot b \qquad (2.53)
$$

We may use 2.53 (the binomial theorem) to calculate products of D operators. For example, $3D_t D_x^2 a \bullet b$ is the co-efficient of $\in^2$ in $(D_t + \in D_x)^3 a \bullet b$.

Because the D_z operator acts linearly, we think of it as an operator of a single variable z, where z is defined by the linear transformation z = t + ∈x. All of our previous results for single variable D operators will hold for D_z. For example, we can write 2.50 in terms of D_z

$$2\cosh\left(y\frac{\partial}{\partial z}\log f\right) = \log\left[\cosh\left(yD_z\right) f \cdot f\right] \tag{2.54}$$

where y is a scalar and f = f(z).

2.5.4 Fundamental Formulae for the Logarithmic Transform

Taylor expanding each side of 2.54 with respect to y and collecting powers of y, we are now able to derive some fundamental formulae for the logarithmic transform.

On the LHS we have

$$2\cosh\left(y\frac{\partial}{\partial z}\right)\log f = 2\left(1+\frac{y^2}{2!}\frac{\partial^2}{\partial z^2}+\frac{y^4}{4!}\frac{\partial^4}{\partial z^4}+\ldots\right)\log f \tag{2.55}$$

We write a Taylor expansion for the RHS in the form

$$\sum_{n=0}^{\infty}\frac{y^n}{n!}f^{(n)}(0)$$

where $f^{(n)}(y) = \frac{\partial^n f}{\partial y^n}$ and $f(y) = \log\left[\cosh\left(yD_z\right)f \cdot f\right]$

We make use of

$$\frac{\partial}{\partial y}\log g(y) = \frac{g'(y)}{g(y)}$$

and use the quotient rule whilst keeping operators and the functions they act on intact. For example, the first few derivatives are

$$f'(x) = \frac{D_z \sinh\left(yD_z\right)f \cdot f}{\cosh\left(yD_z\right)f \cdot f} \quad \Rightarrow \quad f'(0) = 0$$

$$f''(x) = \frac{D_z^2\left[\cosh(yD_z)f \cdot f\right]^2 - D_z^2\left[\sinh(yD_z)f \cdot f\right]^2}{\left[\cosh(yD_z)f \cdot f\right]^2} \quad \Rightarrow \quad f''(0) = \frac{D_z^2 f \cdot f}{f^2}$$

Equating terms on each side in powers of y and using 2.52, we have

$$2\frac{\partial^2}{\partial x^2}\log f = \frac{D_x^2 f \cdot f}{f^2} \tag{2.56}$$

$$2\frac{\partial^2}{\partial x \partial t}\log f = \frac{D_x D_t f \cdot f}{f^2} \tag{2.57}$$

$$2\frac{\partial^4}{\partial x^4}\log f = \frac{D_x^4 f \cdot f}{f^2} - 3\left(\frac{D_x^2 f \cdot f}{f^2}\right)^2 \tag{2.58}$$

2.6 Hirota's Bilinear Equation

Hirota's bilinear form is defined as

$$P(D)\tau \cdot \tau = 0 \qquad (2.59)$$

Where P (D) is a polynomial in the Hirota D-operator. This equation is bilinear in the function τ.

2.6.1 Bilinearising the KdV Equation

Using the D operator, we may now employ the dependent variable transform motivated in section 2.4,

$$u = 2(\log \tau)_{xx} \qquad (2.60)$$

to transform the KdV equation into Hirota's bilinear form .

Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (2.61)$$

Using the chain rule, we re-write this as

$$u_t + 3(u^2)_x + u_{xxx} = 0 \qquad (2.62)$$

Introducing the potential w, defined by

$$u = w_x \qquad (2.63)$$

we may integrate 2.62 once to give

$$w_t + 3(w_x)^2 + w_{xxx} = c \qquad (2.64)$$

where c is an arbitrary constant of integration.

We now make the dependent variable transform

$$w = 2(\log f)_x \qquad (2.65)$$

and 2.64 becomes

$$2(\log f)_{xt} + 3\left[2(\log f)_{xx}\right]^2 + 2[\log f]_{4x} = c$$

Now, using 2.56 -2.58, the KdV equation becomes

$$\frac{D_x D_t f \cdot f}{f^2} + 3\left(\frac{D_x^2 f \cdot f}{f^2}\right)^2 + \frac{D_x^4 f \cdot f}{f^2} - 3\left(\frac{D_x^2 f \cdot f}{f^2}\right)^2 = c \qquad (2.66)$$

Cancelling the 2nd and 4th terms, this simplifies to

$$D_x D_t f \cdot f + D_x^4 f \cdot f = cf^2$$

$$D_x(D_t + D_x^3) f \cdot f = cf^2 \qquad (2.67)$$

Equation 2.67 is the KdV equation in Hirota Bilinear Form if we choose the constant of integration, c to be zero. We will see in the next section that we must set c = 0 to find soliton solutions.

2.7 Hirota's Method

We now have all the information required to use Hirota's method. Here we find soliton solutions of the KdV equation in bilinear form

$$D_x(D_t + D_x^3)\tau \cdot \tau = 0 \tag{2.70}$$

As in the standard perturbation technique, we will expand f as a power series in a small parameter ϵ.

$$\tau = 1 + \epsilon\tau_1 + \epsilon^2\tau_2 + \epsilon^3\tau_3 + ... \tag{2.71}$$

We require P (0) = 0 to find the soliton solutions. Without this condition, we have c = 0 in n 2.67 and we find that the ϵ^0 term in 2.71 gives P (0)1 • 1 = 0. This implies that τ in 2.71 will not be a solution. Thus, we require P (0) =0 and c =0 in order to find soliton solutions.

Substituting 2.71 into 2.70 we obtain

$$[D_x(D_t + D_x^3)]\,(1 + \epsilon\tau_1 + \epsilon^2\tau_2 + \epsilon^3\tau_3 + ...) \cdot (1 + \epsilon\tau_1 + \epsilon^2\tau_2 + \epsilon^3\tau_3 + ...) = 0$$

Matching coefficients of ϵ on both sides of this equation gives

$$\begin{array}{ll} \epsilon: & [D_x(D_t + D_x^3)]\,(\tau_1 \cdot 1 + 1 \cdot \tau_1) \\ \epsilon^2: & [D_x(D_t + D_x^3)]\,(\tau_2 \cdot 1 + \tau_1 \cdot \tau_1 + 1 \cdot \tau_2) \\ \epsilon^3: & [D_x(D_t + D_x^3)]\,(\tau_3 \cdot 1 + \tau_2 \cdot \tau_1 + \tau_1 \cdot \tau_2 + 1 \cdot \tau_3) \end{array} \tag{2.72}$$

where the coeffecient of ϵ^n is $[D_x(D_t + D_x^3)]$ acting on all combinations of $\tau_j \cdot \tau_k$ terms where j + k = n for j, k, n ∈ Z∗ .

Using 2.36 and 2.43 respectively, we can calculate the required Hirota derivatives.

For the ϵ^1 terms, we have

$$D_x^4\tau_n \cdot 1 = D_x^4 1 \cdot \tau_n = (\tau_n)_{xxxx} \tag{2.73}$$

$$D_x D_t \tau_n \cdot 1 = D_x D_t 1 \cdot a = (\tau_n)_{xt} \tag{2.74}$$

Thus, using 2.73 and 2.74, the co-efficient of ϵ^1 is

$$
\begin{aligned}
2D_xD_t\tau_1 \cdot 1 + 2D_x^4\tau_1 \cdot 1 &= 0 \\
\Rightarrow (\tau_1)_{xxxx} + (\tau_1)_{xt} &= 0 \qquad (2.75)
\end{aligned}
$$

which is a linear differential equation for τ_1. The most obvious solution to this equation is

$$\tau_1 = e^{\eta_1} \qquad \eta_1 = p_1x + \Omega_1t + \eta_1^0 \qquad (2.76)$$

where η_1^0 is an arbitrary phase constant for a soliton, which allows an initial condition to be set. We also have $\Omega_1 + P_1^3 = 0$ from the dispersion relation in 2.17. Note that we have chosen an exponential solution as discussed in relation to equation 2.19.

Similarly, the coefficient of ϵ^2 in 2.72 can be re-arranged in terms of τ_1

$$
\begin{aligned}
-\left[D_x(D_t + D_x^3)\right]\tau_1 \cdot \tau_1 &= \left[D_x(D_t + D_x^3)\right](\tau_2 \cdot 1 + 1 \cdot \tau_2) \\
&= 2D_xD_t\tau_2 \cdot 1 + 2D_x^4\tau_2 \cdot 1 \\
&= 2(\tau_2)_{xxxx} + 2(\tau_2)_{xt}
\end{aligned}
$$

where we have used 2.73 and 2.74 to simplify the RHS. We can now substitute the solution for τ_1 into the LHS to obtain

$$-\left[D_x(D_t + D_x^3)\right]e^{\eta_1} \cdot e^{\eta_1} = 2(\tau_2)_{xxxx} + 2(\tau_2)_{xt} \qquad (2.77)$$

Now, making use of 2.45, it is easy to compute the action of D-operators on arbitrary exponentials, e^{η_j} and e^{η_k}, where j, k $\in$ N

$$D_x^mD_t^ne^{\eta_j} \cdot e^{\eta_k} = (p_j - p_k)^m(\Omega_j - \Omega_k)^ne^{\eta_j+\eta_k} \qquad (2.78)$$

We note that when j = k

$$D_x^mD_t^ne^{\eta_j} \cdot e^{\eta_j} = (p_j - p_j)^m(\Omega_j - \Omega_j)^ne^{\eta_j+\eta_j} = 0 \qquad (2.79)$$

This is an important result that comes from the unique action of D-operators on exponentials, as 2.79 implies that the LHS of 2.77 will be zero. Thus, we have,

$$(\tau_2)_{xxxx} + (\tau_2)_{xt} = 0 \qquad (2.80)$$

This equation has exponential solutions, and is also satisfied by the trivial solution $\tau_2 = 0$. If we choose the trivial solution here, then we obtain a non-trivial solution for our perturbation expansion, and the expansion will truncate at τ_1, providing an exact solution to KdV

$$
\begin{aligned}
\tau &= 1 + \epsilon\tau_1 \\
&= 1 + \epsilon e^{\eta_1} \\
&= 1 + \epsilon e^{p_1x+\Omega_1t+\eta_1^0}
\end{aligned}
$$

Since $\in$ is an arbitrary small parameter, we can write $\in = e^{\alpha}$, and absorb $\in$ into the phase constant η_1^0 . Thus, the exact solution that we have found can be written as

$$\tau = 1 + e^{\eta_1} \qquad (2.81)$$

We can now verify that 2.81 is the same as the one soliton solution in 1.13. Applying the logarithmic transform, 2.60, to the solution, 2.81, we find

$$\begin{aligned} u = 2(\log \tau)_{xx} &= 2[\log(1+e^{\eta_1})]_{xx} \\ &= \frac{\partial \eta_1}{\partial x} \frac{2e^{\eta_1}}{(1+e^{\eta_1})^2} \qquad \text{using 2.30} \\ &= \frac{2p_1^2 e^{\eta_1}}{(1+e^{\eta_1})^2} \\ &= \frac{p_1^2}{2} sech^2(\frac{\eta_1}{2}) \qquad \text{using 2.20} \end{aligned}$$

which is exactly the same as the one soliton solution found by direct integration.

We now find the two soliton solution using Hirota's method. We repeat the analysis seen in the one-soliton case, finding the co-efficient of $\in$ as in 2.75

$$(\tau_1)_{xxxx} + (\tau_1)_{xt} = 0$$

Previously, we chose the simplest solution for this equation $\tau_1 = e^{\eta_1}$, . If we now choose a sum of exponentials involving η_1 and η_2, this will also be a solution. Thus, we choose

$$\tau_1 = e^{\eta_1} + e^{\eta_2} \qquad (2.82)$$

where $\eta_i = p_i x + \Omega_i t + \eta_i^0$ and the non-linear dispersion relation is $\Omega_i + p_i^3 = 0$ for $i \in Z+$. The order $\in^2$ equation is still the same as for the one soliton case

$$-\left[D_x(D_t + D_x^3)\right] \tau_1 \cdot \tau_1 = 2(\tau_2)_{xxxx} + 2(\tau_2)_{xt}$$

and substituting the new solution for τ_1 into the LHS, we obtain

$$\begin{aligned} 2(\tau_2)_{xxxx} + 2(\tau_2)_{xt} &= -\left[D_x(D_t + D_x^3)\right](e^{\eta_1} + e^{\eta_2}) \cdot (e^{\eta_1} + e^{\eta_2}) \\ &= -2\left[D_x(D_t + D_x^3)\right] e^{\eta_1} \cdot e^{\eta_2} \\ &= -2(p_1 - p_2)[\Omega_1 - \Omega_2 + (p_1 - p_2)^3] e^{\eta_1 + \eta_2} \qquad (2.83) \end{aligned}$$

where we have used 2.78 and 2.79. Thus, we may choose the solution

$$\tau_2 = a_{12} e^{\eta_1 + \eta_2} \qquad (2.84)$$

where a_{12} is an arbitrary constant. Substituting 2.84 into 2.83 and using 2.78, we can solve for a_{12}

$$a_{12} = \frac{(p1 - p2)^2}{(p1 + p2)^2} \qquad (2.85)$$

Now, we substitute the solutions for τ_1 and τ_2 into the linear differential equation for τ_3 in 2.72,

$$\left[D_x(D_t + D_x^3)\right](\tau_3 \cdot 1 + \tau_2 \cdot \tau_1 + \tau_1 \cdot \tau_2 + 1 \cdot \tau_3) = 0$$
$$\left[D_x(D_t + D_x^3)\right](1 \cdot \tau_3 + \tau_3 \cdot 1) = -\left[D_x(D_t + D_x^3)\right](\tau_2 \cdot \tau_1 + \tau_1 \cdot \tau_2)$$

and using 2.73 and 2.74 to simplify the LHS and 2.78 on the RHS we have

$$\begin{aligned}(\tau_3)_{xxxx} + (\tau_3)_{xt} &= -\left[D_x(D_t + D_x^3)\right](\tau_2 \cdot \tau_1 + \tau_1 \cdot \tau_2) \\ &= -2\left[D_x(D_t + D_x^3)\right] e^{\eta_1+\eta_2}(e^{\eta_1} + e^{\eta_2}) \\ &= 2p_2(\Omega_2 + p_2^3)e^{2\eta_1+\eta_2} + 2p_1(\Omega_1 + p_1^3)e^{2\eta_2+\eta_1} \\ &= 0 \quad \text{by virtue of the dispersion relation } \Omega_i + p_i^3 = 0\end{aligned}$$

Thus, we may choose the trivial solution for $\tau_3 = 0$ here.

The coeffecient of ϵ^4 gives the linear differential equation

$$\left[D_x(D_t + D_x^3)\right](\tau_4 \cdot 1 + \tau_3 \cdot \tau_1 + \tau_2 \cdot \tau_2 + \tau_1 \cdot \tau_3 + 1 \cdot \tau_4) = 0$$

where we have all combinations of τ in which the indices add to 4 (with $\tau_0 = 1$). Setting $\tau_3 = 0$, this becomes

$$\left[D_x(D_t + D_x^3)\right](\tau_4 \cdot 1 + \tau_2 \cdot \tau_2 + 1 \cdot \tau_4) = 0$$

Noting that the inhomogeneous term $[D_x(D_t + D_x^3)](\tau_2 \bullet \tau_2)$ vanishes once again using the D operator property in 2.79.

Thus, we have the familiar homogeneous differential equation for τ_4

$$(\tau_4)_{xxxx} + (\tau_4)_{xt} = 0$$

and once again, we may choose the trivial solution. Following this pattern, we see that we can choose all higher f_j terms to be zero, and we obtain an exact solution for the two soliton by substituting the results for f_1 and f_2 into the perturbation expansion. Thus, the two soliton solution is

$$\tau = 1 + \epsilon(e^{\eta_1} + e^{\eta_2}) + \epsilon^2 a_{12} e^{\eta_1+\eta_2}$$

and once again, we may absorb the $\in$ constants into the phase constants η_j^0 to give

$$\tau = 1 + e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1+\eta_2} \qquad (2.86)$$

This solution describes two solitons travelling to the right. Since $\eta_i = p_i x + \Omega_i t$ describes an inertial reference frame, with velocity proportional to p_i, the speed of each soliton is related to its amplitude via the parameter p_i. By virtue of the D operator property in 2.79, we see that solitons must be of different height for a non-trivial solution.

Hence, in the two soliton solution the taller soliton will travel faster than the smaller one, eventually overtaking it.

By considering the form of the solution at various times, t, it can be seen that solitons keep their shape after the interaction. The a_{12} term accounts for a phase shift after interaction, but the two solitons keep their shape and speed.

2.8 The N-Soliton Solution

Continuing on in the above fashion, we can set $\tau_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}$ to obtain the three-soliton solution, and in general setting $\tau_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \ldots + e^{\eta_n}$ gives the N-soliton solution.

Here we state the n-soliton solution for soliton equations in the form of 2.70 without proof. These are often called KdV-type soliton equations or Hirota equations. The KdV-type equation

$$P(\mathrm{D})\tau \cdot \tau = 0 \qquad (2.87)$$

has an n-soliton solution that can be expressed as

$$\tau = \sum_{\mu_1,\ldots,\mu_n=0,1} \exp\left[\sum_{i=1}^{n} \mu_i\eta_i + \sum_{i<j}^{(n)} A_{ij}\mu_i\mu_j\right] \qquad (2.88)$$

where the first summation is over all possible combinations of

$\mu_1 = 0, 1; \mu_2 = 0, 1; \ldots \mu_n = 0, 1$

and means a summation chosen over all possible pairs i, j from the set i<j

[n]=1, 2, 3....n with the condition that i<j.

Using vector notation and setting

$$\begin{aligned} \boldsymbol{\Omega}_i &= (\Omega_i, P_i, Q_i \ldots) \\ \mathbf{x}_i &= (t, x, y \ldots) \end{aligned}$$ we have, for i, j =1, 2, 3...N

$$\eta_i = \boldsymbol{\Omega}_i \cdot \mathbf{x}_i + constant \tag{2.89}$$

$$P(\boldsymbol{\Omega}_i) = 0 \tag{2.90}$$

and the phase shift is given by

$$a_{ij} = -\frac{P(\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j)}{P(\boldsymbol{\Omega}_i + \boldsymbol{\Omega}_j)} \tag{2.91}$$

The function P (D) is not arbitrary. It must satisfy the Hirota condition given by

$$\sum_{\mu_1,\ldots,\mu_n=0,1} \left(\sum_{i=1}^{N} \sigma_i \boldsymbol{\Omega}_i \right) \prod_{i<j}^{N} P(\sigma_i \boldsymbol{\Omega}_i - \sigma_j \boldsymbol{\Omega}_j) \sigma_i \sigma_j = 0 \tag{2.92}$$

This condition can be found by substituting the n-soliton solution in 2.88 into the Hirota equation in 2.87.

CHAPTER 3

SOLITONIC COLLISIONS FOR GCCKdV EQUATIONS

3.1 *Introduction*

The Kortweg-de Vries (KdV) equation,

$$u_t - 6\,u\,u_x + u_{xxx} = 0\,, \qquad (1)$$

describes the weakly non-linear long waves in the fluids and plasmas,where the wave amplitude u is a function of the scaled "space" x and scaled "time" t.

Based on a (4×4) matrix spectral problem with three potentials, certain NLEEs have been derived. Among

them, the complex coupled KdV (CCKdV) equations, are significant,

$$u_t = \frac{1}{2}u_{xxx} - 3uu_x + 3(|v|^2)_x\,,$$
$$v_t = -v_{xxx} + 3uv_x\,, \qquad (2)$$

where u is a real function and v is a complex function with respect to x and t.

We will consider the following generalized CCKdV (GCCKdV) equations:

$$u_t = \alpha\,(u_{xxx} - 6uu_x) + \beta(|v|^2)_x\,,$$
$$v_t = 2\,\alpha(-v_{xxx} + 3uv_x)\,, \qquad (3)$$

where α and β are both arbitrary real constants. When α = −1 and v = 0, System (3) reduces to Eq. (1); when α = 1/2 and β = 3, System (3) reduces to System (2); when α = 1/2, β = 3 and Im(v) = 0, System (3) reduces to the Hirota–Satsuma equation, which describes the interaction of two long waves with different dispersion relations.

Objective is to find the two-soliton solutions for System (3) and discuss their interactions.

3.2 *Bilinear Forms and One-Soliton Solution*

Through the dependent variable transformations,

$$u = -2\,(\ln f)_{xx}\,, \qquad (4)$$

$$v = \frac{g}{f}\,, \qquad (5)$$

where f(x, t) is a real function with respect to variables x and t, and g(x, t) is a complex one. The bilinear forms of System (3) turn out to be the followings,

$$D_x D_t f\cdot f - \alpha D_x^4 f\cdot f + \beta\, g\, g^* = 0\,, \qquad (6)$$

$$D_t\, g\cdot f + 2\,\alpha D_x^3\, g\cdot f = 0\,, \qquad (7)$$

where * denotes the complex conjugate and DmxDnt is the Hirota's bilinear derivative operator defined by

$$D_x^m D_t^n a\cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \times\ a(x,t)\, b(x',t')\Big|_{x'=x,\ t'=t}\,. \qquad (8)$$

We expand f, g, and g∗ into power series of a small parameter ∈ as

$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \cdots\,, \qquad (9a)$$

$$g = \epsilon\, g_1 + \epsilon^3 g_3 + \epsilon^5 g_5 + \cdots\,, \qquad (9b)$$

$$g^* = \epsilon\, g_1^* + \epsilon^3 g_3^* + \epsilon^5 g_5^* + \cdots \qquad (9c)$$

Substituting Expressions (9) into Eqs. (6) and (7) and collecting the coefficients of the same power of ∈, we have

$$\epsilon^0 : D_x D_t(1\cdot 1) - \alpha D_x^4(1\cdot 1) = 0\,, \qquad (10)$$

$$\epsilon^1 : D_t\,(g_1\cdot 1) + 2\,\alpha D_x^3\,(g_1\cdot 1) = 0\,, \qquad (11)$$

$$\epsilon^2 : 2\,D_x D_t(1\cdot f_2) - 2\,\alpha D_x^4\,(1\cdot f_2) + \beta\, g_1\, g_1^* = 0\,, \qquad (12)$$

$$\epsilon^3 : D_t\,(g_1\cdot f_2 + g_3\cdot 1) + 2\,\alpha D_x^3\,(g_1\cdot f_2 + g_3\cdot 1) = 0, \qquad (13)$$

$$\epsilon^4 : (D_x D_t - \alpha D_x^4)(f_2\cdot f_2 + 1\cdot f_4 + f_4\cdot 1) + \beta\,(g_1\, g_3^* + g_3\, g_1^*) = 0\,, \quad \cdots \qquad (14)$$

In order to obtain one-soliton solutions of System (3), we assume that

$$g_1 = e^{\xi_1}, \quad \xi_1 = \omega_1 t + k_1 x + \xi_1^0, \qquad (15)$$

where the angular frequency ω1, wave number k1, and initial phase ξ01 are the real constants. Inserting Eq. (15) into Expression (11), we can get the dispersion relation

$$\omega_1 = -2\alpha k_1^3. \qquad (16)$$

Substituting Eq. (15) into Expression (12), we can obtain

$$f_2 = \frac{\beta}{6\,\alpha\,(k_1+l_1)^2\,(k_1^2+l_1^2)}\,e^{\xi_1+\xi_1^*}. \qquad (17)$$

Correspondingly, it can be derived that g3 = f4 = g5 =f6 = ⋅⋅=0. So Expressions (9a) and (9b) can be truncated to

$$f = 1 + \epsilon^2 f_2, \qquad (18)$$

$$g = \epsilon\, g_1. \qquad (19)$$

With ∈ = 1, one-soliton solutions for System (3) in the explicit form are

$$u = \frac{\beta^2[\cosh(2\Gamma)+\sinh(2\Gamma)]}{18\alpha^2(k_1+k_1^*)^2(k_1^2+k_1^{*2})^2\left[\frac{\beta\cosh(\Gamma)}{6\alpha(k_1+k_1^*)^2(k_1^2+k_1^{*2})}+\frac{\beta\sinh(\Gamma)}{6\alpha(k_1+k_1^*)^2(k_1^2+k_1^{*2})}+1\right]^2} - \frac{\beta[\cosh(\Gamma)+\sinh(\Gamma)]}{3\alpha(k_1^2+k_1^{*2})\left[\frac{\beta\cosh(\Gamma)}{6\alpha(k_1+k_1^*)^2(k_1^2+k_1^{*2})}+\frac{\beta\sinh(\Gamma)}{6\alpha(k_1+k_1^*)^2(k_1^2+k_1^{*2})}+1\right]}, \qquad (20)$$

$$v = \frac{\cosh(\Gamma)+\sinh(\Gamma)}{\left[\frac{\beta\cosh(\Gamma)}{6\alpha(k_1+k_1^*)^2(k_1^2+k_1^{*2})}+\frac{\beta\sinh(\Gamma)}{6\alpha(k_1+k_1^*)^2(k_1^2+k_1^{*2})}+1\right]^2}, \qquad (21)$$

where $\Gamma = -2t\alpha k_1^3 + xk_1 - 2t\alpha k_1^{*3} + xk_1^* + \xi_1^0 + \xi_1^{0*}$.

In the following part of this section, the influence of α and β on the one-soliton propagation will be investigated. Suppose ξ01 = ξ0∗1 = 0, through Eqs. (20) and (21), we obtain the characteristic line of u and v as the following:

$$-2t\,\alpha\,k_1^3 + x\,k_1 - 2\,t\,\alpha\,k_1^{*3} + x\,k_1^* = 0. \qquad (22)$$

Velocity of the one soliton can be obtained as

$$V = \frac{dx}{dt} = \frac{2\alpha(k_1^3+k_1^{*3})}{k_1+k_1^*}. \qquad (23)$$

Through Eq. (23), we find that if the wave number k1 has the definite value, velocity of the one soliton will be decided only by α. For the influence of α on the one-soliton propagation, Figs. 1(a) and 2(a) show that when α = 1, velocity of the one soliton is V = 2; if α = 2, V = 4, as presented in Figs. 1(b) and 2(b). Comparing Fig. 1(a) [2(a)] with Fig. 1(b) [2(b)], we find that the change of α has the influence on the amplitude of v, but no influence on that of u. For the one-soliton propagation with β changing, comparing Fig. 2(a) with Fig. 2(c), we find that the amplitude of v obviously becomes smaller.

3.3 *Two-Soliton Solutions and Their Interactions*

Similarly, to derive two-soliton solutions, we take

$$g_1 = e^{\xi_1} + e^{\xi_2}, \quad \xi_1 = \omega_1 t + k_1 x + \xi_1^0, \quad \xi_2 = \omega_2 t + k_2 x + \xi_2^0, \tag{24}$$

where the angular frequencys ωi, wave numbers ki, and initial phases ξ0 i (i = 1, 2) are the constants. Substituting Eq. (24) into Expression (11) gives the dispersion relations

$$\omega_i = -2\alpha k_i^3 \quad (i = 1, 2). \tag{25}$$

Inserting Eq. (24) into Expression (12), we can obtain

$$f_2 = z_1\, e^{\xi_1^* + \xi_1} + z_2\, e^{\xi_2^* + \xi_1} + z_3\, e^{\xi_1^* + \xi_2} + z_4\, e^{\xi_2^* + \xi_2}, \tag{26}$$

where

$$z_1 = \frac{\beta}{6\,\alpha(k_1 + k_1^*)^2(k_1^2 + k_1^{*2})}, \tag{27}$$

$$z_2 = \frac{\beta}{6\,\alpha\,(k_1 + k_2^*)^2(k_1^2 + k_2^{*2})}, \tag{28}$$

$$z_3 = \frac{\beta}{6\,\alpha\,(k_2 + k_1^*)^2(k_2^2 + k_1^{*2})}, \tag{29}$$

$$z_4 = \frac{\beta}{6\,\alpha\,(k_2 + k_2^*)^2(k_2^2 + k_2^{*2})}. \tag{30}$$

Substituting Eqs. (24) and (26) into Expression (13), we can obtain

$$g_3 = s_1\, e^{\xi_1^* + \xi_1 + \xi_2} + s_2\, e^{\xi_2^* + \xi_1 + \xi_2}, \tag{31}$$

where

$$s_1 = \frac{\beta\,(k_1 - k_2)^2(k_1^2 + k_2^2)}{6\,\alpha\,(k_1 + k_1^*)^2(k_2 + k_1^*)^2(k_1^2 + k_1^{*^2})(k_2^2 + k_1^{*^2})}, \quad (32)$$

$$s_2 = \frac{\beta\,(k_1 - k_2)^2(k_1^2 + k_2^2)}{6\,\alpha\,(k_1 + k_2^*)^2(k_2 + k_2^*)^2(k_1^2 + k_2^{*^2})(k_2^2 + k_2^{*^2})}. \quad (33)$$

Substituting Eqs. (24), (26), and (31) into Expression (14), we can obtain

$$f_4 = \eta\, e^{\xi_1^* + \xi_2^* + \xi_1 + \xi_2}, \quad (34$$

where

$$\eta = \frac{\beta^2\,(k_1 - k_2)^2(k_1^2 + k_2^2)(k_1^* - k_2^*)^2(k_1^{*^2} + k_2^{*^2})}{36\,\alpha^2\,(k_1 + k_1^*)^2(k_2 + k_1^*)^2(k_1^2 + k_1^{*^2})(k_2^2 + k_1^{*^2})(k_1 + k_2^*)^2(k_2 + k_2^*)^2(k_1^2 + k_2^{*^2})(k_2^2 + k_2^{*^2})}. \quad (35)$$

Correspondingly, it can be obtained that g5 = f6 = g7 = f8 =..= 0 . So Expressions (9a) and (9b) can be truncated into

$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4\,, \quad (36)$$

$$g = \epsilon\, g_1 + \epsilon^3\, g_3\,. \quad (37)$$

With ∈ = 1, two-soliton solutions for System (3) in the explicit form is

$$u = -2\left[\ln(\eta\, e^{\xi_1^* + \xi_2^* + \xi_1 + \xi_2} + z_1\, e^{\xi_1^* + \xi_1} + z_2\, e^{\xi_2^* + \xi_1} + z_3\, e^{\xi_1^* + \xi_2} + z_4\, e^{\xi_2^* + \xi_2} + 1)\right]_{xx}, \quad (38)$$

$$v = \frac{s_1\, e^{\xi_1^* + \xi_1 + \xi_2} + e^{\xi_1} + e^{\xi_2} + s_2\, e^{\xi_2^* + \xi_1 + \xi_2}}{1 + \eta\, e^{\xi_1^* + \xi_2^* + \xi_1 + \xi_2} + z_1\, e^{\xi_1^* + \xi_1} + z_2\, e^{\xi_2^* + \xi_1} + z_3\, e^{\xi_1^* + \xi_2} + z_4\, e^{\xi_2^* + \xi_2}}. \quad (39)$$

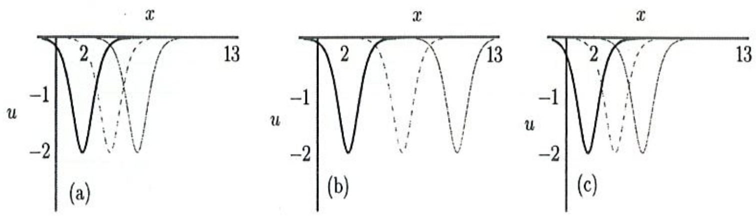

Fig. 1 One-soliton propagation of u via Expression (20) when $t = 0$ (bold solid line), $t = 1$ (dashed line), and $t = 2$ (solid line), (a) with $\alpha = 1$, $\beta = 1$, $k_1 = 1$, $\xi_1^0 = 0$; (b) with $\alpha = 2$, $\beta = 1$, $k_1 = 1$, $\xi_1^0 = 0$; (c) with $\alpha = 1$, $\beta = 2$, $k_1 = 1$, $\xi_1^0 = 0$.

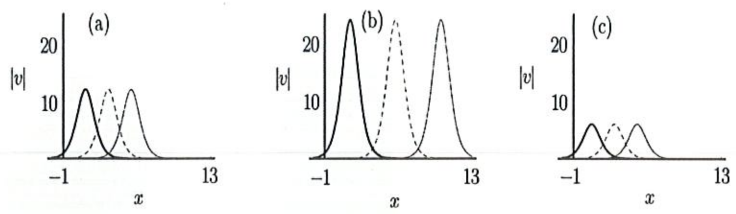

Fig. 2 One-soliton propagation of v via Expression (21) when $t = 0$ (bold solid line), $t = 1$ (dashed line), and $t = 2$ (solid line), (a) with $\alpha = 1$, $\beta = 1$, $k_1 = 1$, $\xi_1^0 = 0$; (b) with $\alpha = 2$, $\beta = 1$, $k_1 = 1$, $\xi_1^0 = 0$; (c) with $\alpha = 1$, $\beta = 2$, $k_1 = 1$, $\xi_1^0 = 0$.

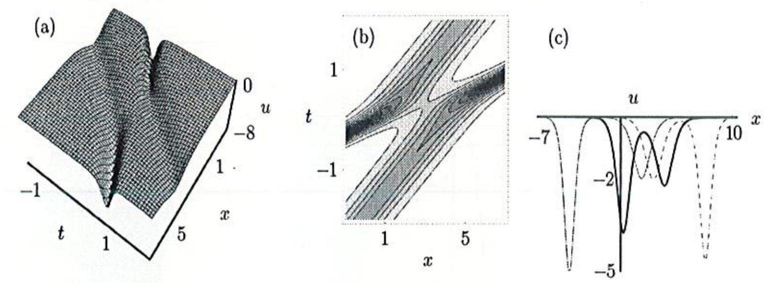

Fig. 3 (a) Two-soliton interactions of u via Expression (38) with $\alpha = 1$, $\beta = 1$, $k_1 = 1$, $k_2 = 1.6$, $\xi_1 = \epsilon = 1$, $\xi_2 = 2$; (b) Contour plot of Fig. 1(a); (c) Wave profiles with $t = -1$ (solid line), $t = 0$ (bold solid line), and $t = 1$ (dashed line).

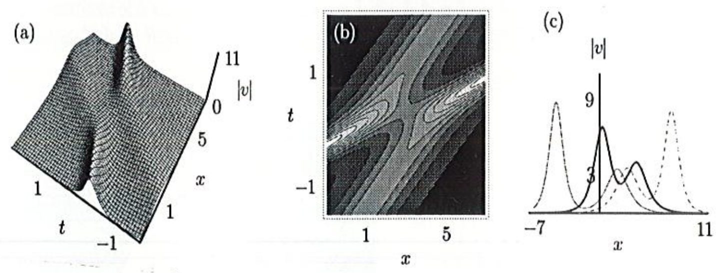

Fig. 4 (a) Two-soliton interactions of v via Expression (39) with $\alpha = 1$, $\beta = 1$, $k_1 = 1$, $k_2 = 1.6$, $\xi_1 = \epsilon = 1$, $\xi_2 = 2$; (b) Contour plot of Fig. 2(a); (c) Wave profiles with $t = -1$ (solid line), $t = 0$ (bold solid line), and $t = 1$ (dashed line).

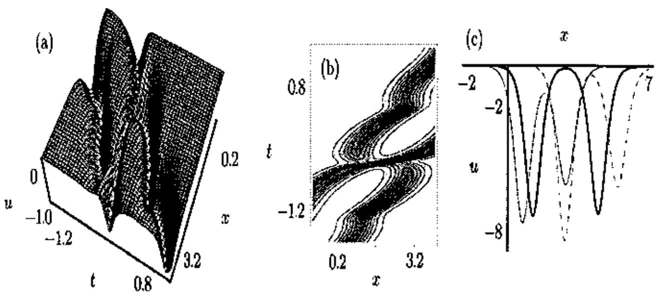

Fig. 5 (a) Two-soliton interactions of u via Expression (38) with $\alpha = 1$, $\beta = 1$, $k_1 = 2 + i$, $k_2 = 1.8 + 0.8i$, $\xi_1 = \xi_2 = ($
$\epsilon = 1$; (b) Contour plot of Fig. 3(a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.$
(dashed line).

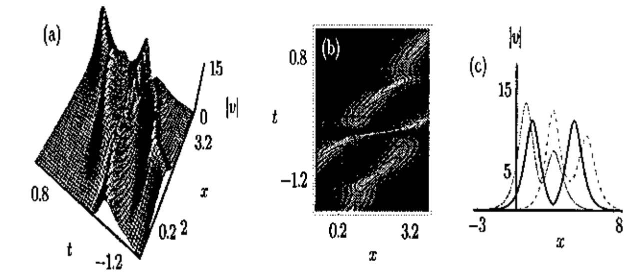

Fig. 6 (a) Two-soliton interactions of v via Expression (39) with $\alpha = 1$, $\beta = 1$, $k_1 = 2 + i$, $k_2 = 1.8 + 0.8i$, $\xi_1 = \xi_2 = ($
$\epsilon = 1$; (b) Contour plot of Fig. (a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.$
(dashed line).

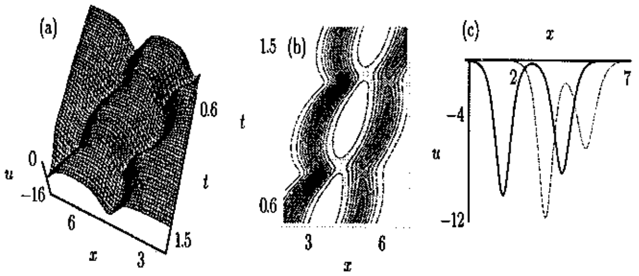

Fig. 7 (a) Two-soliton interactions of u via Expression (38) with $\alpha = 1$, $\beta = 1$, $k_1 = 2 + i$, $k_2 = 2.3 + 1.2i$, $\xi_1 = \xi_2 = ($
$\epsilon = 1$; (b) Contour plot of Fig. 5(a); (c) Wave profiles with $t = 0.1$ (bold solid line), and $t = 0.8$ (solid line).

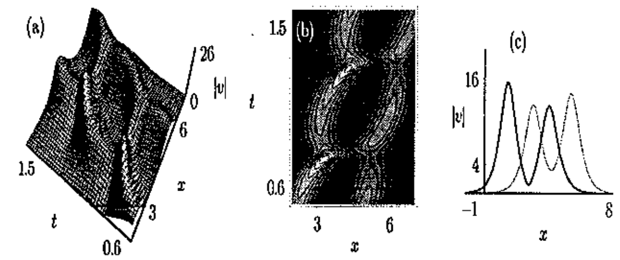

Fig. 8 (a) Two-soliton interactions of v via Expression (39) with $\alpha = 1$, $\beta = 1$, $k_1 = 2 + i$, $k_2 = 2.3 + 1.2i$, $\xi_1 = \xi_2 = ($
$\epsilon = 1$; (b) Contour plot of Fig. 6(a); (c) Wave profiles with $t = 0.1$ (bold solid line), and $t = 0.9$ (solid line).

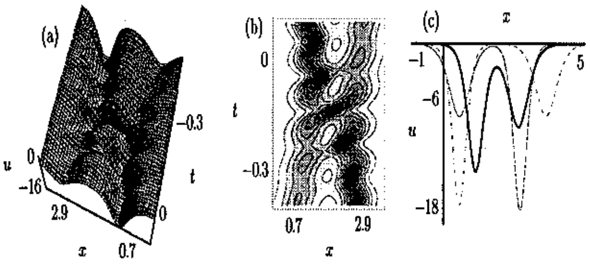

Fig. 9 (a) Two-soliton interactions of u via Expression (38) with $\alpha = 1$, $\beta = 1$, $k_1 = 2+i$, $k_2 = 3+1.8i$, $\xi_1 = \xi_2 = 0$, $\epsilon = 1$; (b) Contour plot of Fig. 7(a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.5$ (dashed line).

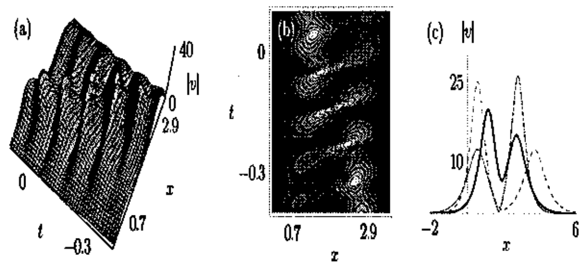

Fig. 10 (a) Two-soliton interactions of v via Expression (39) with $\alpha = 1$, $\beta = 1$, $k_1 = 2+i$, $k_2 = 3+1.8i$, $\xi_1 = \xi_2 = 0$, $\epsilon = 1$; (b) Contour plot of Fig. 8(a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.5$ (dashed line).

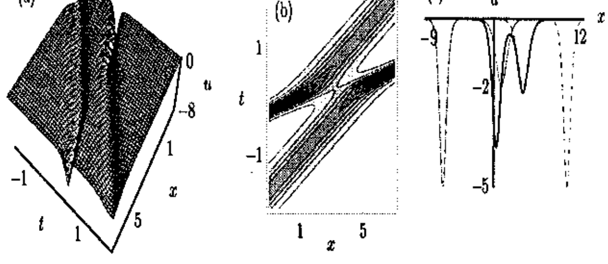

Fig. 11 (a) Two-soliton interactions of v via Expression (38) with $\alpha = 1.5$, $\beta = 1$, $k_1 = 1$, $k_2 = 1.6$, $\xi_1 = \epsilon = 1$, $\xi_2 = 2$; (b) Contour plot of Fig. 11(a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.5$ (dashed line).

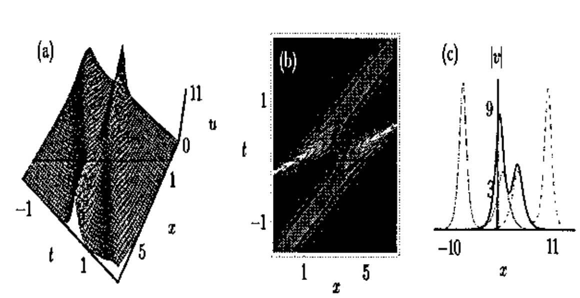

Fig. 12 (a) Two-soliton interactions of v via Expression (39) with $\alpha = 1.5$, $\beta = 1$, $k_1 = 1$, $k_2 = 1.6$, $\xi_1 = \epsilon = 1$, $\xi_2 = 2$; (b) Contour plot of Fig. 12(a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.5$ (dashed line).

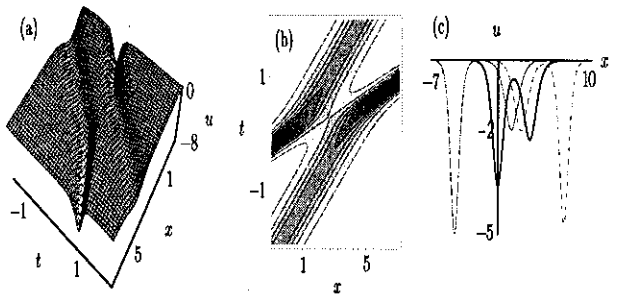

Fig. 13 (a) Two-soliton interactions of v via Expression (38) with $\alpha = 1$, $\beta = 2$, $k_1 = 1$, $k_2 = 1.6$, $\xi_1 = \epsilon = 1$, $\xi_2 = 2$; (b) Contour plot of Fig. 13(a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.5$ (dashed line).

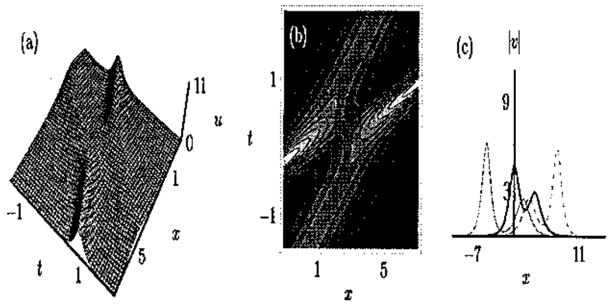

Fig. 14 (a) Two-soliton interactions of v via Expression (39) with $\alpha = 1$, $\beta = 2$, $k_1 = 1$, $k_2 = 1.6$, $\xi_1 = \epsilon = 1$, $\xi_2 = 2$; (b) Contour plot of Fig. 14(a); (c) Wave profiles with $t = -0.5$ (solid line), $t = 0$ (bold solid line), and $t = 0.5$ (dashed line).

3.4 *Numerical Estimations & Discussions*

The following part of this section is devoted to analyzing the two-soliton interactions. Without loss of generality, we suppose that $\alpha = 1$ and $\beta = 1$. Through choosing the different values of the wave numbers $k_1$and k_2 in Eqs. (24), then substituting them into Expressions (38) and (39), we find that the two-soliton propagation trajectories present different forms. When the wave numbers k_1 and k_2 are both real numbers, such as $k_1 = 1$ and $k_2 = 1.6$, two-soliton interactions are described by Figs. 3 and 4. Figures 3 present the two-dark-soliton interactions while Figs. 4 show the bright one. From Figs. 3 and 4, we can find that the two-soliton interactions possess elastic collision features. Two solitons can pass through each other, and their shapes keep unchanged with a phase shift after the separation. The similar phenomena have been investigated in the arterial mechanics.

When the wave numbers $k_1$and k_2 are both complex numbers, we find that the two-soliton interactions present three kinds of different forms. We illustrate those characteristics of interactions with the aid of Figs. 5(6), 7(8), and 9(10). For simplicity, we fix the wave number k1 = 2 + i, only leaving the wave number k_2 to be variable. Figures 5 and 6 show that when the wave number k2 = 1.8+0.8i, two solitons merge and separate from each other nearly twice, then diverge from one another and never interact again.

At the same time, Figs. 5 and 6 also illustrate that the two-soliton interactions have elastic-collision properties, i.e., the shapes of two solitons are variational within the range of interaction and the shapes resume the original ones away from the range of interaction.

When the wave number $k_2 = 2.3 + 1.2i$, with respect to u, the two dark solitons propagation trajectories present the forms of periodic continuous humps, and with respect to v, the two bright solitons propagation trajectories describe the forms of periodic continuous valleys which are shown in Figs. 7(a) and 8(a). During the propagation, two solitons merge and separate from each other periodically. From Figs. 7(c) and 8(c), we can find that two solitons amplitudes increase and decrease alternately. The phenomena that the two bright solitons merge and separate from one another periodically are similar to the cases of bound vector solitons in the optical fibers.

The alternate change of amplitudes is considered as the transformation of energy from one soliton to another. When the wavenumber $k_2 = 3 + 1.8i$, from Figs. 9(a) and 10(a), it can be observed that two-soliton propagation trajectories are fluctuant in the central range of the collision. However, through Figs. 9(b) and 10(b), it is not difficult to find that the fluctuations are analogous before and after the collision except for phase shift. At the same time, we can also find that the shapes resume original ones away from the collision center.

The last part of this section is devoted to analyzing the influence of α and β on the two-soliton interaction, with the real wave numbers. For the influence of α on the two-soliton interaction, comparing Fig. 11(a) [12(a)] with Fig. 3(a) [4(a)], the two-soliton interactions still possess the regular elastic collision features when α turns into 1.5 from 1.

Comparing Fig. 11(c) [12(c)] with Fig. 11(c) [12(c)] with Fig. 3(c) [4(c)], we find that the propagation velocity of two solitons increases with α increasing. For the influence of β on the interaction, when β turns into 2 from 1, comparing Fig. 14(c) [Fig. 13(c)] with Fig. 4(c) [Fig. 3(c)], we find that the amplitude of v clearly decreases, but that of u remains invariable.

CONCLUSION & FUTURE SCOPE

KdV-typed equations have been used in fluids and plasmas. Present work has verified a generalized KdV model. Using the Hirota's bilinear method, we have constructed the bilinear forms and derived the one- and two-soliton solutions of KdV System.

Based on the two-soliton solutions, we have observed that the two-soliton propagation trajectories present different forms by changing the values of the wave numbers k_1 and k_2.

When k_1 and k_2 are both real, for example $k_1 = 1$ and $k_2 = 1.6$, substituting them into Expressions (38) and (39), we can find that the two-soliton interactions possess the regular elastic collision features (as seen in Figs. 3 and 4).

When k_1 and k_2 are complex, we have fixed k_1 = 2+i, only leaving k_2 to be variable and found that the two-soliton propagation trajectories

contain three kinds of patterns: (i) When k_2 = 1.8 + 0.8 i, two solitons merge and separate from one another nearly twice, then diverge from each other and never interact again (as seen in Figs. 5 and 6); (ii) When k_2 = 2.3 + 1.2 i, solitons merge and separate periodically (as seen in Figs. 7 and 8); (iii) When k_2 = 3 + 1.8 i, two-soliton propagation trajectories are fluctuant in the central range of the collision (as seen in Figs. 9 and 10).

Specially, the phenomena that the two bright solitons merge and separate from each other periodically are similar to the bound vector solitons in optical fibers. Meanwhile, we have pointed out the similarity among those complex interaction patterns, which is that the shapes of two solitons resume the original ones away from collision center except for Case (2). Propagation features of the one and two solitons have been investigated with the changes of the α and β.

Comparing Fig. 1(a) [Fig. 2(a), Fig. 11(c), Fig. 12(c)] with Fig. 1(b) [Fig. 2(b), Fig. 3(c), Fig. 4(c)], we have found that the velocities of the one and two solitons increase with α increasing, while the v amplitudes of the one and two solitons increase with α increasing and decrease with β increasing as shown in Fig. 2(a) [Fig. 14(c), Fig. 1(c), Fig. 4(c)].

The results could be expected to be helpful for the study of nonlinear phenomena in fluids and plasmas.

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